Introductory
Category Theory, Grothendieck Topology, and Presheaves

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Abstract

This write up includes a discussion of basic categorical concepts from an introductory perspective, ranging from objects and morphisms to functors and natural transformations. There is then an introduction of some useful categorical constructions, namely the limit and some of its examples (product, pullback) and the notions of opposites and categorical duals. I then take a brief detour to cover the basics of a topology, ending with a subsequent introduction and discussion of Grothendieck topology and presheaves in category theory and topology drawing the necessary links between the corresponding topological and categorical notions. This report was compiled for the 2023 Winter term Directed Reading Program at McGill University. Many thanks to my DRP mentor Lucy Grossman for guidance throughout the term in exploring these topics, to Alexander Grothendieck for inventing the topic that was the focus of my project, to Samuel Eilenberg and Saunders MacLane for basically inventing the whole field of Category theory and writing a good book[5] on the subject, to Emily Riehl for more recently writing another good book[6] on the subject, and to Richard E. Borcherds for giving me a nicely palatable introduction to the basics of Category theory via his online lecture series "Categories for the Idle Mathematician"[2]. Most of what follows, unless specifically cited, I absorbed; amalgamated; and regurgitated in basically my own style drawing from the various sources listed above.

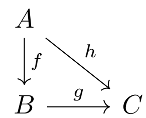
Contents

1 Basics of Category Theory:

1.1 Commutative Diagrams

We draw a lot of pictures to illustrate and prove things in Category Theory. We often will call these commutative diagrams;

Definition 1. A *commutative diagram*, such as:



commutes iff $h = g \circ f$, i.e. any path with the same start and end objects (here A, C respectively) in a commutative diagram should be equivalent.

1.2 Categories

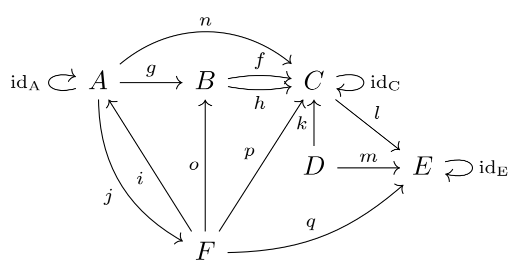
Definition 2. A *category* $\mathcal{C}$ is a collection of *objects* $\mathrm{Obj}(\mathcal{C})$ and *morphisms* $\mathrm{Mor}(\mathcal{C})$, where

(i) composition of morphisms is a sensible notion,

(ii) all objects $X \in \mathrm{Obj}(\mathcal{C})$ have a corresponding identity morphism $\mathrm{id}_{\mathrm{X}} \in \mathrm{Mor}(\mathcal{C})$, and

(iii) for any $X, Y, Z \in \mathrm{Obj}(\mathcal{C})$, $f : X \to Y, g : Y \to Z \in \mathrm{Mor}(\mathcal{C})$ implies $\exists h = g \circ f : X \to Z \in \mathrm{Mor}(\mathcal{C})$ i.e. a category is closed under such composition of morphisms.

A sample (small) category can be interpreted from the below commutative diagram:



(given some assumptions about unwritten identity morphisms and identification of certain arrows with composition morphisms). If you wish, a nice exercise is looking through this sample category and figuring out what these conditions are.[1]

So, at their heart, categories are essentially just a collection of stuff along with relationships between those things that satisfy these special properties. In a commutative diagram (as above), morphisms are typically referred to as arrows.

In summary, we have the main elements that are required to form a category $\mathcal{C}$:

(i) Objects

(ii) Morphisms (arrows between objects)

(iii) A sensible notion of composition of morphisms, which must also be in $\mathcal{C}$

(iv) Identity morphisms exist for all objects in $\mathcal{C}$

[1] each object X should have an identity id_{X}; for commutativity of the whole diagram, $fg = hg = n$, $\exists r = ln = qj : A \to E$, $ri = q$, $gi = o$, $ni = p$, $pj = n$, $p = ho = fo$, $lk = m$, $q = lp$, $ji = \mathrm{id}_{\mathrm{F}}$, $ij = \mathrm{id}_{\mathrm{A}}$ is a probably non-exhaustive list of examples; if commutativity is not imposed then the following composed morphisms all must exist somewhere in the category, but are not necessarily equal.

Remark 1. The general point is to be able to ignore the internal structure of the objects in a category, instead only dealing with the more abstract objects and morphisms between them.

Remark 2. We also have some passing morphism terminology:

- *automorphism* (map to the self), $f : A \to A$
- *homomorphism* (structure-preserving map between structures of the same type), $f : A \to B : A, B \in \mathrm{Obj}(\mathcal{C})$
- *isomorphism* (bijective/invertible homomorphism),
- *monomorphism* (injective homomorphism),
- *epimorphism* (surjective homomorphism)

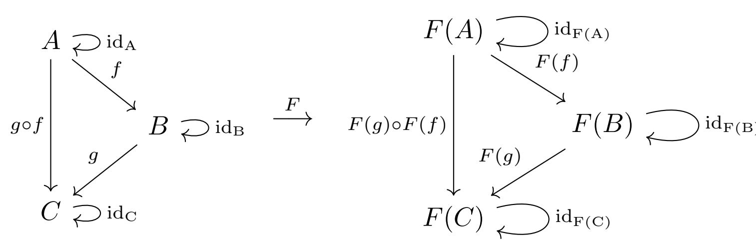
1.3 Functors

Definition 3. If we have two categories $\mathcal{C}, \mathcal{D}$ and some map:

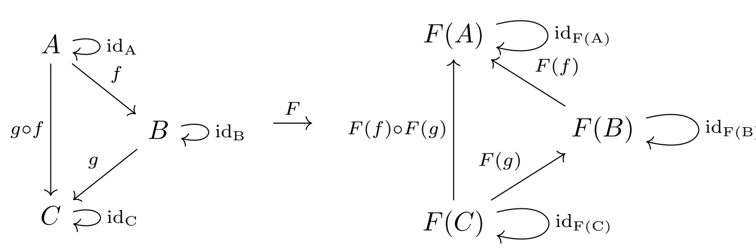
$$\mathcal{C} \xrightarrow{F} \mathcal{D} \tag{1}$$

where F preserves identity and composition of morphisms in the manner of $F(\mathrm{id}_{\mathcal{C}}) = \mathrm{id}_{\mathcal{D}}$ and $F(g \circ f) = F(g) \circ F(f)$, then F is a (covariant) *functor*.

Functors are, in essence, structure preserving maps between categories; they send objects to objects and morphisms to morphisms. A handy visualization of the structure-conserving property is given below, with a (covariant) functor acting on our favourite commutative triangle:



Remark 3. The above-pictured functor is said to be *covariant*, as it preserves arrow direction. If a functor reverses the direction of arrows (and order of morphism composition), then we say it is *contravariant*, i.e.



Note that the contravariant functor can also be defined as its corresponding covariant functor, where one of the categories is made opposite (I'll discuss opposite categories shortly), i.e. if $F : \mathcal{C} \to \mathcal{D}$ is a contravariant functor, then $G : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ and $H : \mathcal{C} \to \mathcal{D}^{\mathrm{op}}$ can both define covariant versions of F.

Remark 4. We can have categories whose objects are categories, and whose morphisms are functors. We could define isomorphism of categories, where if $F : C \to D$ and $G : D \to C$ then $G \circ F = \mathrm{id}_{\mathcal{C}}$ and $F \circ G = \mathrm{id}_{\mathcal{D}}$, but supposedly this is not so useful (it turns out to be too restrictive of a notion).

Rather, we say that functors F, G are equivalences of categories C, D when, if $F : C \to D$ and $G : D \to C$, then $G \circ F \simeq \mathrm{id}_{\mathcal{C}}$ and $F \circ G \simeq \mathrm{id}_{\mathcal{D}}$ (these $G \circ F, F \circ G$ being isomorphic to the respective identity morphisms rather than strictly equal to them).

Remark 5. A common example of a functor is a *forgetful functor*, which takes some collection of objects with additional structure between them and forgets that structure. An example of this is the forgetful functor on the category of groups, which sends each group to its underlying set by "forgetting" the group structure. For this example, there is also a "free group functor" which takes a set S and turns it into the free group on that set.[2] Some other forgetful functors are those which forget the vector space or topological structures in the categories of vector spaces and topological spaces, respectively.

1.4 Natural Transformations

Definition 4. A *natural transformation* is a map between two functors, which conserve the structure of morphism composition. Particularly, if we have *functor categories*, where:

1. objects are some collection of functors,
2. morphisms are the natural transformations between these functors;

then we can for instance take $X, Y \in \mathcal{C}$ with $\mathcal{C}, \mathcal{D}$ categories and $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ functors (and thus objects of a functor category), and if $\exists f : X \to Y$;

$$\begin{array}{c} (F : \mathcal{C} \to \mathcal{D}) \\ \big\downarrow N \\ (G : \mathcal{C} \to \mathcal{D}) \end{array} \quad \text{(where the internal structures of } \mathcal{C}, \mathcal{D} \text{ are preserved as in:)} \quad \begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \big\downarrow N & & \big\downarrow N \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

where if this diagram commutes, then N is a natural transformation.

Remark 6. In fact, natural transformations can be used as the morphisms of a functor category whose objects are functors, as depicted on the left above.

1.5 Higher Category Theory

Different "levels" of categories exist and can be delineated. For instance,

(i) A (small) 0-category can be viewed as a set

(ii) A 1-category is a category whose morphisms form a 0-category

(iii) A 2-category is a category whose morphisms form a 1-category[3]

(iv) and we can generalize further from this construction inductively to construct n-categories with n levels of different morphisms to care about.

1.6 Limits

Definition 5. An object is a *universal object* if it has the *universal property*, which means that it is unique up to a unique isomorphism. In the following diagram demonstrating the definition of a limit, this is represented by the existance of some unique (up to unique isomorphism) map $u : N \to L$, for any other $N, \psi_X, \psi_Y \in D$ which also satisfies the commutative diagram.

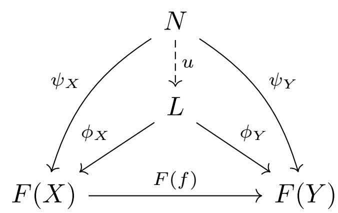
Definition 6. A *limit* is one such universal object. If $F : \mathcal{C} \to \mathcal{D}$ is a functor, and $X, Y \in \mathcal{C}$, for instance, with $f : X \to Y$ a morphism in $\mathcal{C}$, then the limit L is the object in $\mathcal{D}$ which has maps to each element in $\mathcal{D}$. That is to say that there always exists a map $\phi_X : L \to F(X)$ and $\phi_Y : L \to F(Y)$ such that $F(f) \circ \phi_X = \phi_Y$ for L with the universal property, for any such X, Y, f. Note, however, this limit only exists if the category contains limits, which is not necessarily true; it is not always the case that such an object

[2] a free group being the group formed by treating the elements of a set as its generators, and such that the only relations that exist between its generators are those of elements with their inverses (particularly that $\forall g \in G, g \cdot g^{-1} = g^{-1} \cdot g = \mathrm{id_G}$; the group operation applied between an element and its inverse is equal to the group identity element).

[3] and in which we care about this 1-categorical nature, to look inside at the composition of the different levels of morphisms.

and set of morphisms exist.

Generally, if there exists a limit in the target category, we have a family of morphisms ϕ_i that map from some defining universal object of the limit L to all other objects in the target category of the functor F, forming a series of commuting "cones" as pictured.[4]



Definition 7. A *diagram* of shape J in a category $\mathcal{C}$ is a covariant functor $F : J \to \mathcal{C}$; the category J can be viewed as an index category of $\mathcal{C}$, where F indexes both objects and morphisms in $\mathcal{C}$ by objects in J.

Definition 8. Given a diagram $F : J \to \mathcal{C}$, a *cone* to F is an object $N \in \mathcal{C}$ along with a collection of indexed morphisms $\phi_{X_i} : N \to F(X_i)$, where for any other morphism $f : X_i \to X_j \in J$ we have $F(f) \circ \phi_{X_i} = \phi_{X_j} : N \to F(X_j)$.

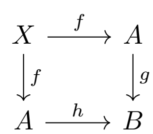
Remark 7. We can see that under these definitions a limit in a category $\mathcal{C}$ is a universal cone of some diagram $F : J \to \mathcal{C}$ from an index category J, i.e. a cone with the universal property.

Example 1. An example of a limit is an *equalizer*; a universal object X with morphism $f : X \to A$, such that if:

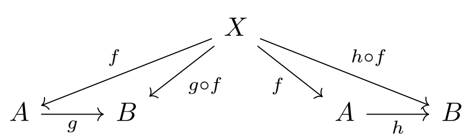
$$X \xrightarrow{f} A \underset{h}{\overset{g}{\rightrightarrows}} B \qquad (2)$$

then $h \circ f = g \circ f$.

Particularly, if we expand this diagram, we get the commutative square:



where although it is not pictured, we can keep in mind that X is here a universal object[5]. Using closure under composition of morphisms, this can be further expanded to:



When the category $\mathcal{C} \ni X, A, B$ we are working with is viewed as the target of a diagram from some index category J, we can therefore see the equalizer (X, f) to be an example of a limit construction based on its diagrammatic nature as a collection of (in this example, two) universal cones.

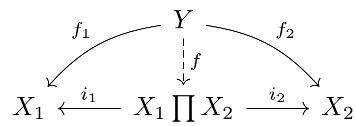
1.6.1 Products

Definition 9. Consider a category C with X_1, X_2 objects in C. Then for any object Y and morphisms $f_1 : Y \to X_1$, $f_2 : Y \to X_2$, the *product* is an object $X_1 \prod X_2$ such that for some (projection) morphisms $i_1 : X_1 \prod X_2 \to X_1$, $i_2 : X_1 \prod X_2 \to X_2$, there exists a unique morphism $f : Y \to X_1 \prod X_2$ such that $f_1 = i_1 \circ f$ and $f_2 = i_2 \circ f$.

i.e. the diagram

[4]this diagram alone should commute; the overall diagram which includes all elements of the category C should have each of its triangles (cones) commuting.

[5]and notice for later that (X, f, f) is also an example of a pullback of (A, A) on B

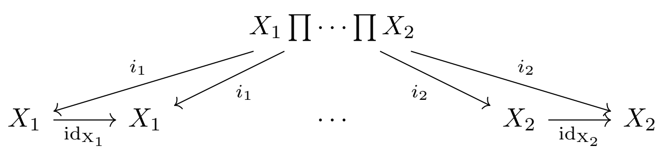


defines the product as the unique (up to unique isomorphism) such object having the universal property, meaning that any other object Y factors through the product uniquely.

Example 2. Let $X_1, X_2 \in \mathbb{R}$, $f_1(Y) = X_1$, $f_2(Y) = X_2$, and $i_1(a, b) = a$, $i_2(a, b) = b$; then for an object $X_1 \coprod X_2 = (X_1, X_2)$, the unique map $f : Y \to X_1 \coprod X_2$ exists where $i_1 \circ f = f_1 : Y \to X_1$, $i_2 \circ f = f_2 : Y \to X_2$ and thus the object $X_1 \coprod X_2 = \{(x_1, x_2) : x_1 \in X_1, x_2 \in X_2\}$ defines a product of X_1, X_2. This is in fact one way to define the *Cartesian product* of sets.

Remark 8. The notion of product between two objects can be generalized to a product between any number of objects.

Remark 9. The product is an example of a limit. We can see this by considering a functor mapping from the [*discrete category* (where the only morphisms are identity morphisms) indexing elements of the product] to [the nearly-discrete category where the objects X_i involved in defining the product live, with the only non-discrete object being the product $X_1 \coprod \cdots \coprod X_2$ and its relevant projection morphisms $i_1, \ldots, i_2$]. If we construct a product (with universal property) on this category, we have the expanded commutative diagram:

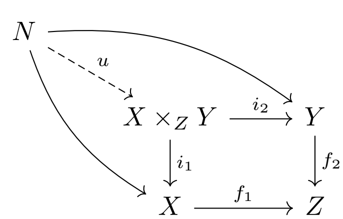


where this full diagram commutes and thus the product also satisfies the diagrammatic form of a limit.

Remark 10. For topological spaces X, Y, the product $X \times Y$ is defined by the space of products of open sets $U \times V : U \in X, V \in Y$. In category theory, we abstract this idea such that we don't need to look inside the sets X, Y, for instance.

1.6.2 Pullbacks

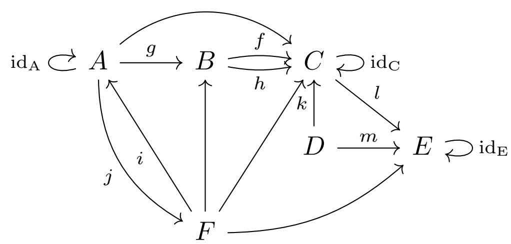
Definition 10. Another example of a limit, the *pullback* $(X \times_Z Y, i_1, i_2)$ is the universal object such that the following diagram commutes, and where the universal property requires that any N factors uniquely through the pullback via some unique isomorphism u:



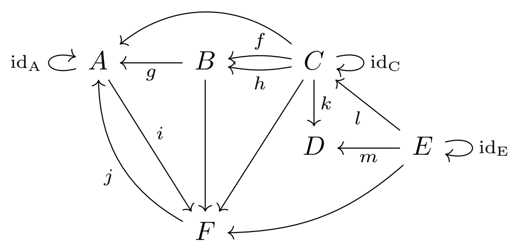
Remark 11. Alternatively notated: the pullback of X, Y on Z is the triple $(P : X \times_Z Y, i_1, i_2)$; a universal object P and two projection morphisms $i_1 : P \to X$ and $i_2 : P \to Y$ such that for some given morphisms $f_1 : X \to Z$ and $f_2 : Y \to Z$ we have $f_1 \circ i_1 = f_2 \circ i_2$.

1.7 Opposite Categories and Categorical Duals

We can now turn out attention to the idea of an *opposite category*. With the definition of a category $\mathcal{C}$, there naturally can be seen to exist an opposite category $\mathcal{C}^{\mathrm{op}}$. It contains exactly the same things as C, except with the direction of all arrows reversed. If we recall the toy category constructed earlier:



then we can see its opposite category is given diagrammatically by:



Definition 11. Given a category $\mathcal{C}$, its *opposite category* $\mathcal{C}^{\mathrm{op}}$ is the same collection of objects and morphisms, but with all arrow directions reversed.

Along with an opposite category comes the notion of a categorical dual construction; particularly

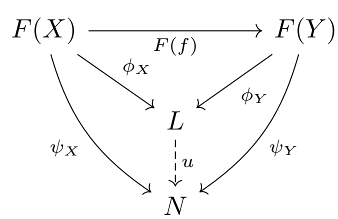
Definition 12. A *dual construction* in a category $\mathcal{C}$ is equivalent to the non-dual construction in its opposite category $\mathcal{C}^{\mathrm{op}}$.[6]

With that in mind, I present the following:

1.8 Colimits

Definition 13. A *colimit* is another universal object, dual to the limit. If $F : \mathcal{C} \to \mathcal{D}$ is a functor, and $X, Y \in \mathcal{C}$, for instance, with $f : X \to Y$ a morphism in $\mathcal{C}$, then the colimit (assuming it exists) is the universal object in $\mathcal{D}$ which is mapped to by to each such element in $\mathcal{D}$ in the manner of $\phi_X : X \to L, \phi_Y : L \to L$[7].

We can have generally a family of morphisms ϕ_i that map objects in C to some defining object L of the colimit, forming a series of inverted cones as pictured. Basically the colimit is the object L pictured below, with the universal property such that any N as pictured factors through L.



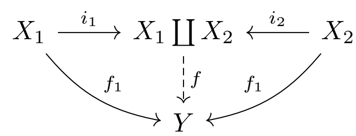
1.8.1 Coproducts

Definition 14. Consider a category C with X_1, X_2 objects in C. Then for any object Y and morphisms $f_1 : X_1 \to Y$, $f_2 : X_2 \to Y$, a *coproduct* is a universal object $X_1 \coprod X_2$ such that for some inclusion morphisms $i_1 : X_1 \to X_1 \coprod X_2$, $i_2 : X_2 \to X_1 \coprod X_2$, there exists a unique morphism $f : X_1 \coprod X_2 \to Y$ such that $f_1 = f \circ i_1$ and $f_2 = f \circ i_2$.

i.e. the diagram

[6] e.g. for a limit in $\mathcal{C}$, the dual construction is a colimit in $\mathcal{C}^{\mathrm{op}}$

[7] the resulting triangular/cone diagrams should commute just as they do with the limit



defines the coproduct as the unique (up to unique isomorphism) universal object.

Example 3. let $X_1, X_2 \subseteq \mathbb{R}$, $f_1(X_1) = Y$, $f_2(X_2) = Y$, and for the collection of subsets $A \subseteq X_1, B \subseteq X_2$ define the collection of morphisms i_1, i_2 by $i_1(A) = \{\forall a \in A : a \text{ if } a \in A \coprod B\}$, $i_2(B) = \{\forall b \in B : b \text{ if } b \in A \coprod B\}$ (i.e. inclusion maps); then for an object $X_1 \coprod X_2$, the unique map f which sends objects to Y by f_1 if they originate from X_1, and by f_2 if they originate from X_2 exists, and by construction of the inclusion maps then $i_1 \circ f = f_1$, $i_2 \circ f = f_2$; thus the object $X_1 \coprod X_2$ defines a coproduct of X_1, X_2 on the category of sets in $\mathbb{R}$.

Remark 12. In fact, this example coproduct is the disjoint union of sets $X_1 \sqcup X_2$, i.e. a set union which distinctly differentiates elements originating from X_1, X_2.

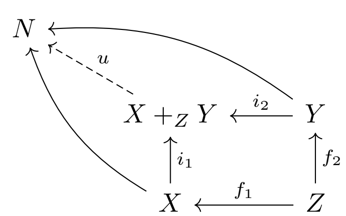
Remark 13. The coproduct is dual to the product.

Remark 14. Interestingly, for abelian groups, the finite coproduct is equal to the product. For the category of abelian groups, the product of groups G, H is the cartesian product of group elements (g, h) with independently acting group operations, and the coproduct is the free product $G * H$ (which constructs the universal group generated by the elements of G and H with both G, H as subgroups). For abelian groups G, H, these are by default normal subgroups and thus $G * H$ is generated by its normal subgroups. This implies we can interpret $G * H$ as simply being the direct sum of the groups $G + H$. This direct sum then has precisely the same form as the cartesian product, since we are here working with only two (i.e. finitely many) components to the product/sum (namely G, H abelian groups, being objects in the category of abelian groups).

1.8.2 Pushforwards (Pushouts)

Remark 15. or, as I think they could be called for consistency, copullbacks. They are, of course, dual to the pullback.

Definition 15. An example of a colimit, the *pushout* $(X +_Z Y, i_1, i_2)$ is the universal object such that the following diagram commutes.[8]



Like a pullback, a pushout is a triple which consists of a universal object P and two morphisms $i_1 : X \to P$, $i_2 : Y \to P$ which complete the commutative square with two morphisms $f_1 : Z \to X$, $f_2 : Z \to Y$ (such that $i_1 \circ f_1 = i_2 \circ f_2$).

[8] the pushout also completes a commutative square in the manner that a pullback does

2 Topological Interlude

2.1 Axioms of a Topology

Definition 16. [3] A *topology* is a construction consisting of a set X and a collection of subsets of X, i.e. $\mathcal{U} = \{S \subset X\}$ such that:

(i) $X, \emptyset \in \mathcal{U}$

(ii) $\mathcal{U}$ is closed under the set operation of *finite intersection*, i.e.

$$S_1, \cdots S_n \in \mathcal{U} \implies S_1 \cap \cdots \cap S_n \in \mathcal{U} \tag{3}$$

(iii) $\mathcal{U}$ is closed under the set operation of *arbitrary union*, i.e.

$$\tilde{\mathcal{U}} \subseteq \mathcal{U} \implies \bigcup_{S \in \tilde{\mathcal{U}}} S \in \mathcal{U} \tag{4}$$

The set $\mathcal{U}$ is a topology on X, and we say $(X, \mathcal{U})$ is a *topological space*. The sets $S \in \mathcal{U}$ are called *open sets*, and their complements in X are *closed sets*.

Remark 16. A topology can be specified either with a collection of open sets or closed sets in X, and either will be equivalent since the collection of closed sets will satisfy the complementary conditions to those defined above (and allow one to recover the collection of open sets if desired).

2.2 Basis of a Topology

Another concept in topology, which I believe builds some intuition for what follows in the construction of a Grothendieck Topology, is that of a basis.

Definition 17. [3] A *basis* of a topology is a collection of subsets of X, $\mathcal{B} \subset 2^X$ (2^X the set of all subsets of X, i.e. its *power set*) such that:

(i) $\emptyset \in \mathcal{B}$

(ii) X can be constructed from the arbitrary union of all elements of the basis, i.e.

$$\bigcup_{S \in \mathcal{B}} S = X \tag{5}$$

(iii) The subset formed by intersection of any two elements of the basis (themselves subsets of X), $B_1 \cap B_2$, is necessarily equal to the union of subsets contained in some sub-collection of subsets $\tilde{\mathcal{B}} \subset \mathcal{B}$, i.e.

$$\forall B_1, B_2 \in \mathcal{B} : \exists \tilde{\mathcal{B}} : \quad B_1 \cap B_2 = \bigcup_{S \in \tilde{\mathcal{B}}} S \tag{6}$$

This all amounts to a sort of generating set of a topology, which is itself closed under arbitrary unions and finite intersections.

3 Grothendieck Topology

Now, one could wonder; if a topology can be relatively abstract on its own, could we further abstract this notion of a topology to a similar construction on an arbitrary category? It turns out that in fact we can, and we have Grothendieck to thank for inventing such a construction.

3.1 The Definition

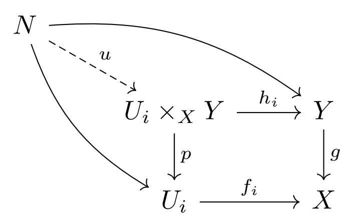
Definition 18. [4][1] A *Grothendieck Topology*, on a category $\mathcal{C}$, is a "collection of (families of (maps))" in $\mathcal{C}$[9], these families referred to as *coverings* or sometimes (emphasizing the analogue to open sets) *open coverings*:

$$\{f_i : U_i \to X\}_{i \in I} \tag{7}$$

such that this collection (indexed by some arbitrary indexing set I) adheres to the axioms:

3.1.1 (T1)

If $\{f_i : U_i \to X\}_{i \in I}$ is a covering of X, and $g : Y \to X$ is a morphism in $\mathcal{C}$ for some Y in $\mathcal{C}$, then the set of projection maps $\{h_i : U_i \times_X Y \to Y\}_{i \in I}$[10] is also a covering (now of Y). This is pictured as:



where if $\{f_i\}_{i \in I}$ is a covering, then $\{h_i\}_{i \in I}$ is also.

This amounts to the notion that we can change the basis for our coverings; we can convert a covering of one object X to a covering another related object Y whenever there exists such a morphism $g : Y \to X$ between them.

3.1.2 (T2)

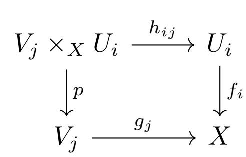
If:

(i) $\{f_i : U_i \to X\}_{i \in I}$ is a covering (of X)

(ii) $\{g_j : V_j \to X\}_{j \in J}$ is a collection of morphisms in $\mathcal{C}$ for some collection of V_j in $\mathcal{C}$

(iii) for any $i \in I$, the collection of projection maps $\{h_{ij} : V_j \times_X U_i \to U_i\}_{j \in J}$ is also a covering (of U_i)

then the collection $\{g_j : V_j \to X\}_{j \in J}$ is in fact a covering (of X). This is pictured as:



where if the collections of morphisms $\{f_i\}_{i \in I}$ and $\{h_{ij}\}_{i \in I, j \in J}$ are each coverings of their respective objects in $\mathcal{C}$, then the collection $\{g_j\}_{j \in J}$ is also a covering in our Grothendieck topology.

This is somewhat of a nesting idea, that compositions of coverings are coverings, relating to the idea that unions of open sets are open in a topology.

[9] U_i, X are objects in $\mathcal{C}$

[10] recall $U_i \times_X Y$ is a pullback

3.1.3 (T3)

If $\{f_i : U_i \to X\}$ is a collection of morphisms such that $\forall U_i$, $\exists f_i$ which admits a section $s : X \to U_i$ such that $f_i \circ s = f_i s = \mathrm{id}_X$, then $\{f_i : U_i \to X\}$ is a covering of X.

Essentially this says that the collection of f_i, which admit sections and subsequently imply composition to the identity map $\mathrm{id_X}$, covers its respective object X in a category.

Note this particular definition works for any category $\mathcal{C}$ as long as $\mathcal{C}$ admits finite limits.

3.2 Showing this can indeed be viewed as a Topology

When we are working in a category with sufficient local smallness, i.e. its collections of objects and morphismscan be locally viewed as a set[11], then we can find coverings of X in a Grothendieck topology for each object X of the category to correspond to the open sets of a classical topology, and find:

(i) The empty set is included in the Grothendieck topology, via any null covering which simply contains no maps. Conveniently, T3 also ensures that any object in a category C is included in the induced topology, since any object will be covered at least by its identity morphism. We can thus naturally try to view $(\mathrm{id}(\mathrm{Obj}(C)), \{coverings\})$ as a topology, with all identity morphisms corresponding to the objects of C contributing to the underlying topological space.

(ii) Closure under finite intersection is satisfied by T1, since we can consider the pullback of coverings, $\{h_i : U_i \times_X Y \to Y\}$ on a covered object X to be like the intersection of the coverings $\{f_i : U_i \to X\}$ on X and any morphisms $g : Y \to X$, and the definition tells us this is also a covering of Y (and hence contained as an open covering in the Grothendieck topology).

So, using this pullback to define the intersection of coverings, we have the second axiom of a topology satisfied.

(iii) Closure under arbitrary union is brought about by $T2$, since the manner of composition/unioning the families of morphisms in the two sort of nested coverings (with $\{h_i : V_j \times_X U_i \to U_i\}$ like an open set "containing" or presiding over the family of coverings $\{f_i : U_i \to X\}$) leading to yet another "composition covering" $g_j : V_j \to X$ (this type of union/composition facilitated by the projection morphism from the pullbacks $V_j \times_X U_i$ to the sets V_j) is akin to the union of open sets in a topology.

So, using this manner of composed morphisms assisted by a projection via a pullback to define the union of coverings, we have the third axiom of a topology satisfied.

We have thus found a rather useful way to construct a notion of topology on a category; such a pair of a Category C equipped with a certain Grothendieck Topology is called a Grothendieck site.

As I've presented it here, this only strictly works with locally small categories (where we can ignore most set-theory paradoxes involving cardinality arising from notions of "sets of all sets", i.e. the collections of objects and morphisms can each be viewed as sets), but can also be generalized to large categories when the right care is taken.

Definition 19. A *Grothendieck site* is a category $\mathcal{C}$ with a specified Grothendieck topology on $\mathcal{C}$.

Remark 17. This notion of choosing a Grothendieck site on a category is sometimes akin to the choice of a classical topology (via the chosen open sets) on a topological space. However, in many cases there is not an exact analogue; a choice of Grothendieck topology should be thought of more as an abstraction of the notions one would like to associate with most classical topology, without necessarily an exact correspondence available from the axioms alone.

[11] when the collection of morphisms is set-like, it usually automatically implies that the collection of objects is also set-like, since there a correspondence between objects and their identity morphisms

3.3 A More Direct Correspondence with Classical Topology

Example 4. Consider the category formed by taking the elements of the power set of some set X[12] as objects, with a collection of set inclusions as the morphisms of the category. Then we have by construction, for any $U \subset V \subset X \in \tau$, inclusion morphisms $U \hookrightarrow V \hookrightarrow X$.

Then with a choice of some (potentially)[13] classical topology on X given by τ[14], let us first define the corresponding Grothendieck topology $\mathcal{U}$ to τ by choosing some open coverings:

$$\mathcal{U} = \{\{f_i : U_i \hookrightarrow S\} : S \in \tau\} \tag{8}$$

for some $U_i \in 2^X$ where each f_i is an inclusion map of $U_i \subseteq S$.

Such a choice of τ, U_i for each $S \in \tau$ and the corresponding coverings $\{f_i\}$ will allow us to generate a Grothendieck topology $\mathcal{U}$ on X. Note that for the direct correspondence of coverings in $\mathcal{U}$ to open sets in τ, (T3) tells us that any "open set" $S \in \tau$ has a corresponding covering given by the (cardinality 1) family of inclusion maps $\{\mathrm{id_S} : S \hookrightarrow S\}$.

With the axioms restricting $\mathcal{U}$, then, this τ can then be seen to be equivalent to a classical topology on X as follows:

(i) We always have identity morphisms in a category, and thus as a specification of (T3) we must have $\emptyset \simeq \{f_i : \emptyset \hookrightarrow \emptyset\} \in \mathcal{U}$ and $X \simeq \{f_i : X \hookrightarrow X\} \in \mathcal{U}$. This ensures the first axiom of a classical topology is satisfied for τ on X, i.e. $\emptyset, X \in \tau$.

(ii) If $\{f_i : U_i \hookrightarrow S\} \in \mathcal{U}$ is a covering of $S \in \tau$, consider the intersection $S \cap_X T$ for some other set $T \subseteq X$. Then by definition of set intersection, $\exists g : T \cap_X S \hookrightarrow S$. Since we have the inclusions $U_i \subseteq S$ and $T \cap_X S \subseteq S$, this also implies another inclusion map for the intersection over S, $T \cap_S U_i \subseteq S$. Then the following diagram of inclusion maps commutes:

$$\begin{array}{ccc} U_i \cap_S T & \xrightarrow{h_i} & T \cap S \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle g} \\ U_i & \xrightarrow{f_i} & S \end{array}$$

and where we can note the set intersection is actually strictly unique here by choice of classical topology, so by default $(U_i \cap_S T, h_i, i) \equiv (U_i \times_S (T \cap S), h_i, i)$ has the universal property and is a pullback.

With this identification of the form of a pullback, we can now invoke (T1): for any such covering $\{f_i\}$ and morphism g, the set of projection maps from the pullback $h_i : U_i \times_S (T \cap S) \to (T \cap S)$ is also a covering of $(T \cap S)$ in $\mathcal{U}$. Therefore, as usual representing the open sets in τ with their coverings in $\mathcal{U}$, we find that in the Grothendieck topology:

$$(\exists \{f_i : U_i \hookrightarrow S\} \in \mathcal{U}) \wedge (\exists g : T \cap S \hookrightarrow S) \qquad \Longrightarrow \qquad \{h_i : U_i \cap_S T \to T \cap S\} \in \mathcal{U} \tag{9}$$

i.e. in direct correspondance to the axiom for a classical topology τ on X;

$$(S \in \tau) \wedge (T \cap S \subseteq S) \qquad \Longrightarrow \qquad T \cap S \in \tau \tag{10}$$

(iii) Finally, let us tackle the arbitrary union axiom. Consider an arbitrary union of sets in τ, $S = \bigcup_{i \in I} U_i$, then for each U_i there exist inclusion morphisms $f_i : U_i \hookrightarrow S$. Particularly, since S as a set covers itself in the classical topology and is a union of sets U_i which also form an open cover of it, then we have some covering of S in the Grothendieck topology. Then if there exist corresponding coverings for each U_i, $\{h_i : (V_j \times_S U_i) \equiv V_{ij} \to U_i\}$, note we can construct a covering of S alternatively via composition of these inclusion maps as follows:

[12] i.e. all subsets of X

[13] the point of this section is to show that choosing a τ choice of open sets according to the Grothendieck topology axioms will result in an equivalent choice of τ using the classical topology axioms on X

[14] recall, this corresponds to the set of all open sets in our choice of classical topology

$$\begin{array}{ccc} V_j \times_S U_i & \xrightarrow{h_i} & U_i \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle f_i} \\ V_j & \xrightarrow{g_i} & S \end{array}$$

for any other sets $V_{ij} \equiv (V_j \times_S U_i)$ in the classical topology which form a covering family of all U_i. This is consistent with the classical notion of an open cover of an arbitrary union in a classical topology τ.

In the other direction, suppose we start with a classical topology;

(T3) Every open set is a cover for itself via the identity inclusion, and these identity inclusion morphisms are the only ones with existing sections in the category formed by these open sets, so we can see (T3) appears somewhat naturally.

(T2) Also, if we have a union of open sets S, it is by definition covered by the open cover formed by its component open sets $\{U_i\}$ since any union of open sets is open by the arbitrary union axiom of classical topology, thus we have a covering $\{f_i : U_i \to S\}$ in our Grothendieck topology. Then, consider an open cover $\{V_j\}$ for each U_i; by inclusion properties, intersection (i.e. the pullback) over S leads to a covering $h_i : V_j \times_S U_i$ for each component open set U_i. It is clear then that in this case the family of all $\{V_j\}$ (viewed as an open cover) must themselves also cover the open set union S, by composition of open covers, and we have arrived at the statement of (T2).

(T3) Finally, if we have an intersection of sets $T \cap S$, there exists an inclusion map to S by nature of set intersection. If we have an open cover $\{U_i\}$ of S, then, by set intersection properties we can intersect each U_i with T and obtain an open cover (since finite intersection of open sets is open in τ) $\{U_i \times_S T\}$ still contained entirely within S, and also by nature of set intersection also within $T \cap S$. This corresponds directly to having a open covering $f_i : U_i \to S$ and an inclusion morphism for the intersection $T \cap S \hookrightarrow S$ implying we have the open covering $U_i \times_S T \to T \cap S$, precisely the statement of (T1).

Hence we have derived the consistency of the Grothendieck topology axioms from the classical topology and its axioms.

4 Presheaves

4.1 Presheaves in Category Theory

Definition 20. A (set-valued) *presheaf* on $\mathcal{C}$ is a (contravariant) functor $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ which satisfies certain functorial properties:

(i) $F(\mathrm{id_A}) \to \mathrm{id_{F(A)}}$ in Set, for any $A \in \mathcal{C}$, i.e. F preserves identities.

(ii) $F(g \circ f) = F(f) \circ F(g)$ (for contravariance) for all inclusion maps $f : V \leftarrow W$ and $g : U \leftarrow V$ with $U, V, W \in C$; i.e.

$$F(U \overset{g}{\hookleftarrow} V \overset{f}{\hookleftarrow} W) = F(U) \overset{F(g)}{\hookrightarrow} F(V) \overset{F(f)}{\hookrightarrow} F(W) \tag{11}$$

4.2 Presheaves in Topology

Definition 21. A *presheaf on a topological space* X is a (contravariant) functor $F : X \to \mathrm{Set}$[15] that satisfies:

(i) For each open set $U \in X$, $F(U) \to \Gamma(U, X)$, i.e. F assigns the set of sections $\Gamma(U, X) = \{s : X \to U\}$ to the inclusion maps $f : U \hookrightarrow X$[16]. Since the objects in Set that we are looking at here are specifically sets of functions, the morphisms we will interest ourselves with are restriction maps of these functions from one domain to another.

(ii) For any open set $V \subset U$ and the corresponding domain restriction map $\mathrm{res}_{U\to V}$ which restrict functions on U to functions on V, the functor F assigns a new restriction map $\mathrm{res}_{F(U)\to F(V)}$ such that composition and identity are conserved under F, i.e.

   (a) For any $W \subset V \subset U$, with $\mathrm{res}_{V\to W} \circ \mathrm{res}_{U\to V} = \mathrm{res}_{U\to W}$, then the composition rule is obeyed: $F(\mathrm{res}_{V\to W} \circ \mathrm{res}_{U\to V}) = F(\mathrm{res}_{V\to W}) \circ F(\mathrm{res}_{U\to V}) = \mathrm{res}_{F(U)\to F(W)}$

   (b) Identity maps are preserved under F: $F(\mathrm{res}_{U\to U}) = \mathrm{res}_{F(U)\to F(U)}$

4.3 The Comparison

Remark 18. We can see the inspiration of the categorical from the topological definition of presheaf by considering the category $\mathcal{C}(X)$ formed from the topology X, whose objects are open sets $U \in X$ and whose morphisms are restriction maps for inclusion between these open sets.

Then the desired functor F defined as a categorical presheaf on $\mathcal{C}(X)$ is one which takes any object (open set) U in $\mathcal{C}(X)^{\mathrm{op}}$, and sends it to the set of sections that relate it to the topological space X $s : X \to U$.

The functor then takes morphisms (i.e. inclusion maps $W \hookrightarrow U$) and sends them to the corresponding restriction maps between functions $\mathrm{res}_{U\to W} : F(U) \to F(W)$. The Functor can be seen to preserve both identities and composition of restriction maps from the categorical definitions. This arrow direction reversal is characteristic of the contravariant nature of the functor F, and can be understood intuitively with the way that if $W \subset U$, functions to U will need to be restricted if we want their range to instead go to W, but conversely any function with range in W will already have range in U.

This example takes the target category of the presheaf to be Set, with its particular functor range being the collection of (sets of (sections of (inclusion maps from open sets in X to X))), together with morphisms between these sets of sections which are the restriction maps of (sections from X to, for instance, some set U) to sections from X to some subset $W \subset U$.

[15]the category of sets

[16]one can see here explicitly the contravariance of the functor F

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