DESCRIPTIVE SETS & INFINITE GRAPHS

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August 3, 2023

Abstract. This write-up consists of a crash course in Descriptive Set Theory and a classical application of the concepts therein to infinite graphs, culminating in a proof of the $\mathbb{G}_0$ Dichotomy. Special attention was paid to ensure it would be comprehensible to an undergraduate student with some familiarity with basic topology, as well as to build intuition, even in complex and intricate parts of proofs.

It was completed as part of McGill University's Directed Reading Program, under the supervision of advisor Sam Murray.

1. A Crash Course in Descriptive Set Theory

We assume a knowledge of basic topology (definitions of topological spaces, continuity, etc.) but no knowledge of Descriptive Set Theory for this write-up. As such, this section serves mainly to bring a curious reader with no experience in the field up to speed, and can be safely skipped by those with some basic knowledge. It also attempts to give a curious reader a taste for the particularities of the field and the insights Descriptive Set Theorists can provide.

1.1. **Perfect Polish Spaces.**

In Descriptive Set Theory, we like to work in a very particular kind of topological space known as a *perfect Polish space*. A perfect Polish space $\mathscr{X}$ is a *perfect*, *separable*, *completely metrizable* topological space. That is,

- a *topological space*,
- on which some *metric* may be defined,
- such that that metric is *complete*, i.e. every Cauchy sequence converges,
- that contains a countable dense subset, so is *separable*,
- and that does not count any singletons as open sets, so has no *isolated points*, and is thus *perfect*.

This is a bit of a large definition, but its not any particularly exotic type of space. $\mathbb{R}$, after all, is a perfect Polish space. However, certain properties of $\mathbb{R}$ make it undesirable to Descriptive Set Theorists, particularly the fact that the space is *connected* – it cannot be written as a union of disjoint nonempty open subsets. For this reason, they turn to some slightly strange spaces.

1.1.1. *The Cantor Space.*

Formally, we can define the Cantor space, notated 2^ω, as the topological space with points in 2^ω and with basic open sets $N(2^\omega, s)$ for $s \in 2^{<\omega}$ defined by $N(2^\omega, s) := \{\sigma \in 2^\omega : s \sqsubseteq \sigma\}$.

To understand what that actually means, imagine an infinitely tall binary tree: the root has two children, labeled 0 and 1, each of those children has two children labeled 0 and 1, ad infinitum. If you traveled down from the root to a depth of n, then the list of the labels of each of the nodes you visited on the way down forms an n-length binary string. If you did this forever, you would get an infinitely long binary string.

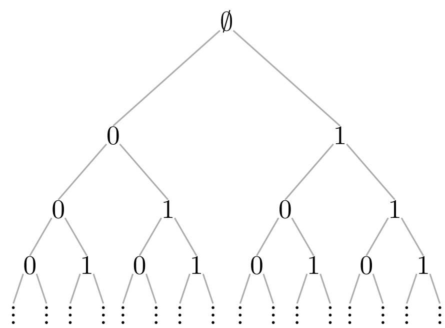


FIGURE 1. The infinite binary tree underlying the Cantor space.

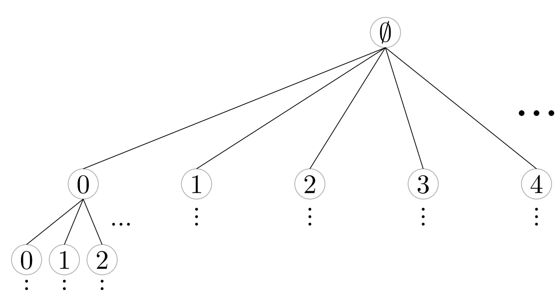
These infinitely long binary strings are in fact the points of our space, and we notate them by 2^ω, with 2 standing in for $\{0,1\}$ and the $\cdot^\omega$ notating that we're considering countably infinite strings (strings indexed by ω).[1] Likewise, we use 2^n to notate the set of binary strings of length n, and $2^{<\omega}$ for finite strings of any length.

To topologize this, for each node, let $s \in 2^{<\omega}$ be the finite string representing the path you take to get to that node. We label the set of all infinite strings that pass through that node

$$N(2^\omega, s) := \{\sigma \in 2^\omega : s \sqsubseteq \sigma\} \subseteq 2^\omega,$$

with $\sqsubseteq$ notating that σ is an extension of s. These are the basic open sets of our space, and so any open set consists of a countable union of these $N(2^\omega, s)$'s.

1.1.2. *The Baire Space.* The Baire space ω^ω is very similar, except instead of taking binary strings we consider strings of natural numbers. Formally, we define it as the space with points in ω^ω and with basic open sets $N(\omega^\omega, x) := \{\xi \in \omega^\omega : x \sqsubseteq \xi\}$ for $x \in \omega^{<\omega}$. This space is extremely important in Descriptive Set Theory, as there is a continuous surjection from the Baire space onto any perfect Polish space, making it in some ways act as a prototype for any other perfect Polish space.



[1]If you have not yet seen ω used like this, think of it as like $\mathbb{N}$. I will also use $i < \omega$ to mean the same as $i \in \omega$. This is because ω is the *first countably infinite ordinal*, ordinals being a sort of extension of $\mathbb{N}$ to infinities. For now, just think of this as a notational quirk.

FIGURE 2. The infinite natural number tree underlying the Baire space.

1.1.3. *Product Spaces.* One nice feature of perfect Polish spaces is that if you take a (countable) product of perfect Polish spaces and give it the natural product topology, this again is a perfect Polish space. This natural topology is exactly what you would expect in the case of finite products—a set is open if and only if it is a product of sets open in their respective spaces—but has a wrinkle in the case of infinite products. In this case, we add the condition that only finitely many of the sets in the product are not trivial, in the sense that they do not contain every point of their space. In other words, where I is countable, $U \subseteq \prod_{i \in I} \mathcal{X}_i$ is open if and only if we can write $U = \prod_{i \in I} U_i$, where $U_i \subseteq \mathcal{X}_i$ is open for all $i \in I$ and $U_i = \mathcal{X}_i$ for all but finitely many $i \in I$.[2]

This gives rise to a few distinctive ways that sets are used in Descriptive Set Theory. The *graph* of a function $f : X \to Y$ is defined as $\{(x, y) : f(x) = y\}$, and it is common for set theorists of all types to identify a function with its graph. By considering the product space, we can consider a function $f : \mathcal{X} \to \mathcal{Y}$ between perfect Polish spaces to be a subset of the product space $\mathcal{X} \times \mathcal{Y}$. In fact, *any relation* between perfect Polish spaces is again a subset of their product space, allowing us to give relations topological properties such as being closed or open.

It also allows us to be somewhat "vague" about the space we are working in. The product space construction ensures that, for the most part, all topological properties of the set S as a subset of $\mathcal{X}$ are equivalent to those of $S \times \mathcal{Y}$ as a subset of $\mathcal{X} \times \mathcal{Y}$. Descriptive Set Theorists will thus often identify these two sets as identical, as they are identical for their purposes.

1.2. **Borel Sets.** Probably the most fundamental concept in Descriptive Set Theory is the concept of Borel sets. These will be our "nice" sets — Borel sets are those sets that can be formed from open sets using some basic operations, and as such they are very well-behaved topologically. If all sets were Borel, the field of Analysis would be a lot simpler.

Formally, in any given perfect Polish space $\mathcal{X}$, the set of Borel sets is the σ-algebra generated by the open sets—i.e, every open set is Borel, compliments of Borel sets are Borel, and countable unions and intersections of Borel sets are Borel.

1.2.1. *Closure Properties of Borel Sets.* The property of being a Borel set (in any arbitrary space) is preserved across a number of operations. In particular, Borel sets are closed under:

- (countable) unions,
- (countable) intersections,
- compliments,
- products,
- continuous inverse images,

[2]If we do not add this condition, we get a topology known as the *box topology.* It coincides with the product topology for finite products of spaces, but diverges in the case of infinite products, having more open sets than the product topology. As the product topology is less fine and still has the properties we want out of a product construction, we consider it the more "natural" construction.

- and a generalized form of continuous inverse images known as continuous substitution, in which for a Borel set $B \supseteq \prod_{i \in I} \mathcal{X}_i$, where $(f_i : \mathcal{X}_i \to \mathcal{Y}_i)_{i \in I}$ are continuous functions, the set
$$\left\{(y_i)_{i \in I} : (f(y_i))_{i \in I} \in B\right\}$$
is Borel.

1.3. **Analytic Sets.** The analytic sets are the Mr. Hyde to the Borel sets' Dr. Jekyll. Originally thought to be one and the same as Borel sets, these sets can be difficult to construct or describe and highly pathological. There are a number of equivalent ways to define them. A set $A \subseteq \mathcal{X}$ is analytic if one of the following equivalent conditions are met:

- It is a continuous image of a Borel set.
- It is either empty or a continuous image of the Baire space.
- It is a continuous image of a closed subset of the Baire space.
- There is a closed set $C \subseteq \omega^\omega \times \mathcal{X}$ such that
$$\text{proj}_{\mathcal{X}}[C] = \{y : \exists x \in \omega^\omega : (x, y) \in C\} = A.$$

1.3.1. *Closure Properties of Analytic Sets.* Analytic sets are closed under some different operations than Borel sets:

- (countable) unions,
- (countable) intersections,
- products,
- continuous images,
- and continuous inverse images.

Note that they are *not* closed under compliments, and indeed by *Suslin's theorem* analytic sets whose compliments are analytic are Borel.

1.4. **Theorems.** We now introduce and prove two basic theorems that will be very important for our journey.

Theorem 1.1. (Separation Theorem). *Let $\mathcal{X}$ be a perfect Polish space, and $A_1, A_2 \subseteq \mathcal{X}$ be two analytic subsets with $A_1 \cap A_2 = \emptyset$. Then there is a Borel set $B \subseteq \mathcal{X}$ such that $B \subseteq A_1$, $A_2 \cap B = \emptyset$.*

Proof. We will notate $N(\omega^\omega, x)$ by N_x in the following proof for brevity.

As discussed above, given that A_1 and A_2 are analytic we may obtain continuous surjections $\varphi : \omega^\omega \to A_1$, $\psi : \omega^\omega \to A_2$. For each $x \in \omega^{<\omega}$, we define $A_1^{(x)} = \varphi[N_x]$ and $A_2^{(x)} = \psi[N_x]$, and consider each pair individually. For $(x, y) \in \omega^n \times \omega^n$ with $n \in \omega$, we call the pair "good" if there exists a Borel $B_{x,y}$ separating $A_1^{(x)}$ and $A_2^{(y)}$ (i.e. $A_1^{(x)} \subseteq B_{x,y}$ and $A_2^{(y)} \cap B_{x,y} = \emptyset$), and "bad" otherwise. Our goal will be to prove that no bad pairs exist.

Note that as $A_1^{(x)}$ is the union of its "children" $A_1^{(x \frown i)}$[3] for $i \in \omega$ and likewise for $A_2^{(y)}$, if every one-step extension $(x \frown i, y \frown j)$ for $i, j \in \omega$ of a pair is good, we then have separations $B_{x \frown i, y \frown j}$ that we can use to construct a separation for $A_1^{(x)}$ and $A_2^{(y)}$:

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[3] $x \frown i$ notates x concatenated with i.

$$\begin{aligned}
A_1^{(x)} &= \bigcup_{i\in\omega} A_1^{(x\frown i)} \subseteq \bigcup_{i\in\omega}\bigcap_{j\in\omega} B_{x\frown i, y\frown j} \\
A_2^{(y)} &= \bigcup_{j\in\omega} A_2^{(t\frown j)} \subseteq \bigcup_{j\in\omega}\left(\bigcup_{i\in\omega} B_{x\frown i, y\frown j}\right)^{\complement} = \bigcup_{j\in\omega}\bigcap_{i\in\omega}\left(B_{x\frown i, y\frown j}\right)^{\complement} \subseteq \bigcap_{i\in\omega}\bigcup_{j\in\omega}\left(B_{x\frown i, y\frown j}\right)^{\complement} \\
&= \left(\bigcup_{i\in\omega}\bigcap_{j\in\omega} B_{x\frown i, y\frown j}\right)^{\complement} \\
\implies A_2^{(y)} &\cap \left(\bigcup_{i\in\omega}\bigcap_{j\in\omega} B_{x\frown i, y\frown j}\right) = \emptyset.
\end{aligned}$$

Contrapositively, if a bad pair (x, y) *did* exist, then there would have to exist some extension $(x \frown i, y \frown j)$ with $i, j \in \omega$ that was also bad. Moreover, at least one parent must be bad. As such, if there are any bad pairs at all then they form a rooted infinite tree; following the bad parents up we get that $(\emptyset, \emptyset)$ must be a bad pair and by following the trail of bad children downwards we find at least one sequence of compatible bad pairs that never ends. Just as we find infinite strings at the "bottom" of infinitely tall trees, by doing the same here we obtain a pair of infinite strings $(\xi, \zeta) \in \omega^\omega \times \omega^\omega$ such that for each $n \in \omega$, $(\xi \restriction n, \zeta \restriction n)$ is bad.[4]

As $A_1 \cap A_2 = \emptyset$, we must have that $\varphi(\xi) \neq \psi(\zeta)$. As $\mathcal{X}$ is metrizable, it is Hausdorff, and so we may obtain open sets U, V such that $\varphi(\xi) \subseteq U$, $\psi(\zeta) \subseteq V$, and $U \cap V = \emptyset$.[5] But then by continuity of φ and ψ, there must be n large enough that

$$\begin{aligned}
A_1^{(\xi\restriction_n)} &= \varphi\left[N_{\xi\restriction_n}\right] \subseteq U \\
A_2^{(\zeta\restriction_n)} &= \psi\left[N_{\zeta\restriction_n}\right] \subseteq V \cap U = \emptyset
\end{aligned}$$

and so U is a Borel set separating $A_1^{(\xi\restriction_n)}$ and $A_2^{(\zeta\restriction_n)}$, contradicting that $(\xi\restriction_n, \zeta\restriction_n)$ is bad! □

Theorem 1.2. (Analytic Perfect Set Theorem). *For any perfect Polish space $\mathcal{X}$ and any analytic subset $A \subseteq \mathcal{X}$, precisely one of the following is true:*

1. *A is countable.*
2. *There is a continuous injection $2^\omega \hookrightarrow A$.*

Proof. Clearly, both cannot be true, as 2^ω is uncountable. We proceed to show one holds.

We will use the fourth equivalent definition of analytic sets from Section 1.3, and so fix a closed set $C \subseteq \omega^\omega \times \mathcal{X}$ such that $\mathrm{proj}_{\mathcal{X}}[C] = A$. We let $(U_i)_{i\in\omega}$ be a countable basis of open sets for $\omega^\omega \times \mathcal{X}$.[6]

[4] $\xi \restriction n$ notates the first n characters of ξ.

[5] More elementarily, we may justify this by recalling $\mathcal{X}$ is metrizable, recovering a metric d, and using it to find open balls with centers $\varphi(\xi)$ and $\psi(\zeta)$ and radius $d(\varphi(\xi), \psi(\zeta))$.

[6] Any separable and metrizable space is *second countable*, meaning it has a countable basis of open sets.

We call a point in a set $S \subseteq \mathcal{X}$ **isolated** if there is an open set U such $U \cap S = \{x\}$, and a set **perfect** if it has no isolated points.[7] The first step of our proof will be to try "sandblasting" away all of the isolated points, leaving a perfect set, which is sometimes called the *perfect kernel*. To do this, we define

$$C^* := C \setminus \bigcup \{U_i : \text{proj}_{\mathcal{X}}[U_i \cap C] \text{ is countable}\}.$$

Noting that $(\text{proj}_{\mathcal{X}}[U_i])_{i\in\omega}$ must be a basis of open sets for $\mathcal{X}$, if $x \in A$ is isolated then some U_i has $\text{proj}_{\mathcal{X}}[U_i \cap C] = \text{proj}_{\mathcal{X}}[U_i] \cap A = \{x\}$.

Now, note that after this process, if $C^* = \emptyset$, then we'll have removed countably many countable sets from C and found ourselves left with nothing, and so C is a countable union of countable sets and thus is itself countable. On the other hand, after this process if $C^* \neq \emptyset$, then we can move on to constructing an injection.

To do this, we recall that ω^ω and $\mathcal{X}$ are both completely metrizable and choose complete metrics $d_{\omega^\omega} : \omega^\omega \times \omega^\omega \to [0, \infty)$ and $d_{\mathcal{X}} : \mathcal{X} \times \mathcal{X} \to [0, \infty)$. We then define

$$\begin{aligned} d : (\omega^\omega \times \mathcal{X}) \times (\omega^\omega \times \mathcal{X}) &\to [0, \infty) \\ ((\xi_1, x_1), (\xi_2, x_2)) &\mapsto \max\{d_{\omega^\omega}(\xi_1, \xi_2), d_{\mathcal{X}}(x_1, x_2)\} \end{aligned}$$

and check that this is also a complete metric on $\omega^\omega \times \mathcal{X}$ that is compatible with its topology. Our goal will be to match up open sets of ω^ω to open sets in $\omega^\omega \times \mathcal{X}$ with uncountable intersection with C^*, and then find our injection using limits.

We start by finding a set to correspond to the open set 2^ω. We choose $y \in C^*$, and choose an arbitrary open ball B with nonzero radius less than 1 and center at y. As this is an open set in $\omega^\omega \times \mathcal{X}$, there must exist $I \subseteq \omega$ such that $\bigcup_{i\in I} U_i = B$. Then $C^* \cap B = C^* \cap \bigcup_{i\in I} U_i = \bigcup_{i\in I}(C^* \cap U_i)$. Each $C^* \cap U_i \neq \emptyset$ must have uncountable projection, as otherwise

$$\begin{aligned} \text{proj}_{\mathcal{X}}[C \cap U_i] &= \text{proj}_{\mathcal{X}}[C^* \cap U_i] \cup \text{proj}_{\mathcal{X}}[(C \setminus C^*) \cap U_i] \\ &= \text{proj}_{\mathcal{X}}[C^* \cap U_i] \cup \text{proj}_{\mathcal{X}}\left[\bigcup\{C \cap U_i : \text{proj}_{\mathcal{X}}(U_i \cap C) \text{ is countable}\}\right] \end{aligned}$$

and so $\text{proj}_{\mathcal{X}}[C \cap U_i]$ is a union of countable sets, and thus also countable, which contradicts that $C^* \cap U_i \neq \emptyset$. As such, B is a union of sets whose intersection with C^* has uncountable projection, and thus also has countable projection after being intersected with C^*. This actually follows for any open set that intersects C^*, and is why finding the perfect kernel was essential.

We then proceed by induction. Let $B_\emptyset := B$ and $y_\emptyset := y$, as well as letting $x_\emptyset$ be the $\mathcal{X}$-coordinate of y. For each $n \in \omega$ and $s \in 2^n$, we assume we have defined B_s, an open ball whose intersection with C^* has uncountable projection. As the projection is uncountable, it certainly contains two distinct points, and so we choose two distinct points $x_{s\frown 0}, x_{s\frown 1} \in \text{proj}_{\mathcal{X}}(C^* \cap B_s)$, then choose arbitrary $y_{s\frown b} \in \text{proj}_{\mathcal{X}}^{-1}[\{x_{s\frown b})\}]$ for $b \in \{0, 1\}$. By choosing them in this way, we have that $\text{proj}_{\mathcal{X}}(y_{s\frown 0}) \neq \text{proj}_{\mathcal{X}}(y_{s\frown 1})$. We then may choose some positive radius $0 < r < 2^{-n}$ and set $B_{s\frown b}$ to be the $\omega^\omega \times \mathcal{X}$ unit ball with center $y_{s\frown b}$ and radius r for $b \in \{0, 1\}$,

[7]Note that this corresponds with the definiton we've previously given in defining perfect Polish spaces when $S = \mathcal{X}$.

choosing r be both small enough that $B_{s\frown b} \subseteq B_s$ for $b \in \{0,1\}$ and smaller than $\frac{1}{3} d_{\mathcal{X}}(x_{s\frown 0}, x_{s\frown 1})$.

Our definition of d then guarantees that the projection of the closure of each ball into $\mathcal{X}$ do not intersect: letting Cl notate closure, $\mathrm{proj}_{\mathcal{X}}[\mathrm{Cl}(B_{s\frown b})] = \{z \in \mathcal{X} : d_{\mathcal{X}}(x_{s\frown b}, x) \leq r\}$ and so for $z_0 \in \mathrm{proj}_{\mathcal{X}}(\mathrm{Cl}(B_{s\frown 0}))$, $d_{\mathcal{X}}(x_{s\frown 0}, z_0) \leq r < \frac{1}{3} d_{\mathcal{X}}(x_{s\frown 0}, x_{s\frown 1})$ and so

$$\begin{aligned} d_{\mathcal{X}}(z_0, x_{s\frown 1}) &\geq d_{\mathcal{X}}(x_{s\frown 0}, x_{s\frown 1}) - d_{\mathcal{X}}(z_0, y_{s\frown 0}) \\ &> 3r - r \\ &= 2r \end{aligned}$$

which implies that $z_0 \notin \mathrm{proj}_{\mathcal{X}}(\mathrm{Cl}(B_{s\frown 1}))$, and likewise switching 0 for 1.

We now define our injection. We do so in two steps; first, we define

$$\begin{aligned} \psi : 2^\omega &\to \omega^\omega \times \mathcal{X} \\ \sigma &\mapsto \lim_{n\to\infty} y_{\sigma\restriction n} \end{aligned}$$

and then from this we define

$$\begin{aligned} \varphi : 2^\omega &\hookrightarrow A \\ \sigma &\mapsto \mathrm{proj}_{\mathcal{X}}(\psi(\sigma)) \end{aligned}$$

We first verify that ψ is well-defined with a simple ε-n_0 argument. For $\varepsilon > 0$ choose $n_0 \in \omega$ with $2^{-n_0} < \varepsilon$. Then for all $n > n_0$, $y_{\sigma\restriction n} \in B_{\sigma\restriction n_0}$ which has radius less than 2^{-n_0}, and so for $n, m > n_0$, $d\big(y_{\sigma\restriction n}, y_{\sigma\restriction m}\big) < 2^{-n_0} < \varepsilon$. By completeness of the metric, the limit then exists. Moreover, as every $y_{\sigma\restriction n} \in C^* \subseteq C$ and C is closed, the limit must be in C, and so its projection must be in $\mathrm{proj}_{\mathcal{X}}[C] = A$. This verifies that the image of φ is contained in A.

We then verify it is an injection. This is also relatively simple; for $\sigma \neq \tau \in 2^\omega$, let n_0 be some index such that $\sigma\restriction_{n_0} \neq \tau\restriction_{n_0}$. Letting Cl denote closure, we then have $\psi(\sigma) \in \mathrm{Cl}\big(B_{\sigma\restriction n_0}\big), \psi(\tau) \in \mathrm{Cl}\big(B_{\tau\restriction n_0}\big)$. As $\mathrm{proj}_{\mathcal{X}}\Big[\mathrm{Cl}\big(B_{\sigma\restriction n_0}\big)\Big] \cap \mathrm{proj}_{\mathcal{X}}\Big[\mathrm{Cl}\big(B_{\tau\restriction n_0}\big)\Big] = \emptyset$, we have that

$$\varphi(\sigma) = \mathrm{proj}_{\mathcal{X}}(\psi(\sigma)) \neq \mathrm{proj}_{\mathcal{X}}(\psi(\tau)) = \varphi(\tau).$$

Finally, we verify this is continuous with a good, old-fashioned, ε-δ argument. Let $\varepsilon > 0$ be arbitrary. We let $d_{2^\omega} : 2^\omega \times 2^\omega \to [0, \infty)$ be the canonical metric on 2^ω, defined by $d_{2^\omega}(\sigma, \tau) := 2^{-\min\{n : \sigma\restriction n \neq \tau\restriction n\}}$. We choose $n_0 \in \omega$ such that $2^{-n_0} < \varepsilon$. Then for $\sigma, \tau \in 2^\omega$ such that $d_{2^\omega}(\sigma, \tau) < 2^{-n_0}$, we have that $s := \sigma\restriction_{n_0} = \tau\restriction_{n_0}$ and so $\psi(\sigma), \psi(\tau) \in \psi[N(2^\omega, s)] = B_s$. As s has length n_0, by our definition B_s has radius less than 2^{-n_0} and so $d(\psi(\sigma), \psi(\tau)) < 2^{-n_0}$. Thus, by our definition of d we have

$$\begin{aligned} \varepsilon > 2^{-n_0} &> d(\psi(\sigma), \psi(\tau)) \\ &\geq d_{\mathcal{X}}(\mathrm{proj}_{\mathcal{X}}(\psi(\sigma)), \mathrm{proj}_{\mathcal{X}}(\psi(\tau))) \\ &= d_{\mathcal{X}}(\varphi(\sigma), \varphi(\tau)), \end{aligned}$$

giving us continuity.

□

2. Infinite Graphs

We now introduce infinite graphs, a favorite topic of discussion for Descriptive Set Theorists. A graph on a set X of vertices can be defined as an irreflexive symmetric relation $G \subseteq X \times X$, where we think of G as being the ordered pairs containing the edges of the graph. This is a little different than the usual definition and notation you see in Graph Theory, but it has some nice properties that will become apparent throughout this write-up.

2.1. **Bestowing graphs with topological properties.** Recall the product space construction from Section 1.1.3. This construction means that a graph G on a perfect Polish space $\mathcal{X}$ is a subset of the perfect Polish space $\mathcal{X} \times \mathcal{X}$ and can be bestowed with topological properties. In fact, properties of graphs on perfect Polish spaces can often be seen as analogues or even generalizations of properties of sets in perfect Polish spaces.

For an example of this, we will consider a theorem that is in some ways a generalization of Theorem 1.1, the Analytic Separation Theorem. For a graph G on $\mathcal{X}$, call a set $S \subseteq \mathcal{X}$ G-independent if for all $x, y \in S$, $(x, y) \notin G$. Note that this is precisely equivalent to the property that $S \times S \cap G = \emptyset$, and so can be thought of as an analog of nonintersection between sets. Then:

Theorem 2.1. *Let G be some analytic graph on $\mathcal{X}$ and $A \subseteq \mathcal{X}$ be an analytic G -independent set. Then, there exists a Borel set $B \subseteq \mathcal{X}$ such that $A \subseteq B$ and B is also G-independent.*

Proof. For disjoint analytic sets A_1, A_2, let $\mathrm{Sep}(A_1, A_2)$ denote the Borel set containing A_1 and disjoint from A_2 that exists by Theorem 1.1. Let $\mathrm{proj}_0 : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ denote the projection to the 0th-coordinate, so $\mathrm{proj}_0[A \times B] = A$.

We start by considering $\mathcal{X}$. It is clearly Borel and contains A, but obviously will not, in general, be G-independent. In any candidate set $B \supseteq A$, there are two ways that $B \times B \cap G$ may fail to be empty:

1. There is an edge in G between $B \setminus A$ and A;
2. There is an edge in G between $B \setminus A$ and $B \setminus A$.

(A is G-independent, so there are no edges in G between A and A.) Our goal is now to eliminate these two types of edges from our cover.

We first handle edges of type 1. We note that each edge of this type appears twice in G, once with the coordinate in A first and once with the coordinate in A last. We use

$$C_1' \coloneqq \mathcal{X} \times A \cap G = (\mathcal{X} \setminus A) \times A \cap G = A^{\complement} \times A \cap G$$

to capture the instance of each type one edge that has its A-coordinate last, and then project it to the $A^{\complement}$-coordinate:

$$C_1 \coloneqq \mathrm{proj}_0[C_1'] = \mathrm{proj}_0[\mathcal{X} \times A \cap G] = \{x \in \mathcal{X} : x \text{ is } G\text{-adjacent to } A.\}$$

We now note that C_1' and thus C_1 are clearly analytic by the closure properties, and so may remove these undesirable elements of $\mathcal{X}$ by separating them:

$$B_1 \coloneqq \mathrm{Sep}(A, C_1)$$

We thus obtain a Borel subset B_1 that still contains A but contains no elements that participate in type one edges.

Now that we have dealt with the type one edges, the type two edges are easy. We take

$$C_2' \coloneqq B_1 \times B_1 \cap G,$$

noting that as we have removed all type one edges, the remaining edges we capture with are all type two, and so neither coordinate is in A. As such, we may take

$$C_2 \coloneqq \mathrm{proj}_0[C_2'] = \{x \in B_1 : x \text{ is } G\text{-adjacent to } B_1\}$$

and be certain that it is disjoint from A. Again, C_2' and thus C_2 is analytic, so we take

$$B \coloneqq \mathrm{Sep}(A, C_2) \cap B_1.$$

This is clearly a Borel set containing A with no elements that are edges of either type, and so we have that B is G-independent. $\square$

We are now ready to introduce the star of this write-up, the $\mathbb{G}_0$ graph.

2.2. **Introducing... The $\mathbb{G}_0$ Graph.** We can formally define the $\mathbb{G}_0$ graph on 2^ω as follows: first, let $D \coloneqq \{\sigma_0, \sigma_1, ...\}$ be a countable dense set in 2^ω, recalling that perfect Polish spaces, including 2^ω, are separable. For each $n \in \omega$, let $s_n \coloneqq \sigma_n \restriction_n$. [8] Note that any given $\sigma \in \omega^\omega$ is "hit" by $(s_n)_{n\in\omega}$ infinitely often, in the sense that for any $n \in \omega$ there is $m > n$ such that $s_m \restriction_n = \sigma \restriction_n$, as this must be true for D to be dense. Now, $(\sigma, \sigma') \in \mathbb{G}_0$ (in other words, σ and σ' are adjacent in $\mathbb{G}_0$) if and only if there exists $n \in \omega$, $\tau \in 2^\omega$, $\{b_0, b_1\} = \{0, 1\}$ such that

$$\sigma = s_n \frown b_0 \frown \tau$$
$$\sigma' = s_n \frown b_1 \frown \tau.$$

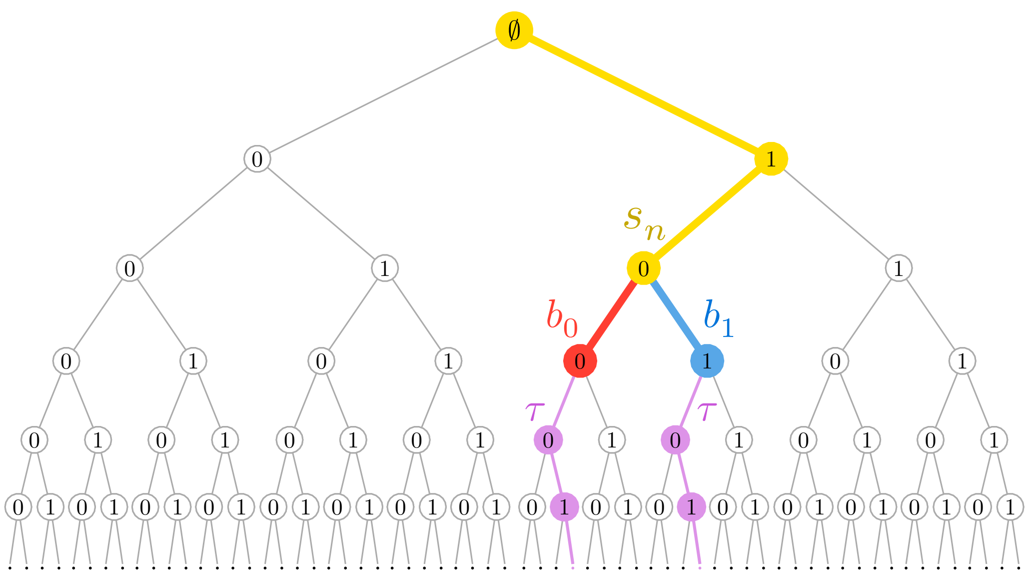

FIGURE 3. An illustration of the conditions for adjacency in $\mathbb{G}_0$; $s_n \frown 0 \frown \tau$ and $s_n \frown 1 \frown \tau$ are adjacent.

[8] $s_0 = \sigma_0 \restriction_0$ is the empty string, $\emptyset$.

In words, we say that σ and σ' are adjacent in $\mathbb{G}_0$ if you can find n such that the first n bits[9] in σ spell out the string s_n, and σ' is identical to σ except the bit at index $n+1$ in σ' has been "flipped" – switched from 1 to 0 or 0 to 1, as compared to σ.

2.2.1. *One interesting property of* $\mathbb{G}_0$. This construction may not seem "natural", to say the least. The conditions for adjacency are pretty strange, not even mentioning the fact that without enumerating a specific dense set, we have not even defined $\mathbb{G}_0$ uniquely.[10] However, there is one property of $\mathbb{G}_0$ that suggests a hidden naturalness to this construction:

Theorem 2.2. *Two vertices $\sigma, \tau \in 2^\omega$ are connected (i.e. there is a path between them) in the $\mathbb{G}_0$ graph if and only if they are* **eventually equivalent**, *i.e. there exists $n_0 \in \omega$ such that for all $n > n_0$, the nth bit of σ is equal to the nth bit of τ.*

Proof. We'll proceed by induction, proving that for all $\tau \in 2^\omega, k \in \omega$, the set

$$C_k(\tau) = \{s \frown \tau : s \in 2^k\}$$

is connected. In the base case, $k = 0$ and so $C_0(\tau) = \{\tau\}$, so we vacuously have connectedness. In the inductive step, assume that $C_k(b \frown \tau)$ is connected for all $\tau \in 2^\omega, b \in \{0,1\}$. We want to show that $C_{k+1}(\tau)$ is connected. Note that

$$\begin{aligned} C_{k+1}(\tau) &= \{s \frown \tau : s \in 2^{k+1}\} \\ &= \{t \frown b \frown \tau : t \in 2^k, b \in \{0,1\}\} \\ &= \{t \frown 0 \frown \tau : t \in 2^k\} \cup \{t \frown 1 \frown \tau : t \in 2^k\} \\ &= C_k(0 \frown \tau) \cup C_k(1 \frown \tau) \end{aligned}$$

and so to show $C_{k+1}(\tau)$ is connected we simply need to find an edge between $C_k(0 \frown \tau)$ and $C_k(1 \frown \tau)$. We can satisfy this with

$$(s_k \frown 0 \frown \tau, s_k \frown 1 \frown \tau) \in \mathbb{G}_0.$$

□

2.3. **Coloring Infinite Graphs.** Once mathematicians have a graph, the first thing they want to do is color it. A **graph coloring in k colors** is usually defined as a function $c : X \to \{1, ..., k\}$ such that no two adjacent vertices $x, y \in X$ have $c(x) = c(y)$. Note that we may equivalently say that a k-coloring of a graph G on X is a partition of X into k sets, each G-independent.

As Descriptive Set Theorists are wont to do, we take this definition, narrow it with topological conditions, and then extend it to infinity. A **Borel ω-coloring** of a graph G on a vertex set X is a partition of X into countably infinitely many sets $\bigsqcup_{i\in\omega} C_i = X$ with each C_i Borel and G-independent.

Now, let $(B_i')_{i\in\omega}$ be a *countable Borel G-independent cover of X*—that is, a countable collection of G-independent Borel sets whose union is X. Then by setting

[9]A bit is a binary digit, i.e. either a 0 or a 1.

[10]Some other definitions do not even specify that $\mathbb{G}_0$ is constructed using an enumeration of a dense set, and instead just fixes some $(s_n)_{n\in\omega}$ with each s_n being of length n and every $\sigma \in \omega^\omega$ being "hit" by an s_n infinitely often in the sense discussed above. I have taken the liberty of specifying that we use a dense set for convenience and concision.

$B_i := B'_i \setminus \bigcup_{j<i} B'_i$ for all $i \in \omega$ we can see that we have constructed a Borel ω-coloring $(B_i)_{i\in\omega}$ according to the definition given in the previous paragraph. For this reason, we take the construction of a Borel G-independent cover as sufficient evidence that G admits a Borel ω-coloring, and sometimes identify the former with the latter.

3. $\mathbb{G}_0$ is not Borel ω-colorable

Now, after learning the definition of Borel ω-colorability, you may have been left wondering whether there are any examples of graphs which are *not* Borel ω-colorable. Perhaps such graphs may defy any simple description like analytic non-Borel sets, or may even be impossible to fully describe, like non-principal ultrafilters. Not so. In fact,

Theorem 3.1. *$\mathbb{G}_0$ is not Borel ω-colorable.*

3.1. **A detour into topology.** To prove this fact, we will need a bit of a detour back into topology.

3.1.1. *Density.* In a topological space $\mathcal{X}$, the closure of a set $S \subseteq \mathcal{X}$ is the intersection of all closed sets containing S, or equivalently the minimal (with regard to $\subseteq$) closed set that contains S. Dually, the interior of a set $S \subseteq \mathcal{X}$ is the union of all of the open sets contained in S, or equivalently the maximal open set contained in S.

Recall that a set $S \subseteq \mathcal{X}$ is dense in $\mathcal{X}$ if its closure is equal to $\mathcal{X}$. Recall that for an open set $U \subseteq \mathcal{X}$, we can topologize U with the subspace topology, in which a set is open in U if it is the intersection of U with a set open in $\mathcal{X}$. Under this definition, we can call a set $S \subseteq \mathcal{X}$ dense in U for U an open set in $\mathcal{X}$ if its closure with regard to the subspace topology on U is equal to U.

We call a set $S \subseteq \mathcal{X}$ nowhere dense if S is not dense in any open set $U \subseteq \mathcal{X}$. Equivalently, a set is nowhere dense if its closure has empty interior.

3.1.2. *Meagre Sets and the Baire Property.* We call a set $S \subseteq \mathcal{X}$ meagre if it is a countable union of nowhere dense sets. Like nowhere-density, meagreness is a measure of smallness and sparcity of a set. Naturally, meagre sets make what is called a σ-ideal — countable unions of meagre sets are meagre and arbitrary subsets of meagre sets are meagre.

We call a set $S \subseteq \mathcal{X}$ comeagre if it is the compliment of a meagre set, or equivalently a countable intersection of sets whose interior is dense. As the dual to meagreness, comeagreness measures "largeness" of a set, and countable intersections and arbitrary supersets of comeagre sets are comeagre.[11]

Now, let $A \triangle B$ be the symmetric difference of A and B, i.e. $A \triangle B = (A \setminus B) \cup (B \setminus A)$. We say a set A has the Baire property and say it is a BP-set if it is an open set away from a meagre set—i.e. there exists an open set U such that $A \triangle U$ is meagre. BP-sets on a given space form a σ-algebra, and so compliments and countable unions and intersections of BP-sets are BP-sets. More-

[11] Despite this being the natural dual to a σ-ideal, I cannot find anyone who calls this a δ-filter.

over, as open sets are clearly also BP-sets (the empty set is meagre), all Borel sets are also BP-sets.

3.1.3. *Baire Spaces.* A Baire space[12] is a topological space in which every meagre set has empty interior, or equivalently, in which every comeagre set is dense. By a theorem known as the *Baire Category Theorem*, any completely metrizable space, including our 2^ω, is a Baire space. Note that this means that a nonempty Baire space cannot be meagre—as a topological space it must be an open set under its own topology, and so equal to its interior, and as a Baire space if the whole space was meagre then that interior would be empty.

3.2. **Proving the property.** With that, we are ready for a very important lemma:

Lemma 3.2. *Any $\mathbb{G}_0$-independent BP-subset of 2^ω is meagre.*

Proof of Lemma. In the following proof, we will notate N_s for $N(2^\omega, s)$. We will proceed by proving the contrapositive: let $S \subseteq 2^\omega$ be nonmeagre and BP. As it is nonmeagre we definitely have that the set is not "small," but we want to find some subspace on which it is "large" enough that it cannot be $\mathbb{G}_0$-independent.

As S is BP, we have that there is some open set U such that $S \triangle U$ is meagre. As U is open, it is a union of countably many basic open sets, and so we can then take some $N_t \subseteq U$, with $t \in 2^{<\omega}$. We can then find some s_n with $t \sqsubseteq s_n$, and so we have $N_{s_n} \subseteq N_t \subseteq U$. Now, $N_{s_n} \setminus S \subseteq S \triangle U$, which is meagre, and so clearly $N_{s_n} \setminus S$ is meagre. If we then consider N_{s_n} as a subspace of 2^ω, then clearly $S \cap N_{s_n} = N_{s_n} \setminus \left(N_{s_n} \setminus S\right)$ is the compliment (with regard to N_{s_n}) of $N_{s_n} \setminus S$, a meagre set, and so $S \cap N_{s_n}$ is comeagre in N_{s_n}. We have found the subspace on which S is large.

Now, to show that S is not $\mathbb{G}_0$-independent, we define an automorphism on N_{s_n}:

$$\begin{aligned} \varphi : \qquad N_{s_n} &\to N_{s_n} \\ s_n \frown 0 \frown \xi &\mapsto s_n \frown 1 \frown \xi \\ s_n \frown 1 \frown \xi &\mapsto s_n \frown 0 \frown \xi. \end{aligned}$$

As it is an automorphism, it must send $S \cap N_{s_n}$, a comeagre set in N_{s_n}, to $\varphi\left(S \cap N_{s_n}\right)$, another comeagre set in N_{s_n}. As they are both comeagre, their intersection must then also be comeagre. As 2^ω, and by extension its isomorphic subspace N_{s_n} is a Baire space, we have that this intersection is dense. And as N_{s_n} is not empty, we have that this intersection is not empty. Unrolling our definitions, if the intersection is not empty then we must have that for some $\xi \in 2^\omega$, $s_n \frown 0 \frown \xi, s_n \frown 1 \frown \xi \in S$. That is precisely the criteria for being $\mathbb{G}_0$-adjacent, and so S contains two $\mathbb{G}_0$-adjacent points, and thus is not $\mathbb{G}_0$-independent. $\square$

With this, we have shown that $\mathbb{G}_0$-independent subsets are small, too small for their union to amount to the whole space, and so the proof follows almost immediately:

[12]I underline "A" to emphasize that I am referring to the property of a space being a Baire space, and not the space ω^ω which we call the Baire space. These are two distinct concepts; a space can be a Baire space without being isomorphic to the Baire space. Thankfully though, the Baire space is in fact a Baire space.

Proof of Theorem 3.1. From here, the proof is straightforward. Finding a Borel ω-coloring of a graph is precisely the same as finding a way of partitioning its vertices into countably many independent Borel sets, each set being a color class. Supposing this is possible for $\mathbb{G}_0$, then as Borel sets are BP, by Lemma 3.2 each of these color classes would be meagre. Thus, 2^ω would be a countable union of meagre sets, and thus would also be meagre. As 2^ω is a Baire space, this means that the interior of 2^ω, which is simply 2^ω again, would be the empty set, implying that $2^\omega = \emptyset$, which is absurd. $\square$

4. The $\mathbb{G}_0$ Dichotomy

We have thus shown that $\mathbb{G}_0$ is not Borel ω-colorable. Neat! But that is not the end. There is a reason we are studying $\mathbb{G}_0$ and not any other graph which is not Borel ω-colorable. We will spend the rest of the write-up proving the property of $\mathbb{G}_0$ I am referring to here, after giving just one more definition:[13] for two graphs G on $\mathcal{X}$ and H on $\mathcal{Y}$, a graph homomorphism is a function $\varphi : \mathcal{X} \to \mathcal{Y}$ such that for $x_1, x_2 \in \mathcal{X}$, if $(x_1, x_2) \in G$, then $(\varphi(x_1), \varphi(x_2)) \in H$.

4.1. Statement.

With that, we introduce the $\mathbb{G}_0$ dichotomy:

Theorem 4.1. ($\mathbb{G}_0$ dichotomy). *For any analytic graph G on ω^ω, precisely one of the following is true:*

1. *G is Borel ω-colorable.*
2. *There exists a continuous graph homomorphism $(2^\omega, \mathbb{G}_0) \to (\omega^\omega, G)$.*

This is the reason we are studying $\mathbb{G}_0$; it is not just any non-Borel-ω-colorable graph, it is the *ur*-non-Borel-ω-colorable graph, the *minimal* non-Borel-ω-colorable graph, and all other such graphs are created in its image. This was first shown by Kechris, Solecki, and Todorcevic in [1].

We can immediately prove that *1* and *2* cannot be true simulatenously from our previous theorem:

Proof of $\neg(1 \wedge 2)$ in Theorem 4.1 . We demonstrate that G cannot admit a continuous graph homomorphism $\varphi : 2^\omega \to \omega^\omega$ and simulatenously be Borel ω-colorable. If we let $(B_0)_{i\in\omega}$ be the coloring, then $(\varphi^{-1}[B_i])_{i\in\omega}$ would constitute a Borel ω-coloring of $\mathbb{G}_0$. We can see this as Borel sets remain Borel under continuous inverse images, and each $\varphi^{-1}[B_i]$ must be $\mathbb{G}_0$ independent. Otherwise we would have some $\sigma_1, \sigma_2 \in \varphi^{-1}[B_i]$ such that $(\sigma_1, \sigma_2) \in \mathbb{G}_0$, implying that $(\varphi(\sigma_1), \varphi(\sigma_2)) \in G$, and thus as $\varphi(\sigma_1), \varphi(\sigma_2) \in B_i$, and thus that B_i is not G-independent, a contradiction. As this implies that $\mathbb{G}_0$ is Borel ω-colorable, this contradicts Theorem 3.1. $\square$

4.2. Goal.

Showing that if *1* is not true, *2 is* will be tougher, and we'll need a strategy—not only to complete the construction of φ, but to prove that what we construct will satisfies our requirements when we're done. We follow the proof outline in [2] and given by Conley, Lecomte, and Miller in [3].

As G is analytic, it is a continuous image of the Baire space, and so we fix a continuous surjection $\Psi : \omega^\omega \to G \subseteq \omega^\omega \times \omega^\omega$. Our goal will be to construct a con-

[13] This is a lie, the following proof contains many definitions, but at the very least you are permitted to forget those as soon as the proof concludes.

tinuous function $\varphi : 2^\omega \to \omega^\omega$ and *witnesses* $\gamma_n : 2^\omega \to \omega^\omega$ for $n \in \omega$; for each $n \in \omega$ and $\tau \in 2^\omega$, γ_n will witness that φ is preserving the edge

$$(s_n \frown 0 \frown \tau, s_n \frown 1 \frown \tau) \in \mathbb{G}_0$$

by satisfying that

$$(\varphi(s_n \frown 0 \frown \tau), \varphi(s_n \frown 1 \frown \tau)) = \Psi(\gamma_n(\tau)) \in G.$$

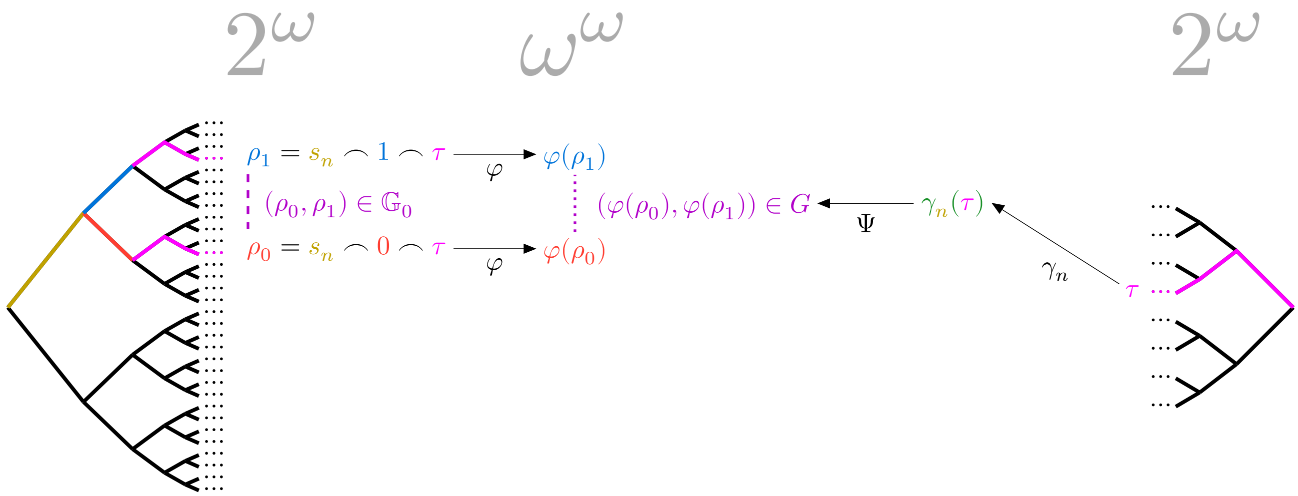

FIGURE 4. A visual of how γ_k witnesses the homomorphism of $\mathbb{G}_0$ into G. We identify two elements of ω^ω with an edge between them in $\mathbb{G}_0$, and send them into ω^ω via φ. We take the s_n and τ that induce this edge, and expect $\gamma_n(\tau)$ to encode the corresponding edge in G via the embedding Ψ.

4.3. **Approximations & Realizations.** We will construct our φ and γ_k's by building up to this goal with more and more detailed finite approximations. In service of this, we define a few different structures.

First, define an (n-)approximation to be a tuple of the form

$$a = \left(n, f^{(a)}, (g_k)_{0 \le k \le n-1}\right)$$

with $f^{(a)} : 2^n \to \omega^n$ acting as a a finite approximation of φ, and each $g_k : 2^{n-(k+1)} \to \omega^n$ acting as a finite approximation of γ_k for $0 \le k \le n-1$. We then say that an approximation $a' = \left(n+1, f^{(a')}, \left(g_k^{(a')}\right)_k\right)$ is a one-step extension of an approximation $a = \left(n, f^{(a)}, \left(g_k^{(a)}\right)_k\right)$ if

1. $f^{(a)}(s) = \left(f^{(a')}(s \frown b)\right) \restriction_n$ for all $s \in 2^n$, $b \in \{0,1\}$;
2. $g_k^{(a)}(s) = \left(g_k^{(a')}(s \frown b)\right) \restriction_n$ for all $0 \le k \le n$, $s \in 2^{n-(k+1)}$, $b \in \{0,1\}$.

These conditions essentially force compatibility, so an extension is a more elaborated version of the approximation it extends.

Second, we define an ($n^{(\alpha)}$-)realization as a tuple

$$\alpha = \left(n^{(\alpha)}, \varphi^{(\alpha)}, \left(\gamma_k^{(\alpha)}\right)_{0 \le k \le n^{(\alpha)}-1}\right)$$

with $n^{(\alpha)} \in \omega$, $\varphi^{(\alpha)} : 2^n \to \omega^\omega$, and $\gamma_k^{(\alpha)} : 2^{n-k-1} \to \omega^\omega$ for $0 \le k \le n-1$ that satisfies the constraint that for all $0 \le k \le n-1$, for all $t \in 2^{n-k-1}$,

$$\left(\varphi^{(\alpha)}(s_k \frown 0 \frown t), \varphi^{(\alpha)}(s_k \frown 1 \frown t)\right) = \Psi\left(\gamma_k^{(\alpha)}(t)\right). \tag{$*$}$$

For an n-approximation a, we say that a realization α realizes a if the two are compatible in the sense that

1. $n^{(\alpha)} = n$;
2. $(\forall s \in 2^n)\ f^{(a)}(s) \sqsubseteq \varphi^{(\alpha)}(s)$;
3. $(\forall 0 \le k \le n-1)\ \left(\forall t \in 2^{n-k-1}\right)\ g_k^{(a)}(t) \sqsubseteq \gamma_k^{(\alpha)}(t)$.

The way it extends an approximation's outputs into ω^ω and is bound by Equation ($*$) is the reason why the above structure is called a *realization*—it realizes it in the full Baire space rather than a finite simulacrum of it. We also note that given a realization α, we may derive the approximation a by setting $n \coloneqq n^{(\alpha)}$, $f^{(a)}(s) \coloneqq \varphi^{(\alpha)}(s) \restriction_n$, and $g_k^{(a)}(t) \coloneqq \gamma_k^{(\alpha)}(t) \restriction_n$. We call a the approximation derived from α.

4.3.1. *Realizations within Y.* Let $Y \subseteq \omega^\omega$ be arbitrary, at least for now. We call say that a realization α is a realization within Y if the image of $\varphi^{(\alpha)}$ is contained within Y. If α realizes a and is a realization within Y, we say that it realizes a within Y, and for an approximation a admitting a realization within Y we say that a is realizable within Y.

4.4. **The Y-Lemma.** We now come to what I call *the Y-lemma*. This lemma is extremely important to make the proof of Theorem 4.1 work, but is also very hard to motivate, seeming extremely arbitrary on first blush. For this reason, we introduce the lemma now, will motivate it when we use it, and will prove it at the end of the write-up. Behold, the Y-lemma:

Lemma 4.2. (The Y-Lemma). *For any graph G on ω^ω, (precisely) one of the following is true:*

1. *G is Borel ω-colorable.*
2. *There exists some Borel $Y \subseteq \omega^\omega$ such that for all $n \in \omega$, for any n-approximation a that is realizable within Y, for all $s \in 2^n$,*

$$A_s^{(a)}(Y) \coloneqq \left\{\varphi^{(\alpha)}(s) : \alpha \text{ realizes } a \text{ within } Y\right\}$$

is not G-independent.

For now, recall Theorem 1.2, the Analytic Perfect Set Theorem, in which an analytic set is either countable or admits continuous injection from 2^ω. That theorem shares some direct parallels to Theorem 4.1, the $\mathbb{G}_0$ dichotomy, with Borel ω-colorability being analogous to countability and $\mathbb{G}_0$ analogous to 2^ω. The first step of the proof of Theorem 1.2 was to show that either the analytic set was countable or it had some *perfect kernel* we could find by sandblasting off counterexamples. The Y-lemma will serve a similar purpose in our proof of the $\mathbb{G}_0$ dichotomy, showing that if G is not Borel ω-colorable then we should be able to find some subset of ω^ω

in which the edges are dense enough that a property useful for constructing our homomorphism emerges.

4.5. **Proof of the $\mathbb{G}_0$-dichotomy.** We are now finally ready for proof of the $\mathbb{G}_0$-dichotomy.

Proof of $\neg 1 \Rightarrow 2$ *in Theorem 4.1.* We treat Lemma 4.2, "the Y-lemma", as a surprise tool that will help us later, and invoke it to obtain a Y satisfying its criteria. Our current goal is to find $(a_n)_{n\in\omega}$ such that each a_n is an n-approximation realizable within Y, and each a_{n+1} is a one-step extension of a_n. To do this, we proceed by induction.

In our base case, we set a_0 to the trivial 0-approximation, defined by

$$f^{(a_0)} : 2^0 = \{\emptyset\} \to \omega^0 = \{\emptyset\}$$
$$\emptyset \mapsto \emptyset$$

and $\left(g_k^{(a_0)}\right)_{0\le k\le n-1} = \emptyset$. By choosing arbitrary $\xi \in Y$, we may construct a realization α_0 within Y by

$$\varphi^{(\alpha_0)} : 2^0 \to \omega^\omega$$
$$\emptyset \mapsto \xi,$$

thus proving that a_0 is realizable within Y.

Now, we consider the inductive step moving from n to $n+1$. We will be given an n-approximation a_n realizable within Y and attempt to construct an approximation $n+1$-approximation realizable within Y. To do this, we will need to take a closer look at approximations and realizations.

Consider the approximation a_n. By identifying $s \in 2^n$ with the basic open set $N(2^\omega, s)$ and likewise with $x \in \omega^n$ and $N(\omega^\omega, x)$ we can imagine $f^{(a_n)} : 2^n \to \omega^n$ as a "fuzzy" version of the $\varphi : 2^\omega \to \omega^\omega$ we are trying to construct that sends neighborhoods to neighborhoods rather than points to points. We imagine two neighborhoods $r_1, r_2 \in 2^n$ as having a "fuzzy edge" between them if for all $\tau \in 2^\omega$, $r_1 \frown \tau$ is adjacent to $r_2 \frown \tau$ in $\mathbb{G}_0$. Just as each edge in $\mathbb{G}_0$ corresponds to an s_k and a tail $\tau \in 2^\omega$, each fuzzy edge ends up corresponding to an s_k and a tail $t \in 2^{n-k-1}$, and $g_k^{(a_n)}$ maps these edges to neighborhoods of witnesses in ω^n.

When we go from a_n to its realization α_n, it is like choosing one point in each neighborhood $f^{(a_n)}(s)$ to map $\varphi^{(\alpha_n)}(s)$ for each string s, as if the outputs of $f^{(a_n)}$ have suddenly snapped into focus. The $\gamma_k^{(\alpha_n)}$'s are likewise tasked with finding witnesses in the neighborhoods outlined by the $g_k^{(a_n)}$'s to map each fuzzy edge to. With their outputs now concrete points in ω^ω and specific witnesses, we can check that our realization is valid. We constrain all realizations to satisfy Equation $(*)$: $\gamma_k^{(\alpha_n)}(t)$ should witness the edge in G corresponding to the fuzzy one between $s_k \frown 0 \frown t$ and $s_k \frown 1 \frown t$ for $t \in 2^{n-k-1}$ by identifying an element of ω^ω that corresponds to it via the surjection $\Psi : \omega^\omega \to G$.

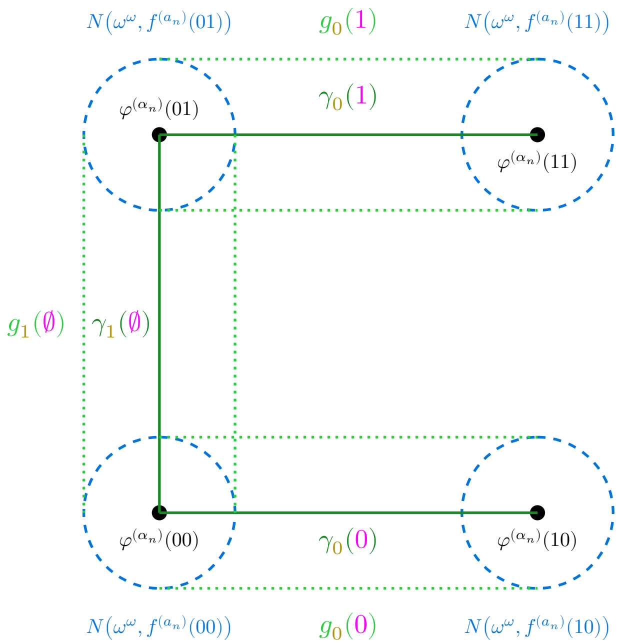

FIGURE 5.
A visualization of the approximation and realizations a_n and α_n in the Baire space, when $n = 2$ and $s_0 = \emptyset$, $s_1 = 0$. The dashed blue circles and dotted light green shapes represent the fuzzy neighborhoods and edges of a_n, while the black dots and dark green lines represent points and edges of the realization. The label on the dark green edges gives the element of ω^ω that the edge is witnessed with, while the light green fuzzy edges are labeled with the neighborhood of potential witnesses identified by the approximation.

In going from a_n to a_{n+1}, each neighborhood splits: for each $N\big(\omega^\omega, f^{(a_n)}(s)\big)$ identified by a_n, we identify $N\Big(\omega^\omega, f^{(a_{n+1})}(s \frown b)\Big)$ for $b = 0, 1$ in a_{n+1}. If r_1 and r_2 had a fuzzy edge in a_n, then $r_1 \frown b$ and $r_2 \frown b$ have a fuzzy edge for $b = 0, 1$ in a_{n+1}, effectively doubling the number of fuzzy edges. Finally, there is one brand new fuzzy edge, between $s_n \frown 0$ and $s_n \frown 1$. To construct a realization α_{n+1} of a_{n+1}, then, we need to choose double the number of points and find witnesses for one more than double the number of edges.

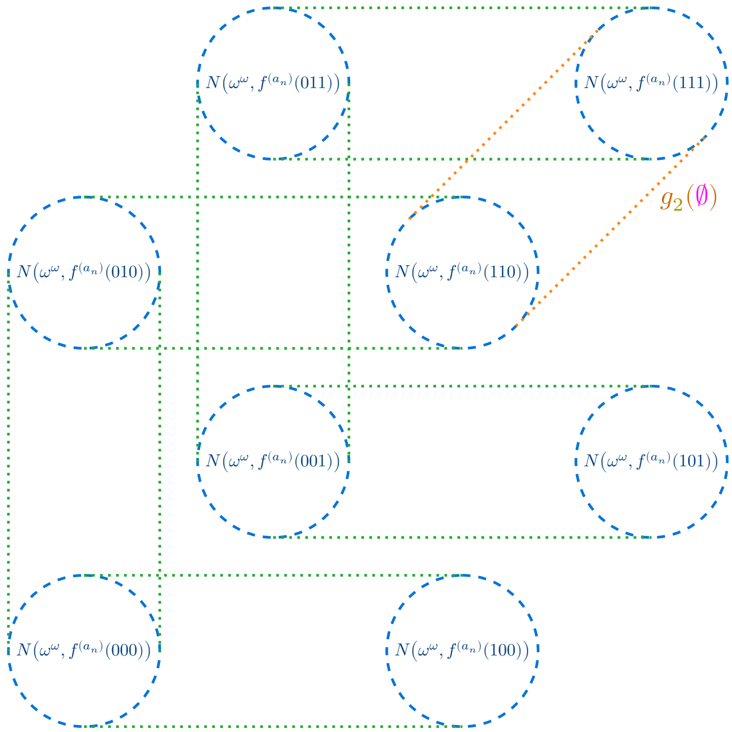

FIGURE 6. A visualization of a_{n+1} in the Baire space when $n = 2$, $s_0 = \emptyset$, $s_1 = 0$, $s_2 = 10$. Note how each fuzzy edge in a_n (see Figure 5) corresponds to two here, as well as the brand new one, colored in orange, which corresponds to the edge between $s_n \frown 0 \frown \emptyset$ and $s_n \frown 1 \frown \emptyset$.

How do we find double the number of points and edges? By starting with two distinct realizations! The Y-lemma (Lemma 4.2) tells us that the Y we choose at the beginning of this proof is such that $A^{(a_n)}_{s_n}(Y)$ is not G-independent, which in particular means that there are at least two distinct realizations within Y of a_n. Letting α_n, α'_n be some pair of distinct realizations in $R^{(a_n)}(Y)$, we can construct

$$\begin{aligned} \varphi^{(\alpha_{n+1})} :\ & 2^{n+1} \to \omega^\omega \\ & s \frown 0 \mapsto \varphi^{(\alpha_n)}(s) \\ & s \frown 1 \mapsto \varphi^{(\alpha'_n)}(s) \end{aligned}$$

and

$$\begin{aligned} \gamma_k^{(\alpha_{n+1})} :\ & 2^{n-k} \to \omega^\omega \\ & t \frown 0 \mapsto \gamma_k^{(\alpha_n)}(t) \\ & t \frown 1 \mapsto \gamma_k^{(\alpha'_n)}(t). \end{aligned}$$

Each neighborhood has split in two, and we have delegated the copy ending with 0 to be handled by α_n and the copy ending with 1 to be handled by α'_n. We can check that this satisfies Equation $(*)$ for $0 \leq k \leq n-1$.

All that remains is to define $\gamma_n^{(\alpha_{n+1})} : 2^{n+1-n-1} = 2^0 = \{\emptyset\} \to \omega^\omega$. For this definition to satisfy Equation $(*)$, we need that

$$\Big(\varphi^{(\alpha_{n+1})}(s_n \frown 0), \varphi^{(\alpha_{n+1})}(s_n \frown 1)\Big) = \Big(\varphi^{(\alpha_n)}(s_n), \varphi^{(\alpha'_n)}(s_n)\Big) = \Psi\Big(\gamma_n^{(\alpha_{n+1})}(s_n)\Big).$$

This is where the final property given by Lemma 4.2 comes in. As $A_{s_n}^{(a_n)}(Y)$ is not G-independent, we may choose α_n and α'_n such that $\varphi^{(\alpha_n)}(s_n)$ and $\varphi^{(\alpha'_n)}(s_n)$ are G-adjacent. We then take some

$$\xi \in \Psi^{-1}\big[\Big(\varphi^{(\alpha_n)}(s_n), \varphi^{(\alpha'_n)}(s_n)\Big)\big]$$

and set

$$\begin{aligned} \gamma_n^{(\alpha_{n+1})} : 2^0 &\to \omega^\omega \\ \emptyset &\mapsto \xi. \end{aligned}$$

We thus have constructed an $n+1$-realization α_{n+1} within Y, and may derive the corresponding $n+1$-approximation a_{n+1}, which is obviously realizable (by α_{n+1}) within Y. It is easy to check that a_{n+1} is then also a one-step extension of a_n, completing the induction.

We now have defined $(a_n)_{n\in\omega}$ and $(\alpha_n)_{n\in\omega}$ as desired, with the former consisting of one-step extensions, each realizable, and the latter being their corresponding realizations. We then define

$$\begin{aligned} \varphi : 2^\omega &\to \omega^\omega \\ \sigma &\mapsto \lim_{n\in\omega} f^{(a_n)}(\sigma\restriction_n) \\ &= \lim_{n\in\omega} \varphi^{(\alpha_n)}(\sigma\restriction_n) \end{aligned}$$

and likewise define

$$\begin{aligned} \gamma_k : 2^\omega &\to \omega^\omega \\ \tau &\mapsto \lim_{n\in\omega} g_{k(\tau\restriction_{n-k-1})}^{(a_n)} \\ &= \lim_{n\in\omega} \gamma_{k(\tau\restriction_{n-k-1})}^{(\alpha_n)} \end{aligned}$$

noting that in each definition the first limit is a limit of compatible finite strings that increase in length to infinity, while the second is a topological limit, using the topology on ω^ω. It is easy to verify that these coincide.

All that remains is to verify that φ is continuous and that the witnesses work correctly. Let $\sigma \in 2^\omega$ be arbitrary and consider $\xi = \varphi(\sigma)$. Let $M \subseteq \omega^\omega$ be an arbitrary neighborhood of ξ. By definition of a neighborhood, we can find an open U such that $\xi \in U \subseteq M$, and then by definition of openness in a metric space we have that there is some open ball $N(\omega^\omega, \xi\restriction_n)$ such that $N(\omega^\omega, \xi\restriction_n) \subseteq U \subseteq M$. Then for the neighborhood $N(2^\omega, \sigma\restriction_n)$, we have that

$$\begin{aligned}\varphi[N(2^\omega, \sigma\restriction_n)] &= \{\varphi((\sigma\restriction_n)\frown\tau) : \tau\in 2^\omega\}\\ &\subseteq \left\{f^{(a_n)}(\sigma\restriction_n)\frown\zeta : \zeta\in\omega^\omega\right\}\\ &= N\big(\omega^\omega, f^{(a_n)}(\sigma\restriction_n)\big)\\ &= N(\omega^\omega, \varphi(\sigma)\restriction_n)\\ &= N(\omega^\omega, \xi\restriction_n)\\ &\subseteq M\end{aligned}$$

We thus have continuity.

Finally, we check the witnesses. For $n\in\omega$, $\tau\in 2^\omega$, we have that

$$\begin{aligned}(\varphi(s_n\frown 0\frown\tau), \varphi(s_n\frown 1\frown\tau)) &= \Big(\lim_{i\in\omega}\varphi^{(\alpha_i)}(s_n\frown 0\frown\tau\restriction_i), \lim_{i\in\omega}\varphi^{(\alpha_i)}(s_n\frown 1\frown\tau\restriction_i)\Big)\\ &= \lim_{i\in\omega}\big(\varphi^{(\alpha_i)}(s_n\frown 0\frown\tau\restriction_i), \varphi^{(\alpha_i)}(s_n\frown 1\frown\tau\restriction_i)\big)\\ &= \lim_{i\in\omega}\Psi\Big(\gamma_n^{(\alpha_i)}(\tau\restriction_{i-n-1})\Big)\\ &= \Psi\Big(\lim_{i\in\omega}\gamma_n^{(\alpha_i)}(\tau\restriction_{i-n-1})\Big)\\ &= \Psi(\gamma_n(\tau)).\end{aligned}$$

This completes the proof. □

4.6. **Proving the Y-lemma.** Our last remaining goal is to prove the Y-lemma. In order to do this, we need to prove one more (sub)lemma:

Lemma 4.3. *For all $n\in\omega$, $s\in 2^n$, n-approximations a, and $Y\subseteq\omega^\omega$,*

$$A_s^{(a)}(Y) := \left\{\varphi^{(\alpha)}(s) : \alpha \textit{ realizes } a \textit{ within } Y\right\}$$

is analytic.

An easy way to prove this is to find some reasonable representation of realizations in a perfect Polish space such that

$$R^{(a)}(Y) := \{\alpha : \alpha \text{ realizes } a \text{ within } Y\}$$

is Borel, and such that the operation $\alpha\mapsto\varphi^{(\alpha)}(s)$ is a projection. We then end have that $A_s^{(a)}(Y)$ is a continuous image of a Borel set, and thus analytic. Descriptive Set Theorists will usually take this step for granted, as it is fairly trivial to them that the definition of a realization given in Section 4.3 admits such a representation.

For the sake of demonstrating an explicit example of such a representation and how to construct one and for those who are new to Descriptive Set Theory and do not find this totally intuitive, we reproduce these details in the following section. Those willing to take Lemma 4.3 for granted may skip to Section 4.6.2.

4.6.1. *Topologizing realizations.* We start by finding a reasonable represenation of n-realizations in some product space, recalling the product space construction in Section 1.1.3. Fixing n, we note that each $\varphi^{(\alpha)} : 2^n\to\omega^\omega$ may be fully specified with 2^n distinct elements of ω^ω, and likewise for each $0\le k\le n-1$, $\gamma_k^{(\alpha)} : 2^{n-k-1}\to\omega^\omega$ may be fully specified with 2^{n-k-1} elements of ω^ω. As such, by fixing some assignment of coordinates, each α may be represented as an element of the product space formed by producting ω^ω with itself $2^n + \sum_{k=0}^{n-1} 2^{n-k-1} = 2^{n+1} - 1$ times, which we will notate by $(\omega^\omega)^{2^{n+1}-1}$.

We then fix some such arbitrary assignment of coordinates and notate these with $[\![\cdot]\!]$ in such a manner that for any $\alpha \in (\omega^\omega)^{2^{n+1}-1}$, we have:

- For $s \in 2^n$, $\mathrm{proj}_{[\![\varphi,s]\!]}(\alpha) = \varphi^{(\alpha)}(s)$;
- For $0 \le k \le n-1$, $t \in 2^{n-k-1}$, $\mathrm{proj}_{[\![\gamma,k,t]\!]}(\alpha) = \gamma_k^{(\alpha)}(t)$.

With our representation specified, we are ready to prove that $R^{(a)}(Y)$ is Borel:

Lemma 4.4. *For any n-approximation a and $Y \subseteq \omega^\omega$, $R^{(a)}(Y)$ is Borel.*

Proof of Lemma 4.4. We recall that for a tuple

$$\alpha = \left(n, \varphi^{(\alpha)}, \left(\gamma_k^{(\alpha)}\right)_k\right)$$

the conditions for it to realize a are that

1. $n^{(\alpha)} = n$;
2. $(\forall s \in 2^n)\ f^{(a)}(s) \sqsubseteq \varphi^{(\alpha)}(s)$;
3. $(\forall 0 \le k \le n-1)\ \left(\forall t \in 2^{n-k-1}\right)\ g_k^{(a)}(t) \sqsubseteq \gamma_k^{(\alpha)}(t)$;
4. $(\forall 0 \le k \le n-1)\ \left(\forall t \in 2^{n-k-1}\right)\ \left(\varphi^{(\alpha)}(s_k \frown 0 \frown t), \varphi^{(\alpha)}(s_k \frown 1 \frown t)\right) = \Psi\left(\gamma_k^{(\alpha)}(t)\right)$.

The first is irrelevant as n is fixed, but we are going to have to find topological ways of ensuring that *2*, *3* and *4* are satisfied.

For *2* and *3*, we note that for $x \in \omega^n$, $\xi \in \omega^\omega$, $x \sqsubseteq \xi$ is equivalent to $\xi \in N(\omega^\omega, x)$, where $N(\omega^\omega, x)$ is the basic open set of ω^ω consisting of all strings extending x, as discussed in Section 1.1.2. As such, we can reprhase the former as

$$\begin{aligned} &(\forall s \in 2^n)\ f^{(a)}(s) \sqsubseteq \varphi^{(\alpha)}(s) \\ \iff &(\forall s \in 2^n)\ \mathrm{proj}_{[\![\varphi,s]\!]}(\alpha) = \varphi^{(\alpha)}(s) \in N\big(\omega^\omega, f^{(a)}(s)\big) \\ \iff &\alpha \in \bigcap \mathrm{proj}^{-1}_{[\![\varphi,s]\!]}\big[N\big(\omega^\omega, f^{(a)}(s)\big)\big] \qquad (2') \end{aligned}$$

and the latter as

$$\begin{aligned} &(\forall 0 \le k \le n-1)\ \left(\forall t \in 2^{n-k-1}\right)\ g_k^{(a)}(t) \sqsubseteq \gamma_k^{(\alpha)}(t) \\ \iff &(\forall 0 \le k \le n-1)\ \left(\forall t \in 2^{n-k-1}\right)\ \mathrm{proj}_{[\![\gamma,k,t]\!]}(\alpha) \in N\left(\omega^\omega, g_k^{(a)}(t)\right) \\ \iff &\alpha \in \bigcap_{0 \le k \le n-1} \bigcap_{t \in 2^{n-k-1}} \mathrm{proj}^{-1}_{[\![\gamma,k,t]\!]}\left[N\left(\omega^\omega, g_k^{(a)}(t)\right)\right] \qquad (3') \end{aligned}$$

We note that in each case our restrictions are intersections of inverse projections of basic open sets, and so both respresent Borel sets.

For *4* we need to further invoke relations. As discussed earlier, any relation on perfect Polish spaces lives in a product space. The only relation we need for this is the equalty relation, $\mathrm{Eq}_{\mathcal{X}} = \{(x, x) : x \in \mathcal{X}\}$. We can see that this relation is Borel, as we may construct it by finding a metric $d_{\mathcal{X}} : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ and then taking $\mathrm{Eq}_{\mathcal{X}} = d_{\mathcal{X}}^{-1}[\{0\}]$, noting that d is continuous.

We will sometimes denote $(x, y) \in \mathrm{Eq}_{\mathcal{X}}$ by $\mathrm{Eq}_{\mathcal{X}}(x, y)$. We then recall that Borel sets are closed under continuous substitution; that is, if f, g are continuous, and

$R \in \mathcal{X} \times \mathcal{X}$ is a Borel relation, then $\{(x, y) : R(f(x), g(y))\}$ is also Borel. This finally allows us to state *4* topologically:

$$
\begin{aligned}
&(\forall 0 \leq k \leq n-1)\left(\forall t \in 2^{n-k-1}\right)\left(\varphi^{(\alpha)}(s_k \frown 0 \frown t), \varphi^{(\alpha)}(s_k \frown 1 \frown t)\right) = \Psi\Big(\gamma_k^{(\alpha)}(t)\Big) \\
\iff &(\forall 0 \leq k \leq n-1)\left(\forall t \in 2^{n-k-1}\right)\Big(\mathrm{proj}_{[\![\varphi, s_k \frown 0 \frown t]\!]}(\alpha), \mathrm{proj}_{[\![\varphi, s_k \frown 1 \frown t]\!]}(\alpha)\Big) = \Psi\Big(\mathrm{proj}_{[\![\gamma,k,t]\!]}(\alpha)\Big) \\
\iff &\alpha \in \left\{\alpha : \bigcap_{0 \leq k \leq n-1} \bigcap_{t \in 2^{n-k-1}} \mathrm{Eq}\Big(\Big(\mathrm{proj}_{[\![\varphi, s_k \frown 0 \frown t]\!]}(\alpha), \mathrm{proj}_{[\![\varphi, s_k \frown 1 \frown t]\!]}(\alpha)\Big), \Psi\Big(\mathrm{proj}_{[\![\gamma,k,t]\!]}(\alpha)\Big)\Big)\right\} \qquad (4')
\end{aligned}
$$

as the projections and Ψ are all continuous functions, this property of continuous substitution gives us that this last restriction is also Borel.

Our last restriction is that of being a realization **within $\boldsymbol{Y}$**. This is trivial, as this restriction simply requires that every coordinate corresponding to an output of φ lies within Y, which may be enforced by intersecting with $\prod_{i \in 2^{n+1}-1} Z_i$ where Z_i is equal to Y when $i \in \{[\![\varphi, s]\!] : s \in 2^n\}$ and ω^ω otherwise.

Intersecting this restriction with our sets for $2'$, $3'$, $4'$, we will have constructed $R^{(a)}(Y)$ from Borel sets and Borel-preserving operations, proving that $R^{(a)}(Y)$ is Borel. ☐

With that, we may show Lemma 4.3 trivially:

Proof of Lemma 4.3. For a an n-approximation, we translate our logical definition of $A_s^{(a)}(Y)$ to a topological definition:

$$
\begin{aligned}
A_s^{(a)}(Y) &= \left\{\varphi^{(\alpha)}(s) : \alpha \text{ realizes } a \text{ within } Y\right\} \\
&= \left\{\mathrm{proj}_{[\![\varphi, s]\!]}(\alpha) : \alpha \in R^{(a)}(Y)\right\} \\
&= \mathrm{proj}_{[\![\varphi, s]\!]}\left[R^{(a)}(Y)\right]
\end{aligned}
$$

Then, immediately from the above rewriting, Lemma 4.4, definition of analytic sets, and continuity of projections, we have that $A_s^{(a)}(Y)$ is analytic. ☐

4.6.2. *Proof of the Y-lemma.* We are now finally ready to prove the Y-lemma:

Proof of Lemma 4.2. We let $P(Y)$ be the set of all approximations realizable within Y. As discussed in Section 4.4, we will follow a similar strategy to that used in the proof of Theorem 1.2, in which we sandblast off counterexamples, until either nothing or something is left.

We let $Y_0 = \omega^\omega$ in our base case. In our step case, we given Y_ι, and if for all $a \in P(Y_\iota)$, $s \in 2^n$, $A_s^{(a)}(Y_\iota)$ is not G-independent, we set $Y := Y_\iota$ and halt; we have accomplished our goal. Otherwise, we choose a counterexample $a_\iota \in P(Y_\iota)$, $s \in 2^{<\omega}$ such that $A_s^{(a_\iota)}(Y_\iota)$ is G-independent. By Lemma 4.3 we have that this set is analytic, and so we use Theorem 2.1 to find a Borel $B_\iota \subseteq \omega^\omega$ such that $A_s^{(a_\iota)}(Y_\iota) \subseteq B_\iota$ and B_ι is G-independent. We then set $Y_{\iota+1} := Y_\iota \setminus B_\iota$ and continue.

Aside from halting due to finding a Y that satisfies our requirements, there is a second condition under which this process halts; if $P(Y_\iota) = \emptyset$. As the trivial 0-approximation $(0, \emptyset \mapsto \emptyset, \emptyset)$ always has a Y-realization so long as $Y \neq \emptyset$, then this occurs if and only if $Y_\iota = \emptyset$.

Now, note that if on step ι we have not halted, then we have found $a_\iota \in P(Y_\iota)$ and a B_ι such that $A_{s(Y_\iota)}^{(a_\iota)} \subseteq B_\iota$. We thus have that the image of every realization in Y_ι of a_ι intersects B_ι, and so is not a $Y_\iota \setminus B_\iota = Y_{\iota+1}$-realization. As such,

$a_\iota \notin P(Y_{\iota+1})$. The upshot of this is that any particular approximation may only appear once in the sequence $(a_\iota)_\iota$ that our procedure is constructing, and so it is enumerating some subset of P, the set of all n-approximations. As P is only countably large (each n-approximation is fully determined by $2^{n+1}-1$ elements of ω) we are able to put a bound on the number of steps this takes.

Unfortunately, we are not able to show that this process halts after finitely many steps. Worse, the traditional induction trick of taking $Y_\omega := \bigcap_{\iota\in\omega} Y_\iota$ does not work here either: while $(a_\iota)_{\iota\in\omega}$ will certainly enumerate some subset of P, it is entirely possible that $P \setminus \{a_\iota : \iota \in \omega\}$ is not empty and thus that P_ω contains yet more counterexamples. Fortunately, with *transfinite induction* we may remedy this.

An introduction to transfinite induction is out of scope for this write-up, but we may think of is ω as like the step number representing the limit of all steps whose number is finite, a step that you can never get to in any finite amount of time. ω is a *limit ordinal*, and for any such limit ordinal λ, we assume that we have completed every step $\iota < \lambda$ and set $Y_\lambda := \bigcap_{\iota<\lambda} Y_\iota$, allowing us to continue inducting.[14]

Now that we have extended this induction transfinitely, we can be certain that it halts—maybe not at a finite step, or even step ω, but it certainly must stop before ω_1, the *first uncountable ordinal*. Otherwise, we would have an enumeration $(a_\iota)_{\iota\in\omega_1}$ of a subset of P indexed by an uncountable ordinal, which would imply that P is uncountably large, and we know that it is not. As such, we can be sure that at some ordinal $\eta < \omega_1$, we either have that Y_η fulfils our requirements or that $\emptyset = Y_\eta = \omega^\omega \setminus \bigcup_{\iota\in\eta} B_\eta$. Then as $\bigcup_{\iota\in\eta} B_\iota = \omega^\omega$ and each set in the union is Borel and G-independent, this constitutes a Borel ω-coloring of G, which we have assumed G does not admit. As such, this is a contradiction, and we may assume that our process ends by finding a satisfactory Y.

Finally, as Y is formed by taking ω^ω and subtracting a countable union of Borel sets, Y is Borel. □

4.7. **Bonus Round.** The canonical version of The $\mathbb{G}_0$ Dichotomy is a little more general than the one given in Theorem 4.1:

Theorem 4.5. ($\mathbb{G}_0$ Dichotomy, General Statement). *For any perfect Polish space $\mathcal{X}$, for any analytic graph G on $\mathcal{X}$, precisely one of the following is true:*

1. *G is Borel ω-colorable.*
2. *There is a continuous graph homomorphism $(2^\omega, \mathbb{G}_0) \to (\mathcal{X}, G)$.*

This follows directly from our Theorem 4.1, using a property of ω^ω we will state but not prove here:

Proof. Given any $\mathcal{X}$, we may fix a continuous surjection $\psi : \omega^\omega \hookrightarrow \mathcal{X}$ such that for some closed $C \subseteq \omega^\omega$, $\psi\restriction_C : C \to \mathcal{X}$ is a bijection.

We then clearly have that

$$\begin{aligned} \rho : C \times C &\to \mathcal{X} \times \mathcal{X} \\ (x_1, x_2) &\mapsto (\psi(x_1), \psi(x_2)) \end{aligned}$$

[14]If this is the first time you have seen transfinite induction you may find this strange or hand-wavy, but this form of induction is in fact very rigorous.

is a bijection. We then set $G' := \rho^{-1}[G]$ and obtain G', an analytic graph on ω^ω, and apply Theorem 4.1.

In the case that G' is Borel ω-colorable, we let $(B'_i)_{i\in\omega}$ be such a coloring, and obtain a Borel ω-coloring of G by $B_i := \psi[B'_i \cap C]$ for $i \in \omega$. Each of these sets is Borel because isomorphic images of Borel sets are Borel, and each is independent as if $\psi(\xi_1)$ and $\psi(\xi_2)$ are adjacent in G with $\psi(\xi_1), \psi(\xi_2) \in B_i$, then $(\xi_1, \xi_2) = \rho^{-1}(\psi(\xi_1), \psi(\xi_2)) \in \rho^{-1}[G] = G'$.

Otherwise, we obtain a continuous graph homomorphism $\varphi : 2^\omega \to \omega^\omega$ of $\mathbb{G}_0$ into G'. $\psi \circ \varphi : 2^\omega \to \mathcal{X}$ is then also continuous. Moreover, if $(\sigma_1, \sigma_2) \in \mathbb{G}_0$, then as φ is a graph isomorphism we have $(\varphi(\sigma_1), \varphi(\sigma_2)) \in G'$. Finally, by definition of G' we have that $(\psi \circ \varphi(\sigma_1), \psi \circ \varphi(\sigma_2)) = \rho(\varphi(\sigma_1), \varphi(\sigma_2)) \in \rho(G') = G$ and so $\psi \circ \varphi$ is a continuous graph homomorphism of $\mathbb{G}_0$ into G. $\square$

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