Grassmann Varieties

Ratnadha Kolhatkar

Department of Mathematics and Statistics,

McGill University, Montréal

Québec, Canada

August, 2004

A thesis submitted to the Faculty of Graduate Studies and Research

in partial fulfillment of the requirements of the degree of

Master of Science

Abstract

In this thesis we study the simplest types of generalized Grassmann varieties. The study involves defining those varieties, understanding their local structures, calculating their Zeta functions, defining cycles on those varieties and studying their cohomology groups.

We begin with the classical Grassmannian $G(d,n)$ and then study a special type of the Grassmannian, namely the Lagrangian Grassmannian. For a field k and a subring $R \subset \mathrm{End}(k^n)$ we study the generalized Grassmann variety $G(R;d,n)$ which is the set of all d-dimensional subspaces of k^n that are preserved under R. We study the local structure of the generalized Grassmann scheme $F_R := G(R;d,n)$ and its zeta function in some particular cases. We study closely the example of a quadratic field when R is the ring of integers.

Résumé

Dans cette thèse, nous étudions les plus simples variétés de Grassmann genéralisées. Cette étude consiste à définir ces variétés, à comprendre leur structure locale, à calculer leurs fonctions Zêta, à définir des cycles sur ces variétés et à étudier leurs groupes de cohomologie.

Nous commençons avec la variété de Grassmann classique $G(d, n)$ et ensuite, nous étudions spécialement la variété de Grassmann Lagrangienne. Pour un corps k donné et un sous-anneau $R \subset \mathrm{End}(k^n)$, nous étudions la variété de Grassmann généralisée $G(R; d, n)$, c'est-à-dire l'ensemble de tous les sous-espaces de k^n de dimension d qui sont préservés par R. Nous étudions la structure locale du schéma de Grassmann généralisé $F_R := G(R; d, n)$ et, dans quelques cas particuliers, sa fonction Zêta. Nous étudions en détail l'exemple d'un corps quadratique lorsque R est l'anneau des entiers.

Acknowledgments

Many people have helped me over past two years and I am thankful to all of them. First of all, I would like to express my deep gratitude to my supervisors Prof. Eyal Goren and Prof. Peter Russell for patiently guiding me throughout the course of my thesis. I am also thankful to them for the kind financial support they provided.

Special thanks to my father for encouraging me to pursue my studies in Canada. Thanks to my family and my friends for their constant love and moral support.

Je remercie mes amis Québécois surtout Frédéric, Jean-Sébastien, Marc-Hubert et Éloi qui sont toujours de bons et attentifs amis pour moi. Finalement, je suis reconnaissante d'avoir été exposée à ce nouveau monde, à une variété d'experiences et aux gens d'ici. Mon séjour à Montréal m'a donné de merveilleuses opportunités d'apprendre et de réaliser beaucoup de choses importantes dans la vie.

Table of Contents

Introduction

Let us start with the classical Grassmann variety $G(d, n)$, which is the set of all d-dimensional subspaces of a vector space V of dimension n. The same set can be considered as the set of all $(d-1)$-dimensional linear subspaces of the projective space $\mathbb{P}^{n-1}(V)$. In that case we denote it by $G^{\mathbb{P}}(d-1, n-1)$.

In Chapter 1 we see that $G(d, n)$ defines a smooth projective variety of dimension $d(n-d)$. It is quite interesting to note that the number of $\mathbb{F}_q$-rational points of $G(d, n)$ equals the standard q-binomial coefficient $\binom{n}{d}_q$ that can be expressed as a polynomial in powers of q. Consequently, the Zeta function of $G(d, n)$ is easy to calculate and we see that all odd Betti numbers of the Grassmannian are zero. The Euler characteristic of $G(d, n)$ comes out to be the usual binomial coefficient $\binom{n}{d}$.

The Schubert calculus is introduced thereafter to understand the cohomology ring of the Grassmannian, namely $H^*(G^{\mathbb{P}}(d, n)(\mathbb{C}); \mathbb{Z})$. Schubert calculus helps us solve many enumerative problems such as : How many lines in 3-space in general intersect 4 given lines? The subject is studied quite intensively in [7, 11, 17, 12, 20]. We mainly follow [12] to develop the basic notions of the subject and state without proof many results like the Basis Theorem, Giambelli's formula and Pieri's formula. Then we study the cohomology ring $H^*(G^{\mathbb{P}}(1, 3)(\mathbb{C}); \mathbb{Z})$ in detail.

In the last few sections of Chapter 1 we see that the construction of the classical Grassmannian has a natural extension to the category of schemes. Indeed the Grassmann scheme $G_{\mathbb{Z}}(d, n)$ represents the Grasmann functor g : (rings) $\rightarrow$ (sets) given by

$$g(T) = \{T\text{-submodules } K \subset T^n \text{ that are rank } d \text{ direct summands of } T^n \}.$$

We show the representability of the Grasmann functor following [6]. The Basis Theorem of the Schubert calculus states that the Schubert cycles generate the cohomology ring $H^*(G^{\mathbb{P}}(d,n)(\mathbb{C});\mathbb{Z})$. As one of its applications we compute the Zeta function of the Grassmannian using the information of cohomology groups in characteristic zero and get the information of the cohomology groups in characteristic p. Finally, we compute the Zeta function of the Grassmann scheme $G_{\mathbb{Z}}(d,n)$ which comes out to be a product of Riemann Zeta functions.

In Chapter 2 we discuss a special type of Grassmannian, $L(n,2n)$, called the Lagrangian Grassmannian; it parametrizes all n-dimensional isotropic subspaces of a $2n$-dimensional symplectic space. A lot of symplectic geometry can be found in [14] and [2]. The Lagrangian Grassmannian $L(n,2n)$ is a smooth projective variety of dimension $\frac{n(n+1)}{2}$. We then give a similar treatment to the Lagrangian Grassmannian as to the classical Grassmannian and compute its Zeta function, Euler characteristic etc.

The Schubert calculus for Lagrangian Grassmannians is discussed for example in [17, 22]. We mostly follow [22]. Using the Basis Theorem for the Lagrangian Garssmannian we compute the dimensions of the cohomology groups $H^i(L(n,2n);\mathbb{Z})$. We then study the representability of the Lagrangian Grassmann functor, which is a functor $l : (\text{rings}) \to (\text{sets})$ given by

$$l(T) = \{\text{isotropic } T\text{-submodules } K \subset T^{2n} \text{ that are rank } n \text{ direct summands of } T^{2n}\}.$$

Finally we compute the Zeta function of the Lagrangian Grassmann scheme $L_{\mathbb{Z}}(n,2n)$.

In Chapter 3 we begin with the following set up. Let $0 < d < n$ be integers, $R \subseteq M_n(\mathbb{Z})$ be a ring, and consider the functor $f_R : (\text{rings}) \to (\text{sets})$ given by

$$f_R(T) = \{T\text{-submodules } K \subset T^n \text{ that are } R\text{-invariant rank } d \text{ direct summands of } T^n\}$$

We show that f_R is representable by a scheme F_R. We study the local structure of the scheme F_R and its Zeta function in some examples.

We consider in detail the example of a quadratic field $L = \mathbb{Q}(\sqrt{D})$ where D is a squarefree integer. Let R be the ring of integers in L. Let $R_1 = R \otimes \overline{\mathbb{F}}_p$. We concentrate on the case $D \equiv 2, 3 \pmod 4$, study the scheme $F_{R_1}(\overline{\mathbb{F}}_p)$ and compute its Zeta function.

Chapter 1

Grassmann Varieties

In Chapter 1 we discuss in detail the classical Grasssmannian, first as a variety and then as a scheme. In section 1.1 we discuss the construction of the Grassmannian as an algebraic variety. We also study an affine cover of the Grassmannian. Section 1.2 discusses the Zeta function of these varieties. In section 1.3 we give an introduction to the Schubert calculus. This leads us to understand the cohomology ring of the complex Grassmannian $G(d,n)(\mathbb{C})$ with integer coefficients. In section 1.4 we describe how the construction of the classical Grassmannian has a natural extension to the category of schemes. We will also talk on the representability of the Grassmann functor and the Zeta function of the Grassmann scheme.

1.1 Grassmann varieties

1.1.1 The Grassmannian, basic notions

Recall the construction of a projective space over field k. The projective space $\mathbb{P}^n(k)$ is defined as the collection of all lines in k^{n+1}. Equivalently, it is the set of all hyperplanes in k^{n+1}. This construction gives rise to a natural question : Why not consider the set of subspaces in k^{n+1} of arbitrary dimension? The construction of Grassmannians has its origin in answering this question. Classically we define the Grassmannian as follows.

Definition 1.1. Let V be a vector space of dimension $n \geq 2$ over field k. Let $0 < d < n$ be an integer. Then the Grassmannian $G(d,n)$ over k is defined as the set of all d-dimensional subspaces of V i.e.

$$G(d,n)(k) = \{W \mid W \text{ is a } k\text{-subspace of } V \text{ of dimension } d\}.$$

Alternately, $G(d,n)$ can be considered as the set of all $(d-1)$-dimensional linear subspaces of the projective space $\mathbb{P}^{n-1}(k)$. If we think of the Grassmannian this way, we denote it by $G^{\mathbb{P}}(d-1,n-1)$. The simplest example of the Grassmannian is $G(1,n)$ which is the set of all 1-dimensional subspaces of the vector space V which is nothing but the projective space on V.

1.1.2 Review of some exterior algebra

Let R be a commutative ring with unity and let M be an R- module. For each natural number r let

$$T^r(M) = \begin{cases} R & \text{if } r = 0, \\ M \otimes_R T^{r-1}(M) & \text{otherwise.} \end{cases}$$

Thus $T^r(M) = \underbrace{M \otimes_R \cdots \otimes_R M}_{r \text{ times}}$. The tensor product is assosiative and we have a bilinear map $T^r(M) \times T^s(M) \to T^{r+s}(M)$ by which we can define a ring structure on the direct sum

$$T(M) := \bigoplus_{r=0}^{\infty} T^r(M).$$

In fact, $T(M)$ is an R-algebra. It is called the **tensor algebra** of M over R. Let us denote by $\mathcal{A}_n(M)$ the submodule in $T^n(M)$ generated by the elements of the type

$x_1 \otimes \cdots \otimes x_n$ where $x_i = x_j$ for some $i \neq j$. We define

$$\bigwedge^n M := T^n(M)/\mathcal{A}_n(M).$$

Also define the **exterior algebra** of M as the direct sum

$$\bigwedge M := \bigoplus_{n=0}^{\infty} \bigwedge^n M.$$

Let I be the ideal in $T(M)$ generated by $\{x \otimes x \mid x \in M\}$. Then we have

$$\bigwedge M = T(M)/I.$$

If $w \in \bigwedge^r M$ with, $w = u + \mathcal{A}_r(M)$ and $w' \in \bigwedge^s M$, with $w' = u' + \mathcal{A}_s(M)$, we define

$$w \wedge w' = u \otimes u' + \mathcal{A}_{r+s}(M)$$

as an element of $\bigwedge^{r+s} M$.

Notation : Let $u_1, \ldots, u_n \in M$. The element $u_1 \otimes \cdots \otimes u_n + \mathcal{A}_n(M)$ is denoted by $u_1 \wedge \cdots \wedge u_n$.

One has the following Lemma.

Lemma 1.2. [9, Corollary 10.16] *Let $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ be two families of vectors of M related by a matrix $A = (a_{ij})_{n\times n}$ of coordinate change, which means, $(v_1, \ldots, v_n) = (u_1, \ldots, u_n)(a_{ij})_{n\times n}$. Then,*

$$v_1 \wedge \cdots \wedge v_n = \det(A) \cdot u_1 \wedge \cdots \wedge u_n.$$

1.1.3 The Plücker map and coordinates

We can embed $G(d, n)$ in the projective space $\mathbb{P}(\bigwedge^d V)$ as follows. Let U be a subspace of V of dimension d with a basis $\{u_1, \cdots, u_d\}$. Define $P(U)$ as the point of the projective space $\mathbb{P}(\bigwedge^d V)$ which is determined by $u_1 \wedge \cdots \wedge u_d$. The map P,

$$P : G(d, n) \to \mathbb{P}(\bigwedge^d V),$$

is called the **Plücker map**. By Lemma 1.2, P is a well defined map. Since the wedge product $u_1 \wedge \cdots \wedge u_d \wedge u = 0$ if and only if $u \in U$, it follows that P is injective. Thus, via P, we may consider $G(d,n)$ as a subset of $\mathbb{P}(\bigwedge^d V)$. Let $\{e_1, \ldots, e_n\}$ be a basis for V. Then the canonical basis for $\bigwedge^d V$ is given by

$$\{e_{i_1} \wedge \cdots \wedge e_{i_d} \mid 1 \leq i_1 < \cdots < i_d \leq n\}.$$

Let U be a d-dimensional subspace of V with a basis $\{u_1, \ldots, u_d\}$. For $1 \leq i \leq d$, let $u_j = \sum_{i=1}^{n} a_{ij} e_i$. Then the coordinates of $P(U) = u_1 \wedge \cdots \wedge u_d$ are called the **Plücker coordinates**. These are nothing but the $\binom{n}{d}$ minors of the matrix $(a_{ij})_{\substack{1\leq i\leq n \\ 1\leq j\leq d}}$.

1.1.4 Examples

Example 1.3. *The Grassmannian $G(1,n)$: Let U be the space spanned by the vector $u_1 = a_1e_1 + \cdots + a_ne_n$. The Plücker coordinates are the maximal minors of the matrix $(a_1, \cdots, a_n)$. Therefore the Grassmannian $G(1,n) \cong \mathbb{P}^{n-1}$ and the Plücker map P sends U to $(a_1 : \cdots : a_n)$.*

Example 1.4. *The Grassmannian $G(2,4)$: Let $\{e_1, e_2, e_3, e_4\}$ be a basis for V. The canonical basis for $\bigwedge^2 V$ is given by*

$$\{e_1 \wedge e_2,\ e_1 \wedge e_3,\ e_1 \wedge e_4,\ e_2 \wedge e_3,\ e_2 \wedge e_4,\ e_3 \wedge e_4\}.$$

Let $\{u_1, u_2\}$ be a basis for $U \in G(2,4)$ with

$$u_1 = a_{11}e_1 + a_{21}e_2 + a_{31}e_3 + a_{41}e_4, \quad u_2 = a_{12}e_1 + a_{22}e_2 + a_{32}e_3 + a_{42}e_4.$$

Then,

$$\begin{aligned} u_1 \wedge u_2 = {} & (a_{11}a_{22} - a_{12}a_{21})e_1 \wedge e_2 + (a_{11}a_{32} - a_{31}a_{12})e_1 \wedge e_3 \\ & + (a_{11}a_{42} - a_{41}a_{12})e_1 \wedge e_4 + (a_{21}a_{32} - a_{31}a_{22})e_2 \wedge e_3 \\ & + (a_{21}a_{42} - a_{41}a_{22})e_2 \wedge e_4 + (a_{31}a_{42} - a_{41}a_{32})e_3 \wedge e_4. \end{aligned}$$

So the Plücker coordinates are

$$(a_{11}a_{22} - a_{12}a_{21},\ a_{11}a_{32} - a_{31}a_{12},\ a_{11}a_{42} - a_{41}a_{12},\ a_{21}a_{32} - a_{31}a_{22},$$
$$a_{21}a_{42} - a_{41}a_{22},\ a_{31}a_{42} - a_{41}a_{32}).$$

We will denote these coordinates by $x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}$ *respectively. One observes that these are indeed the* 2×2 *minors of the matrix*

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix}.$$

1.1.5 The Grassmannian as an algebraic variety

We observed that $G(d, n)$ can be embedded in the projective space $\mathbb{P}(\bigwedge^d V)$ via the Plücker map P. The goal of this section is to show that the image is a closed subset of $\mathbb{P}^N$ where $N = \binom{n}{d} - 1$.

Definition 1.5. Let $w \in \bigwedge^d V$. Let $v \in V$, $v \neq 0$. We say that v **divides** w if there exists $u \in \bigwedge^{d-1} V$ such that $w = v \wedge u$.

We have the following Lemma.

Lemma 1.6. *Let* $w \in \bigwedge^d V$*. Let* $v \in V$*,* $v \neq 0$*. Then* v *divides* w *if and only if the wedge product* $w \wedge v = 0$*.*

Proof. Clearly if v divides w, say $w = u \wedge v$, then $w \wedge v = u \wedge v \wedge v = 0$. To see the other direction let $\{e_1, e_2, \ldots, e_n\}$ be a basis of V with $e_1 = v$. The canonical basis for $\bigwedge^d V$ is given by:

$$\{e_{i_1} \wedge \cdots \wedge e_{i_d} \mid 1 \leq i_1 < \cdots < i_d \leq n\}.$$

Let $e_{i_1} \wedge \cdots \wedge e_{i_d} = e_{i_1, i_2, \ldots, i_d}$. Any $w \in \bigwedge^d V$ can be written as

$$w = \sum_{1 \leq i_1 < \cdots < i_d \leq n} a_{i_1, \ldots, i_d} e_{i_1, i_2, \ldots, i_d}.$$

Then

$$v \wedge w = \sum_{i_1 < \cdots < i_d} a_{i_1,\ldots,i_d} e_1 \wedge e_{i_1,i_2,\ldots,i_d}.$$

So we see that $v \wedge w = 0$ if and only if $a_{i_1,\ldots,i_d} = 0$ for every $i_1, \ldots, i_d$ with $1 < i_1$ i.e. the vector $e_1 = v$ divides w. □

Using above lemma we can show that the collection of all vectors $v \in V$ dividing a fixed vector $w \in \bigwedge^d V$ is a subspace of V. Indeed, if v_1, v_2 divide w then

$$(v_1 + v_2) \wedge w = v_1 \wedge w + v_2 \wedge w = 0,$$

which implies that $v_1 + v_2$ divides w. And also $v \wedge w = 0$ implies that $av \wedge w = 0$ for any scalar a.

Definition 1.7. We say that $w \in \bigwedge^d V$ is **totally decomposable** if there exist linearly independent vectors $v_1, \ldots, v_d \in V$ so that $w = v_1 \wedge \cdots \wedge v_d$.

Lemma 1.8. *Let $w \in \bigwedge^d V$. Then w is totally decomposable if and only if the space of vectors dividing it is d-dimensional.*

Proof. Let $w \in \bigwedge^d V$ be a totally decomposable vector. Let $w = v_1 \wedge \cdots \wedge v_d$ for some linearly independent $v_i \in V$. Then by lemma 1.6 the space of vectors dividing w is given by

$$U = \{v \in V \mid v_1 \wedge \cdots \wedge v_d \wedge v = 0\}.$$

Thus $v \in U$ if and only if it is linearly dependent with the vectors $v_1, \ldots, v_d$, i.e. U has a basis $\{v_1, \ldots, v_d\}$. Conversely let U be d-dimensional subspace of V with a basis $\{v_1, \ldots, v_d\}$. Extend this to a basis $\{v_1, \ldots, v_d, v_{d+1}, \ldots, v_n\}$ for V. Then we can write $w \in \bigwedge^d V$ as

$$w = \sum_{1 \le i_1 < \cdots < i_d \le n} a_{i_1,\ldots,i_d} v_{i_1,i_2,\ldots,i_d}.$$

For all $j = 1, \dots, d$ we have $v_j \wedge w = 0$. Then,

$$\begin{aligned} v_j \wedge w &= \sum_{1 \leq i_1 < \cdots < i_d \leq n} a_{i_1,\dots,i_d} v_j \wedge v_{i_1,i_2,\dots,i_d} \\ &= \sum_{\substack{1 \leq i_1 < \cdots < i_d \leq n \\ i_r \neq j}} a_{i_1,\dots,i_d} v_j \wedge v_{i_1,i_2,\dots,i_d}. \end{aligned}$$

The right hand side of the above equation is zero if and only if $a_{i_1,\dots,i_d} = 0$ unless some $i_r = j$. Thus,

$$v_1 \wedge w = \cdots = v_d \wedge w = 0$$

if and only if $a_{i_1,\dots,i_d} = 0$ unless $\{1, \dots, d\} \subset \{i_1, \dots, i_d\}$. Then we have

$$w = a_{i_1,\dots,i_d} v_1 \wedge \cdots \wedge v_d.$$

□

Lemma 1.9. *Let $w \in \bigwedge^d V$. Let*

$$\varphi_w : V \to \bigwedge^{d+1} V$$

be the linear map given by

$$\varphi_w(v) = w \wedge v.$$

Then w is totally decomposable if and only if $\operatorname{Ker}(\varphi_w)$ has dimension d.

Proof. The proof of this lemma follows from the last two lemmas. Note that the kernel of φ_w is given by $\operatorname{Ker}(\varphi_w) = \{v \in V \mid \varphi_w(v) = w \wedge v = 0\}$. This by Lemma 1.6 is the space of vectors dividing w. And by Lamma 1.8, w is totally decomposable if and only if this space has dimension d. □

Theorem 1.10. *The image of $G(d, n)$ via the Plücker map P is an algebraic set of the projective space $\mathbb{P}^N = \mathbb{P}(\bigwedge^d V)$.*

Proof. We observe that $P(G(d,n))$ is the set of all totally decomposable vectors w in $\bigwedge^d V$. By the Lamma 1.9, it can be identified with the set of vectors $w \in \bigwedge^d V$ such that $\dim(\mathrm{Ker}(\varphi_w)) = d$. Equivalently the rank of the map φ_w is $n-d$. Now the map $\bigwedge^d V \to \mathrm{Hom}(V, \bigwedge^{d+1} V)$ sending w to φ_w is linear, that is, the entries of the matrix $\varphi_w \in \mathrm{Hom}(V, \bigwedge^{d+1} V)$ are homogeneous coordinates on $\mathbb{P}(\bigwedge^d V)$. Thus the subset $P(G(d,n)) \subset \mathbb{P}(\bigwedge^d V)$ can be considered as the subvariety defined by the vanishing of $(n-d+1) \times (n-d+1)$ minors of this matrix. □

Unfortunately the equations we get by the above method do not generate the homogeneous ideal of the Grassmannian. To work out this ideal we have to work a bit further.

Lemma 1.11. [9, p.18-19] *Let V be a vector space over k of dimension n with V^* as the dual space. Let $0 < d < n$ be an integer. We have a nondegenerate pairing*

$$\bigwedge^{d} V \times \bigwedge^{n-d} V \to \bigwedge^{n} V \cong k,$$

inducing an isomorphism

$$\bigwedge^{n-d} V \cong (\bigwedge^{d} V)^* = \bigwedge^{d} V^*.$$

Thus, we can identify $\bigwedge^d V$ naturally (up to scalar multiplication) with the exterior power $\bigwedge^{n-d} V^*$ of the dual space.

Now given $w \in (\bigwedge^d V)$ let w^* be the corresponding vector in $\bigwedge^{n-d} V^*$. This gives us a linear map

$$\psi_w : V^* \to \bigwedge^{n-d+1} V^*,$$

which sends v^* to $w^* \wedge v^*$. By the same argument $w \in \bigwedge^d V$ is totally decomposable if and only if the map ψ_w has rank d.

Moreover the kernel of φ_w is precisely the annihilator of the kernel of ψ_w. Take the transpose maps

$$\varphi_w^t : \bigwedge^{d+1} V^* \to V^* \quad \text{and} \quad \psi_w^t : \bigwedge^{n-d+1} V \to V,$$

whose images annihilate each other. Thus, a vector $w \in G(d,n)$ if and only if for every pair $\alpha \in \bigwedge^{d+1} V^*$ and $\beta \in \bigwedge^{n-d+1} V$,

$$\Xi_{\alpha,\beta}(w) := \left\langle \varphi_w^t(\alpha)\ ,\ \psi_w^t(\beta) \right\rangle = \varphi_w^t(\alpha)[\psi_w^t(\beta)] = 0.$$

The $\Xi_{\alpha,\beta}(w)$ are quadratic polynomials and they are called the **Plücker relations**. It turns out that they do generate the homogeneous ideal of the Grassmannian. This ideal is called the **Plücker ideal**.

Example 1.12. *The ideal defining the Grassmannian $G(2,4)$.*
As before let $\{e_1, e_2, e_3, e_4\}$ be a basis for V. The canonical basis for $\bigwedge^2 V$ is given by:

$$B = \{e_1 \wedge e_2,\ e_1 \wedge e_3,\ e_1 \wedge e_4,\ e_2 \wedge e_3,\ e_2 \wedge e_4,\ e_3 \wedge e_4\}.$$

Also the natural basis for $\bigwedge^3 V$ is given by

$$B_1 = \{e_1 \wedge e_2 \wedge e_3,\ e_1 \wedge e_2 \wedge e_4,\ e_1 \wedge e_3 \wedge e_4,\ e_2 \wedge e_3 \wedge e_4\}.$$

If $w = \sum a_{ij} e_i \wedge e_j$, then $\varphi_w : V \to \bigwedge^3 V$ sending v to $v \wedge w$ is given by the following matrix

$$\begin{pmatrix} a_{23} & -a_{13} & a_{12} & 0 \\ a_{24} & -a_{14} & 0 & a_{12} \\ a_{34} & 0 & -a_{14} & a_{13} \\ 0 & a_{34} & -a_{24} & a_{23} \end{pmatrix}.$$

Thus, the variety $G(2,4)$ is defined by the ideal I generated by all 3×3 subdeterminants of the above matrix, namely by the entries of the matrix of the adjoint of the above matrix.

To find the homogeneous ideal defining $G(2,4)$ one observes that $w \in \bigwedge^2 V$ where V is a vector space over field k, $\operatorname{char}(k) \neq 2$, is totally decomposable if and only if $w \wedge w = 0$ and in the case when $V \cong k^4$ we get exactly one quadratic Plücker relation.

Lemma 1.13. *Let k be a field, $\operatorname{char}(k) \neq 2$. Let V be a 4-dimensional vector space over the field k. Then a vector $w \in \bigwedge^2 V$ is totally decomposable if and only if the corresponding Plücker coordinates satisfy the relation $x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0$.*

Proof. Let $w \in \bigwedge^2 V$ be totally decomposable. Let $w = v_1 \wedge v_2$. Then

$$w \wedge w = v_1 \wedge v_2 \wedge v_1 \wedge v_2 = -v_1 \wedge v_2 \wedge v_2 \wedge v_1 = 0.$$

We can write w as

$$w = a_{12}e_1 \wedge e_2 + a_{13}e_1 \wedge e_3 + a_{14}e_1 \wedge e_4 + a_{23}e_2 \wedge e_3 + a_{24}e_2 \wedge e_4 + a_{34}e_3 \wedge e_4.$$

Then by simple computation we get

$$w \wedge w = 2(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})e_1 \wedge e_2 \wedge e_3 \wedge e_4.$$

Thus $w \wedge w = 0$ implies that $a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0$. Therefore, if w is totally decomposable then it satisfies $x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0$. Conversely, let

$$w = a_{12}e_1 \wedge e_2 + a_{13}e_1 \wedge e_3 + a_{14}e_1 \wedge e_4 + a_{23}e_2 \wedge e_3 + a_{24}e_2 \wedge e_4 + a_{34}e_3 \wedge e_4$$

be a vector satisfying

$$a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0. \tag{1.1}$$

Then $w \wedge w = 0$. Now we want to show that w is totally decomposable. For this we consider the following different cases.

1. Suppose first that $a_{12} \neq 0, a_{13} \neq 0$. Then using equation 1.1 we can show that

$$w = \left(a_{12}e_1 + \frac{a_{23}a_{12}}{a_{13}}e_2 + \frac{a_{23}a_{14} - a_{13}a_{24}}{a_{13}}e_4\right) \wedge \left(e_2 + \frac{a_{13}}{a_{12}}e_3 + \frac{a_{14}}{a_{12}}e_4\right).$$

2. Let $a_{12} = 0 = a_{13}$. Then equation 1.1 yields $a_{14}a_{23} = 0$. So we have $a_{14} = 0$ or $a_{23} = 0$ or both are zero. If in this case $a_{14} = 0 = a_{23}$ we can write w as

$$w = a_{24}e_2 \wedge e_4 + a_{34}e_3 \wedge e_4 = (a_{24}e_2 + a_{34}e_3) \wedge e_4.$$

If $a_{14} = 0, a_{23} \neq 0$ then we can decompose w as

$$w = (a_{23}e_2 - a_{34}e_4) \wedge \left(e_3 + \frac{a_{24}}{a_{23}}e_4\right).$$

If $a_{14} \neq 0, a_{23} = 0$, w can be written as

$$w = (a_{14}e_1 + a_{24}e_2 + a_{34}e_3) \wedge e_4.$$

So w is totally decomposable.

3. If $a_{12} = 0, a_{13} \neq 0$, equation 1.1 gives us that $a_{13}a_{24} = a_{14}a_{23}$ and w can be decomposed as

$$w = (a_{13}e_1 + a_{23}e_2 - a_{34}e_4) \wedge \left(e_3 + \frac{a_{14}}{a_{13}}e_4\right).$$

4. If $a_{13} = 0, a_{12} \neq 0$, equation 1.1 gives us that $a_{12}a_{34} = -a_{14}a_{23}$ and w in this case can be decomposed as

$$w = (a_{12}e_1 - a_{23}e_3 - a_{24}e_4) \wedge \left(e_2 + \frac{a_{14}}{a_{12}}e_4\right).$$

Thus we see that in all the cases w is totally decomposable. □

1.1.6 An affine covering of the Grassmannian

Abstractly the Grassmannian $G(d, n)$ can be considered as a union of open sets each isomorphic to the affine space $\mathbb{A}^{d(n-d)}$. To see this, let $\Gamma \subset V$ be a fixed subspace of V of dimension $n-d$. Let $\{e_{i_1}, \ldots, e_{i_{n-d}}\}$ be a basis for Γ. Let $\lambda = P(\Gamma) = e_{i_1} \wedge \cdots \wedge e_{i_{n-d}}$ be the image of Γ via the Plücker map P. We can view λ as a linear form on

$\mathbb{P}(\bigwedge^d V)$ as follows. For $v \in \bigwedge^d V$ define $\lambda(v) := v \wedge \lambda \in \bigwedge^n V \cong k$. We can check that whether $\lambda(v)$ is zero or not is well defined and writing everything in terms of coordinates we get that λ is a homogeneous polynomial of degree 1. We then get the affine variety

$$\begin{aligned} U_\Gamma &= [\mathbb{P}(\bigwedge^d V) - Z(\lambda)] \cap G(d,n) \\ &= \{\, P(K) \mid K \in G(d,n), P(K) \wedge \lambda \neq 0 \,\}. \end{aligned}$$

Now $P(K) \wedge \lambda \neq 0$ means that K is spanned by d elements none of which is linearly dependent with $e_{i_1}, \ldots, e_{i_{n-d}}$. So we can find a basis of V with first d elements spanning K and the remaining elements spanning Γ. Therefore $V = K \bigoplus \Gamma$. Conversely suppose that $V = K \bigoplus \Gamma$. Then we have such a basis for K as these basis vectors together with $e_{i_1}, \ldots, e_{i_{n-d}}$ are linearly independent. So, $P(K) \wedge \lambda \neq 0$.

Proposition 1.14. *Let $\Gamma \subset V$ be a fixed subspace of dimension $n-d$. Fix a subspace K_0 of V such that $V = \Gamma \oplus K_0$. Then in the above notations, U_Γ is given by*

$$U_\Gamma \cong Hom\,(K_0, \Gamma) \cong k^{d(n-d)}.$$

Proof. For $\varphi \in \mathrm{Hom}(K_0, \Gamma)$ we associate to it its graph $\{(t, \varphi(t)) \mid t \in K_0\}$ which is a d-dimensional subspace of V. Also given $K \in G(d,n)$ such that $K \oplus \Gamma = V$ we can see that K arises as a graph of some $\varphi \in \mathrm{Hom}(K_0, \Gamma)$. If $w \in K_0$ there exists unique $u \in \Gamma$ such that $(w, u) \in K$. Then we define $\varphi(w) = u$. Thus we can identify the set U_Γ with $\mathrm{Hom}\,(K_0, \Gamma)$. Moreover, the identification $U_\Gamma \cong \mathrm{Hom}\,(K_0, \Gamma) \cong k^{d(n-d)}$ respects the Zariski topology, i.e.,

$$U_\Gamma \cong \mathrm{Hom}\,(K_0, \Gamma) \cong \mathbb{A}^{d(n-d)}.$$

□

We now see this condition in terms of coordinates. Let $\{e_1, e_2, \ldots, e_n\}$ be a basis for $V \cong k^n$ and let Γ be spanned by $\{e_{d+1}, \ldots, e_n\}$. Then if $K \in U_\Gamma$ and if K has basis

$\{v_1,\ldots,v_d\}$ with $v_j = \sum_{i=1}^n a_{ij}e_j$, then the first $d\times d$ minors of the matrix (a_{ij}) are nonzero. As $P(K)$ does not depend on the choice of the basis, the basis $\{v_1,\ldots,v_d\}$ may be chosen so that the matrix (a_{ij}) has the form

$$(a_{ij}) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ b_{1,1} & \ldots & & b_{1,d} \\ \vdots & \vdots & & \vdots \\ b_{n-d,1} & \ldots & & b_{n-d,d} \end{pmatrix}.$$

Thus, any $K \in U_\Gamma$ can be represented as the column space of the unique matrix of the above form, the entries $b_{i,j}$ of this matrix give the bijection between U_Γ and $k^{d(n-d)}$.

Corollary 1.15. *The dimension of the Grassmannian $G(d,n)$ is $d(n-d)$.*

Proof. Since the Grassmannian $G(d,n)$ can be covered by open sets isomorphic to the affine space $\mathbb{A}^{d(n-d)}$, an immediate consequence is that

$$\dim(G(d,n)) = d(n-d).$$

□

1.2 The number of points in $G(d,n)(\mathbb{F}_q)$

Let k be a perfect field. The Galois Group $\Gamma = \mathrm{Gal}(\overline{k}/k)$ acts on the projective space $\mathbb{P}^n(\overline{k})$ as follows. For $\sigma \in \Gamma$ and $(a_0 : a_1 : \cdots : a_n) \in \mathbb{P}^n(\overline{k})$, we define

$$\sigma(a_0 : \cdots : a_n) = (\sigma(a_0) : \cdots : \sigma(a_n)).$$

The action is well defined since $\forall \lambda \in k^*$ we have

$$\begin{aligned}\sigma(\lambda a_0 : \cdots : \lambda a_n) &= (\sigma(\lambda a_0) : \cdots : \sigma(\lambda a_n)) \\ &= (\sigma(\lambda)\sigma(a_0) : \cdots : \sigma(\lambda)\sigma(a_n)) \\ &= \sigma(a_0 : \cdots : a_n).\end{aligned}$$

One can easily verify that

1. $\mathrm{Id}(a_0 : \cdots : a_n) = (a_0 : \cdots : a_n)$,

2. $\sigma_1\sigma_2(a_0 : \cdots : a_n) = \sigma_1(\sigma_2(a_0 : \cdots : a_n))$.

Lemma 1.16. *The Galois group* $\Gamma = \mathrm{Gal}(\overline{k}/k)$ *acts on* $\mathbb{P}^n(\overline{k})$ *and the fixed points are precisely the points in* $\mathbb{P}^n(k)$ *i.e.*

$$\{u = (a_0 : \cdots : a_n) \in \mathbb{P}^n(\bar{k}) \mid \sigma(u) = u,\ \forall\ \sigma \in \Gamma\} = \mathbb{P}^n(k).$$

Proof. Suppose that for $\sigma \in \Gamma$,

$$\sigma(a_0 : \cdots : a_n) = (a_0 : \cdots : a_n).$$

Then for every σ there is a λ_σ such that

$$\sigma(a_i) = \lambda_\sigma a_i, \quad i = 0, \cdots, n.$$

Without loss of generality let $a_0 \neq 0$. Then for $\sigma \in \Gamma$ we have

$$\sigma(a_i) = \frac{\sigma(a_0) \cdot a_i}{a_0} \quad \text{for} \quad i = 0, 1, \cdots, n.$$

Therefore, we have

$$\frac{a_i}{a_0} = \sigma\left(\frac{a_i}{a_0}\right), \forall \sigma \in \Gamma,$$

that is,

$$\frac{a_i}{a_0} \in k\ \forall i = 0, \cdots, n.$$

So we get

$$(a_0 : a_1 : \cdots : a_n) = (1 : a_1/a_0 : \cdots : a_n/a_0) \in \mathbb{P}^n(k).$$

Thus, the Galois group $\Gamma = \mathrm{Gal}(\bar{k}/k)$ acts on $\mathbb{P}^n(\bar{k})$ and the fixed points are precisely the points in $\mathbb{P}^n(k)$. □

On similar lines we will now consider the action of the Galois group on the Grassmannian $G(d,n)$ and use that to calculate the number of points of $G(d,n)(\mathbb{F}_q)$.

1.2.1 Action of the Galois group on the Grassmannian

Without loss of generality suppose that the n-dimensional vector space V is $(\bar{k})^n$. Then $G(d,n)$ is the collection of all d-dimensional subspaces of $(\bar{k})^n$ and $\Gamma = \mathrm{Gal}(\bar{k}/k)$ acts on it as follows. For $U \in G(d,n)$ and $\sigma \in \Gamma$ define

$$\sigma(U) = \{\sigma(x_1, x_2, \ldots, x_n) \mid (x_1, \ldots, x_n) \in U\},$$

where,

$$\sigma(x_1, x_2, \ldots, x_n) = (\sigma(x_1), \ldots, \sigma(x_n)).$$

It is easy to verify that if U has a basis $\{v_1, v_2, \ldots, v_d\}$ then $\sigma(U)$ is again a d-dimensional subspace of $(\bar{k})^n$ with a basis $\{\sigma(v_1), \ldots, \sigma(v_d)\}$. We therefore get an action of Γ on $G(d,n)(\bar{k})$. We can also think of $G(d,n)$ as embedded in the projective space $\mathbb{P}^N = \mathbb{P}(\bigwedge^d V)$ via the Plücker map $P : G(d,n) \to \mathbb{P}^N$ and we may consider the action of Γ on it as induced by the action on the projective space. Note that the two actions of Γ on $G(d,n)$ are compatible, this means, for $\sigma \in \Gamma, U \in G(d,n)$, we have,

$$\sigma(P(U)) = P(\sigma(U)).$$

Definition 1.17. We say that $U \in G(d,n)$ is **Γ-invariant** if $\sigma(U) = U$ for all $\sigma \in \Gamma$.

Lemma 1.18. *A subspace $U \in G(d,n)(\bar{k})$ is Γ-invariant if and only if U has a basis $\{w_1, w_2, \ldots, w_d\}$ with each $w_i \in k^n$.*

Proof. Clearly if the subspace U has a basis $\{w_1, w_2, \ldots, w_d\}$ with each $w_i \in k^n$ then U is Γ-invariant. Now let U be a d- dimensional subspace of V spanned by the vectors $\{v_1, v_2, \cdots, v_d\}$ such that $\sigma(U) = U,\ \forall\ \sigma \in \Gamma$. We prove that there is a basis $\{w_1, w_2, \ldots, w_d\}$ of U such that

$$\forall \sigma \in \Gamma\,,\ \sigma(w_i) = w_i, \quad i = 1, 2, \cdots, d.$$

As $\sigma(U) = U$, $\exists\, A(\sigma) \in \mathrm{GL}(d, \bar{k})$ such that

$$\sigma \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix} = A(\sigma) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix}.$$

Then we have,

$$\begin{aligned} A(\sigma\tau) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix} &= \sigma \left[\tau \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix} \right] = \sigma \left[A(\tau) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix} \right] \\ &= \sigma[A(\tau)] \sigma \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix} = \sigma[A(\tau)] A(\sigma) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix}. \end{aligned}$$

So we have $A(\sigma\,\tau) = [\sigma A(\tau)]\, A(\sigma)$, i.e., $\{A(\sigma)\}$ is a 1-cocycle and using the result that $H^1(\Gamma, GL(n, \bar{k}))$ is the identity [18, p.159] we get that the 1-cocycle $\{A(\sigma)\}$ splits. Therefore, there exists $B \in \mathrm{GL}(d, \bar{k})$ such that $B = (\sigma B)\, A(\sigma)$. Now let

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_d \end{pmatrix} = B \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix}.$$

Then we have

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_d \end{pmatrix} = B \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix} = (\sigma B) \cdot A(\sigma) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix} = (\sigma B) \cdot \sigma \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix} = \sigma \left[B \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix} \right] = \sigma \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_d \end{pmatrix}.$$

So for all $\sigma \in \Gamma\,,\ \sigma(w_i) = w_i\,,\ \ i = 1, 2, \cdots d$, which implies that U has a basis $\{w_1, w_2, \cdots, w_d\}$ with $w_i \in k^n$ (as $(\bar{k})^\Gamma = k$). □

We now use these results to calculate the cardinality of $G(d,n)(\mathbb{F}_q)$.

Proposition 1.19. *The number of points of $G(d,n)(\mathbb{F}_q)$ is given by*

$$|G(d,n)(\mathbb{F}_q)| = \frac{f(n)}{f(d)\cdot f(n-d)\cdot q^{d(n-d)}}.$$

where $f(n) = (q^n-1)(q^n-q)\cdots(q^n-q^{n-1})$.

Proof. Let $k = \mathbb{F}_q$. Then we have $|\ G(d,n)(k)\ | = |\ [G(d,n)(\overline{k})]^{\Gamma}\ |$ which is the number of d-dimensional subspaces of $(\overline{k})^n$ that are Γ- invariant. Let J denote the collection of all ordered bases $\{v_1, v_2, \dots, v_d\}$ with each $v_i \in k^n$. Then J defines an open subset of $(k^n)^d$. By Lemma 1.18 it follows that to compute the number of subspaces which are Γ-invariant one can compute the number of elements of J and take into account how many different ordered bases give rise to the same element of $G(d,n)$. Let $U \in G(d,n)$. The cardinality of $G(d,n)(\mathbb{F}_q)$ is given by

$$\frac{\text{number of points of } J}{\text{number of ordered bases for each } U}.$$

The number of ordered bases for each U is $|\mathrm{GL}(d,k)|$. So we get

$$|G(d,n)(\mathbb{F}_q)| = \frac{|J|}{|\mathrm{GL}(d,k)|}.$$

Now we find $|J|$. The general linear group $\mathrm{GL}(n,k) = \mathrm{Aut}(k^n)$ acts naturally on J and the action is transitive. The stabilizer of $X = \{e_1, \dots, e_d\}$ has the block matrix of the form

$$\begin{pmatrix} I_d & * \\ 0 & \mathrm{GL}(n-d,k) \end{pmatrix}.$$

Hence,

$$|J| = \frac{|\mathrm{GL}(n,k)|}{|\mathrm{stabilizer}(X)|} = \frac{1}{q^{d(n-d)}} \cdot \frac{|\mathrm{GL}(n,k)|}{|\mathrm{GL}(n-d,k)|}.$$

Then we have

$$|G(d,n)(\mathbb{F}_q)| = \frac{|\mathrm{GL}(n,\mathbb{F}_q)|}{|\mathrm{GL}(d,\mathbb{F}_q)|\cdot|\mathrm{GL}(n-d,\mathbb{F}_q)|\cdot q^{d(n-d)}} = \frac{f(n)}{f(d)\cdot f(n-d)\cdot q^{d(n-d)}},$$

where $f(n) = (q^n-1)(q^n-q)\cdots(q^n-q^{n-1})$. □

1.2.2 The Zeta function of the Grassmannian

Let X be a smooth projective variety over $k = \mathbb{F}_q$. The Zeta function of X is defined by

$$Z(X,t) := \exp\left(\sum_{r=1}^{\infty} N_r.\frac{t^r}{r}\right) \in \mathbb{Q}[[t]],$$

where N_r is the number of points of X defined over $\mathbb{F}_{q^r}$. Let X be a non-singular projective variety of dimension n. Then the Weil conjectures [8, Appendix C], proven by Deligne and Dwork, concerning $Z(X,t)$ are:

1. **Rationality** : $Z(X,t)$ is a rational function in t.

2. **Functional equation** : $Z(X,t)$ satisfies the functional equation namely,

$$Z\left(X, \frac{1}{q^n t}\right) = \pm q^{\frac{nE}{2}} t^E Z(X,t),$$

   where E is the Euler characteristic of X which can be defined as the self intersection number of the diagonal $\triangle \subset X \times X$.

3. **Riemann hypothesis** : We can write

$$Z(X,t) = \frac{P_1(t) \cdot P_3(t) \cdots P_{2n-1}(t)}{P_0(t) \cdot P_2(t) \cdots P_{2n}(t)},$$

   where $P_0(t) = 1 - t$, $P_{2n}(t) = 1 - q^n t$ and for each $1 \leq i \leq 2n-1$, $P_i(t)$ is a polynomial with integer coefficients which can be written as

$$P_i(t) = \prod_{j=1}^{b_i} (1 - \omega_{ij} t),$$

   where ω_{ij} are algebraic integers with $|\omega_{ij}| = q^{i/2}$. Given $Z(X,t)$, these conditions uniquely determine the polynomials $P_i(t)$.

4. **Cohomological interpretation**: Define the i-th Betti number b_i of X as the degree of the polynomial $P_i(t)$ where $P_i(t)$ is as in 3. Then we have the

Euler characteristic, $E = \sum(-1)^i b_i$. Suppose now that X is obtained from a variety Y defined over an algebraic number ring R, by reduction modulo a prime ideal $\mathfrak{p}$ of R. Then b_i is equal to the i-th Betti number of the topological space $Y_h = Y \otimes_R \mathbb{C}$, i.e., b_i is the rank of the singular cohomology group $H^i(Y_h;\mathbb{Z})$.

As seen before, the Grassmannian $G(d,n)$ can be embedded into the projective space $\mathbb{P}(\bigwedge^d V)$ via the Plücker map. Recall that $G(d,n)$ can be covered by open affine spaces of dimension $d(n-d)$, so it is a smooth projective variety of dimension $d(n-d)$ which may be considered over any finite field $\mathbb{F}_q$. We now calculate the Zeta function of some Grassmannians over $\mathbb{F}_q$. We will also see the rationality of the Zeta function and the functional equation in a few examples.

Example 1.20. *Projective space* $G(1,n+1) = \mathbb{P}^n$. *One has*

$$|\mathbb{P}^n(\mathbb{F}_q)| = 1 + q + q^2 + \cdots + q^n,$$

and so,

$$N_r = |\mathbb{P}^n(\mathbb{F}_{q^r})| = 1 + q^r + q^{2r} + \cdots + q^{nr},$$

$$Z(t) := Z(\mathbb{P}^n_{\mathbb{Z}} \otimes \mathbb{F}_q, t) = \exp\left(\sum_{r=1}^{\infty}(1 + q^r + \cdots + q^{nr})\frac{t^r}{r}\right).$$

Taking logarithm on both sides and using the formula : $\ln(1-t) = -t - t^2/2 - t^3/3 - \ldots,$ *we get,*

$$\begin{aligned} \ln[Z(t)] &= \sum_{r=1}^{\infty}(1 + q^r + ... + q^{nr})\frac{t^r}{r} \\ &= -\ln(1-t) - \ln(1-qt) - \cdots - \ln(1-q^n t). \\ &= -\ln[(1-t)\cdots(1-q^n t)]. \end{aligned}$$

It follows that

$$Z(\mathbb{P}^n_{\mathbb{Z}} \otimes \mathbb{F}_q, t) = \frac{1}{(1-t)(1-qt)\cdots(1-q^n t)}.$$

We see that $P_i(t) = 1$ *for all odd* i *and* $P_{2i}(t) = 1 - q^i t$ *for* $i = 0, 1, 2, \cdots, n$*. The degree of* $P_i(t)$ *is zero for* i *odd and 1 for* i *even; odd Betti numbers are zero and the even Betti numbers are equal to* 1*. The Euler characteristic is* $E = \sum b_i = n+1$*. We now verify the functional equation*

$$\begin{aligned} Z\left(\mathbb{P}^n_{\mathbb{Z}} \otimes \mathbb{F}_q, \frac{1}{q^n t}\right) &= \frac{1}{(1-1/q^n t)(1-q/q^n t)\cdots(1-q^n/q^n t)} \\ &= \frac{q^n t \cdot q^{n-1} t \cdots qt \cdot t}{(1-t)(1-qt)\cdots(1-q^n t)} \\ &= q^{n(n+1)/2} \cdot t^{n+1} \\ &= q^{n \cdot E/2} \cdot t^E \cdot Z(\mathbb{P}^n \otimes \mathbb{F}_q, t). \end{aligned}$$

Thus, the functional equation is verified. Also note that the numbers $b_0, b_1, \cdots b_n$ *match with the Betti numbers of the complex projective space* $\mathbb{P}^n(\mathbb{C})$ *and the number* $E = n+1$ *matches with the Euler characteristic of* $\mathbb{P}^n(\mathbb{C})$*.*

Example 1.21. *The Grassmannian* $G(2,4) \otimes \mathbb{F}_q$*. By the general formula, the dimension of* $G(2,4)$ *is* 4*. We first calculate* N_r*. By Proposition 1.19,*

$$\begin{aligned} |G(2,4)(\mathbb{F}_q)| &= \frac{(q^4-1)(q^4-q)(q^4-q^2)(q^4-q^3)}{(q^2-1)^2(q^2-q)^2 q^4} \\ &= (q^2+1)(q^2+q+1) = 1 + q + 2q^2 + q^3 + q^4, \end{aligned}$$

and so

$$N_r = 1 + q^r + 2q^{2r} + q^{3r} + q^{4r}.$$

It follows that

$$\begin{aligned} Z(G(2,4) \otimes \mathbb{F}_q, t) &= \exp\left(\sum_{r=1}^{\infty} (1 + q^r + 2q^{2r} + q^{3r} + q^{4r}) \frac{t^r}{r}\right) \\ &= \frac{1}{(1-t)(1-qt)(1-q^2t)^2(1-q^3t)(1-q^4t)}. \end{aligned}$$

We see that $Z(t)$ *is a rational function in* t*. The polynomial* $P_i(t) = 1$ *for all odd* i*. We have,* $P_0(t) = 1 - t,\ P_2(t) = 1 - qt,\ P_4(t) = (1-q^2t)^2,\ P_6(t) = 1 - q^3 t,\ P_8(t) = 1 - q^4 t.$

The Betti numbers b_i are zero for all odd i and $b_0 = 1,\ b_2 = 1,\ b_4 = 2,\ b_6 = 1,\ b_8 = 1$. The Euler characteristic $E = \sum b_i = 6$. We now verify the functional equation for $X = G(2,4) \otimes \mathbb{F}_q$

$$\begin{aligned} Z\left(X, \frac{1}{q^4 t}\right) &= \frac{1}{(1-1/q^4t)(1-q/q^4t)(1-q^2/q^4t)^2(1-q^3/q^4t)(1-q^4/q^4t)} \\ &= q^4t \cdot q^3t \cdot (q^2t)^2 \cdot qt \cdot t \cdot Z(X,t) \\ &= q^{12} \cdot t^6 \cdot Z(X,t) \\ &= q^{nE/2} t^E \cdot Z(X,t), \end{aligned}$$

and the functional equation is verified.

Example 1.22. *The Grassmannian $G(2,5) \otimes \mathbb{F}_q$. We have*

$$\begin{aligned} |G(2,5)(\mathbb{F}_q)| &= \frac{(q^5-1)(q^5-q)(q^5-q^2)(q^5-q^3)(q^5-q^4)}{(q^2-1)(q^2-q)(q^3-1)(q^3-q)(q^3-q^2)q^6} \\ &= 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6, \end{aligned}$$

and so,

$$N_r = 1 + q^r + 2q^{2r} + 2q^{3r} + 2q^{4r} + q^{5r} + q^{6r}.$$

It follows that

$$Z(G(2,5) \otimes \mathbb{F}_q, t) = \exp\left(\sum_{r=1}^{\infty} (1 + q^r + 2q^{2r} + 2q^{3r} + 2q^{4r} + q^{5r} + q^{6r}) \frac{t^r}{r}\right),$$

and by similar calculations we get

$$Z(G(2,5) \otimes \mathbb{F}_q, t) = \frac{1}{(1-t)(1-qt)(1-q^2t)^2(1-q^3t)^2(1-q^4t)^2(1-q^5t)(1-q^6t)}.$$

1.2.3 The general case $G(d,n) \otimes \mathbb{F}_q$

By proposition 1.19 we get

$$\begin{aligned} N_r &= |G(d,n)(\mathbb{F}_{q^r})| \\ &= \frac{(q^{nr}-1)(q^{nr}-q^r)\cdots(q^{nr}-q^{(n-1)r})}{(q^{dr}-1)\cdots(q^{dr}-q^{(d-1)r}) \cdot (q^{(n-d)r}-1)\cdots(q^{(n-d)r}-q^{(n-d-1)r}) \cdot q^{rd(n-d)}}. \end{aligned}$$

For simplicity set $q^r = l$. So we have

$$N_r = \frac{(l^n-1)(l^n-l)\cdots(l^n-l^{n-1})}{(l^d-1)\cdots(l^d-l^{d-1})\cdot(l^{n-d}-1)\cdots(l^{n-d}-l^{n-d-1})\cdot l^{d(n-d)}}$$

Multiplying and dividing by $l^{d(n-d)}$ and simplifying we get

$$N_r = \frac{(l^n-1)(l^{n-1}-1)\cdots(l^{n-d+1}-1)}{(l^d-1)(l^{d-1}-1)\cdots(l-1)}.$$

This is the usual **Gaussian binomial coefficient or l-binomial coefficient** $\binom{n}{d}_l$ and it can be interpreted as a polynomial in l. To be more precise

$$\binom{n}{d}_l = \sum_{i=0}^{d(n-d)} b_i l^i.$$

where the coefficient b_i of l^i is the number of distinct partitions of i elements that fit inside a rectangle of size $d \times (n-d)$. For a detailed discussion on the Gaussian binomial coefficient refer to [1, section 13.5]. We illustrate this with examples.

Example 1.23. *Find the Gaussian binomial coefficient* $\binom{4}{2}_l$.
Suppose $\binom{4}{2}_l = b_0 + b_1 l + b_2 l^2 + b_3 l^3 + b_4 l^4$. *We summarize the number of partitions of* i *for* $i = 0, 1, 2, 3, 4$ *in the following table.*

i	*admissible partitions of* i	b_i = *number of admissible partitions*
0	$\{\ \}$	1
1	$\{1\}$	1
2	$\{\{2\},\{1,1\}\}$	2
3	$\{2,1\}$	1
4	$\{2,2\}$	1

Hence we get

$$\binom{4}{2}_l = 1 + l + 2l^2 + l^3 + l^4,$$

i.e., $N_r = 1 + q^r + 2q^{2r} + q^{3r} + q^{4r}$. *Note that this calculation matches with the calculation done before while calculating the Zeta function of* $G(2,4) \otimes \mathbb{F}_q$.

Example 1.24. *Find the Gaussian binomial coefficient* $\binom{5}{2}_l$.
Suppose $\binom{5}{2}_l = b_0 + b_1 l + b_2 l^2 + b_3 l^3 + b_4 l^4 + b_5 l^5$. *We summarize the number of allowed partitions of* i *for* $i = 0, 1, 2, 3, 4, 5, 6$ *in the following table*

i	*admissible partitions of* i	b_i = *number of admissible partitions*
0	$\{\}$	1
1	$\{1\}$	1
2	$\{\{2\},\{1,1\}\}$	2
3	$\{\{2,1\},\{1,1,1\}\}$	2
4	$\{\{2,2\},\{2,1,1\}\}$	2
5	$\{\{2,2,1\}\}$	1
6	$\{\{2,2,2\}\}$	1

Hence we have

$$\binom{5}{2}_l = 1 + l + 2l^2 + 2l^3 + 2l^4 + l^5 + l^6,$$

i.e., $N_r = 1 + q^r + 2q^{2r} + 2q^{3r} + 2q^{4r} + q^{5r} + q^{6r}$. *Again this calculation matches with the calculation done before while calculating the Zeta function of* $G(2,5) \otimes \mathbb{F}_q$.

Example 1.25. *Find the Gaussian binomial coefficient* $\binom{6}{3}_l$.
Suppose

$$\binom{6}{3}_l = b_0 + b_1 l + b_2 l^2 + b_3 l^3 + b_4 l^4 + b_5 l^5 + b_6 l^6 + b_7 l^7 + b_8 l^8 + b_9 l^9.$$

We summarize the number of allowed partitions of i *for* $i = 0, 1, \ldots, 9$ *in the following table*

i	*admissible partitions of* i	b_i = *number of admissible partitions*
0	$\{\}$	1
1	$\{1\}$	1
2	$\{\{2\},\{1,1\}\}$	2
3	$\{\{3\},\{2,1\},\{1,1,1\}\}$	3
4	$\{\{3,1\},\{2,2\},\{2,1,1\}\}$	3
5	$\{\{2,2,1\},\{3,2\},\{3,1,1\}\}$	3
6	$\{\{2,2,2\},\{3,2,1\},\{3,3\}\}$	3
7	$\{\{3,2,2\},\{3,3,1\}\}$	2
8	$\{\{3,3,2\}\}$	1
9	$\{\{3,3,3\}\}$	1

Hence we get

$$\binom{6}{3}_l = 1 + l + 2l^2 + 3l^3 + 3l^4 + 3l^5 + 3l^6 + 2l^7 + l^8 + l^9,$$

i.e., $N_r = 1 + q^r + 2q^{2r} + 3q^{3r} + 3q^{4r} + 3q^{5r} + 3q^{6r} + 2q^{7r} + q^{8r} + q^{9r}$.

Now we consider the general case. Regarding l as a formal variable, it is possible to express the coefficient N_r of any Grassmannian $G(d,n) \otimes \mathbb{F}_q$ as

$$N_r = \sum_{i=0}^{d(n-d)} b_i l^i.$$

where $b_i = b_i(d,n,l)$ can be found as explained before and the Zeta function of the Grassmannian then comes out to be

$$Z(G(d,n) \otimes \mathbb{F}_q, t) = \frac{1}{(1-t)^{b_0}(1-qt)^{b_1} \cdots (1-q^{d(n-d)}t)^{b_{d(n-d)}}}.$$

From this we observe that all the odd Betti numbers of the Grassmannians are zero. As we shall see in section 1.4.3, the numbers b_i here are the even topological Betti numbers of the complex Grassmannian $X(\mathbb{C}) = G(d,n)(\mathbb{C})$, i.e., $b_i = \dim H_{2i}(X(\mathbb{C}), \mathbb{Z})$. The odd Betti numbers of $X(\mathbb{C})$ are zero.

1.2.4 Euler characteristic of the Grassmannian

Consider the Grassmannian $G(d,n)$. As seen in the last section, the odd Betti numbers of the Grassmannian are zero and the even Betti numbers are related to the Gaussian binomial coefficient by

$$\binom{n}{d}_l = \sum_{i=0}^{d(n-d)} b_i l^i.$$

Putting $l=1$ in the above expression we immediately get the Euler characteristic of the Grassmannian as

$$E = \sum_{i=0}^{d(n-d)} b_i = \binom{n}{d}_1.$$

Referring to [1] section 13.5, Theorem 1, the Gaussian binomial coefficient $\binom{n}{d}_1$ is the usual binomial coefficient $\binom{n}{d}$. Hence the Euler characteristic of $G(d,n)$ is $\binom{n}{d}$.

1.3 Schubert calculus

Let $G^{\mathbb{P}}(d,n)$ be the set of all d-dimensional subspaces (or d-planes) of the n dimensional complex projective space $\mathbb{P}^n$ i.e. in our old notation, $G^{\mathbb{P}}(d,n) = G(d+1,n+1)$. Now onwards we always refer to the projective space $\mathbb{P}^n$ over the complex numbers $\mathbb{C}$. Let $N = \binom{n+1}{d+1} - 1$. As seen before there is a natural way of associating a point of $\mathbb{P}^N$ to a d-plane $L \in G^{\mathbb{P}}(d,n)$ and the coordinates of L regarded as elements of $\mathbb{P}^N$ are called the Plücker coordinates. This embedding of $G^{\mathbb{P}}(d,n)$ into $\mathbb{P}^N$ makes it into a manifold of dimension $(d+1)(n-d)$. The Schubert Calculus describes the cohomology ring of $G^{\mathbb{P}}(d,n)$ say with integer coefficients when the base field is $\mathbb{C}$. The subject started with a typical enumerative problem: How many lines in 3-space in general, intersect 4 given lines? The answer to this question lies in finding the degree of some Schubert cycles. The fundamental theorem of Schubert calculus also helps understand the generalization of Bézout's theorem. We now develop important notions of the subject.

1.3.1 Schubert conditions and Schubert varieties.

We are interested in finding a necessary and sufficient condition for a d-plane in the projective space $\mathbb{P}^n$ to intersect a given sequence of linear spaces of $\mathbb{P}^n$ in a prescribed way. Let $\underline{\mathrm{A}} : A_0 \subset A_1 \subset \cdots \subset A_d$ be a strictly increasing sequence of $d+1$ linear spaces of $\mathbb{P}^n$. Such a sequence is called a **flag**. Let $\dim A_i = a_i$ for each i. If we take A_i to be consisting of all points in $\mathbb{P}^n$ of the form $(x_0 : x_1 : \cdots : x_i : 0 : 0 : 0 : \cdots : 0)$ then we call $\underline{\mathrm{A}}$ the **standard flag**.

Definition 1.26. A d-plane L in $\mathbb{P}^n$ is said to satisfy the **Schubert condition** defined by the flag $\underline{\mathrm{A}}$ if $\dim(A_i \bigcap L) \geq i \quad \text{for all} \quad i = 0, 1, \cdots, d.$

Thus a d-plane satisfying the Schubert conditions with respect to the flag $\underline{\mathrm{A}}$ intersects A_0 at least in a point, A_1 at least in a line etc., and it lies in A_d. It can be seen that the condition $\dim(A_i \bigcap L) \geq i$ for $i = 0, \cdots, d$ is satisfied if and only if the Plücker coordinates of the d-plane L satisfy certain linear relations in addition to the quadratic relations. Indeed the collection of all such planes defines a variety. For the proof of this refer to [12, p.1066-1070].

Definition 1.27. The collection of all d-planes in $G^{\mathbb{P}}(d, n)$ satisfying the Schubert condition with respect to a given flag $\underline{\mathrm{A}}$ defines a projective variety. It is known as the **Schubert variety** $\Omega(\underline{\mathrm{A}})$ corresponding to the flag $\underline{\mathrm{A}}$.

In fact this variety is the intersection of a linear subspace of $\mathbb{P}^n$ with $G^{\mathbb{P}}(d, n)$. The dimension of the Schubert variety $\Omega(\underline{\mathrm{A}})$ with $\underline{\mathrm{A}}$ as above is $\sum_{i=0}^{d}(a_i - i)$. For the proof of this fact refer to [12, p.1071].

Example 1.28. *Let A_0 be a line in $\mathbb{P}^3$. Let $A_1 = \mathbb{P}^3$. Let $\underline{A} : A_0 \subset A_1 = \mathbb{P}^3$ be a flag in $\mathbb{P}^3$. Then $\Omega(\underline{A})$ is the set of all lines L in $\mathbb{P}^3$ such that* $\dim(L \cap A_0) \geq 0$ *and* $\dim(L \cap \mathbb{P}^3) \geq 1$*. As $L \cap \mathbb{P}^3 = L$, the second condition is automatically satisfied and $\Omega(\underline{A})$ is the set of all lines L in $\mathbb{P}^3$ that intersect the line A_0.*

Example 1.29. *Suppose that* $\dim(A_i) = i \quad \forall i = 0, 1, \cdots, d$. *Then* $\Omega(\underline{A})$ *consists of the single* d*-plane* A_d.

Example 1.30. *Suppose that* $\dim(A_i) = i \quad \forall i = 0, 1, \cdots, d-1$. *Let* $\dim(A_d) = d+r$. *Then* $\Omega(\underline{A})$ *consists of all* d*-dimensional subspaces of* $G^{\mathbb{P}}(d, n)$ *which contain* A_{d-1} *and which are contained in* A_d *and such a set is isomorphic to* $\mathbb{P}^r$.

Example 1.31. *Suppose that* $\dim(A_i) = n-d+i \ \forall i = 0, 1, \cdots, d$. *Then, the Schubert variety* $\Omega(\underline{A})$ *is* $G^{\mathbb{P}}(d, n)$.

1.3.2 Some cohomology theory for a topological space and Schubert cycles

We recall some singular homology theory. For the details of the subject one can refer to [10, Chapters 2 and 3]. An n-simplex is the smallest convex set in $\mathbb{R}^m$ containing $n+1$ points $v_0, v_1, \ldots, v_n$ that do not lie in a hyperplane of dimension less than n. The points v_i are the vertices of the simplex, and the simplex itself will be denoted by $[v_0, \cdots, v_n]$. The standard n simplex is given by

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1, \quad t_i \geq 0 \ \forall \ i\}.$$

A singular n-simplex in a topological space X is a continuous map $\sigma : \Delta^n \to X$. Let $C_n(X)$ be the free abelian group with basis consisting of the set of all singular n-simplices in X. Elements of $C_n(X)$ are called singular n-chains. These are formal sums $\sum_i n_i \sigma_i, \ n_i \in \mathbb{Z}$, almost all zero and $\sigma_i : \Delta^n \to X$. The boundary of the n-simplex $[v_0, \cdots, v_n]$ consists of the various $(n-1)$-dimensional simplices $[v_0, \ldots, \widehat{v_i}, \ldots, v_n]$, where the symbol hat over v_i indicates that this vertex is deleted from the sequence $v_0, \cdots, v_n$. The boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ is defined by

$$\partial_n(\sigma) = \sum_i (-1)^i \, \sigma| \, [v_0, \ldots, \widehat{v_i}, \ldots, v_n].$$

We have $\partial_n \cdot \partial_{n+1} = 0$ and we define the n-th homology group of X by

$$H_n(X) = \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1}.$$

We now define the cohomology of a space.

Definition 1.32. Let X be a topological space and G be an abelian group. We define the group $C^n(X;G)$ of singular n-cochains with coefficients in G to be the dual group $\operatorname{Hom}(C_n(X);G)$ of the singular chain group $C_n(X)$. The coboundary map $\delta^n : C^n(X;G) \to C^{n+1}(X;G)$ is the dual of the map between n chains and we have $\delta^n \cdot \delta^{n-1} = 0$. Elements of $\operatorname{Ker} \delta^n$ are called n**-cocycles** and the elements of $\operatorname{Im}\ \delta^n$ are called n**-coboundaries**. We define the cohomology group $H^n(X;G)$ as the quotient $\operatorname{Ker}\ \delta^n / \operatorname{Im} \delta^{n-1}$.

Definition 1.33. Cup product: We consider the cohomology with coefficients in a ring R (e.g. in $\mathbb{Z}$). For cochains $\phi \in C^k(X;R)$, $\psi \in C^l(X,R)$, the cup product $\phi \cup \psi \in C^{k+l}(X;R)$ is the cochain whose value on a singular simplex $\sigma : \Delta^{k+l} \to X$ is given by

$$(\phi \cup \psi)(\sigma) = \phi(\sigma|[v_0, \ldots, v_k])\ \psi(\sigma|[v_k \ldots\ v_{k+l}]).$$

The cup product of cochains is bilinear and associative. It can be shown that the cup product of cochains induces a cup product of cohomology classes namely

$$H^k(X;R) \times H^l(X;R) \to H^{k+l}(X;R).$$

This product is bilinear, associative and distributive since at the level of cochains the product has these properties. If R has an identity element, there is an identity element for the cup product. Note that the cup product is, in general, not commutative. Instead it is anti-commutative. If $\phi \in C^k(X;R)$, $\psi \in C^l(X,R)$ then one has (in cohomology)

$$\phi \cup \psi = (-1)^{kl} \psi \cup \phi.$$

Definition 1.34. Cap product: Let X be a topological space. Let R be the coefficient ring. For for $\sigma : \Delta^k \to X$ and $\phi \in C^l(X;R), k \geq l$, define an R-bilinear cap product $\cap : C_k(X;R) \times C^l(X;R) \to C_{k-l}(X;R)$ by

$$\sigma \cap \phi = \phi(\sigma|[v_0, \ldots, v_l])\sigma|[v_l, \ldots, v_k].$$

This induces a cap product in homology and cohomology namely

$$H_k(X;R) \times H^l(X;R) \to H_{k-l}(X;R),$$

which is R-linear in each variable.

Definition 1.35. The cohomology ring: Define the cohomology ring $H^*(X;R)$ as the graded ring $H^*(X;R) := \bigoplus_{n\geq 0} H^n(X;R)$. Elements of $H^*(X;R)$ are finite sums $\sum_i \alpha_i$ with $\alpha_i \in H^i(X;R)$. We define the product $(\sum_i \alpha_i).(\sum_i \beta_i) = \sum_{i,j} \alpha_i \cup \beta_j$. This makes $H^*(X,R)$ into a ring with identity if R has identity.

Poincaré Duality: The Poincaré duality relates the homology and the cohomology groups of a compact oriented triangulated n-manifold X in dimension k and $n-k$. The cohomology groups form a graded ring with respect to cup product and the homology groups form a module over the cohomology ring by means of cap product. The canonical map $H^i(X;R) \to H_{n-i}(X;R)$ taking α to $\alpha \cap [X]$ is an isomorphism. This map is called the **Poincaré duality map**. When X is a non-singular complex projective variety of dimension n, it is an oriented real $2n$-manifold and the group $H_{2n}(X;R)$ has a canonical generator $[X]$. A closed subvariety V of dimension k of a projective variety X determines a class $[V]$ in $H_{2k}(X;R)$ and, by Poincaré duality, we have the class in $H^{2c}(X;R) = H_{2k}(X;R)$, where c is codimension of V in X. Thus, if X is smooth proper over $\mathbb{C}$ and V is a subvariety of codimension k then, there exists associated to V a cohomology class $\eta(Y) \in H^{2k}(X;\mathbb{Z})$. This map extends by linearity to cycles.

Applying this to Schubert varieties we see that $\Omega(\underline{A})$ defines a cohomology class in the cohomology ring $H^*(G^{\mathbb{P}}(d,n);\mathbb{Z})$. The cohomology class of $\Omega(\underline{A})$ in $H^*(G^{\mathbb{P}}(d,n);\mathbb{Z})$ is called a **Schubert cycle**. Although the variety $\Omega(\underline{A})$ depends on the choice of the flag $\underline{A}$, it can be shown that [12, p. 1070] the cohomology class of $\Omega(\underline{A})$ depends only on the integers $a_i = \dim A_i$. So we denote the class of $\Omega(\underline{A})$ by $\Omega(a_0,\dots,a_d) = \Omega(\underline{a})$ where $\underline{a}$ is defined by integers $a_i = \dim A_i$, $0 \le a_0 < a_1 < \cdots < a_d \le n$.

We now state the fundamental theorem of Schubert calculus which asserts that the Schubert cycles completely determine the cohomology of $G^{\mathbb{P}}(d,n)$.

Theorem 1.36. *The Basis Theorem* (as stated in [12, p. 1071]) *Considered additively, $H^*(G^{\mathbb{P}}(d,n);\mathbb{Z})$ is a free abelian group and the Schubert cycles $\Omega(a_0,\dots,a_d)$ form a basis. Each integral cohomology group $H^{2p}(G^{\mathbb{P}}(d,n);\mathbb{Z})$ is a free abelian group and the Schubert cycles $\Omega(\underline{a})$ with $[(d+1)(n-d) - \sum_{i=0}^{d}(a_i - i)] = p$ form a basis. Each cohomology group $H^r(G^{\mathbb{P}}(d,n);\mathbb{Z})$, with r odd, is zero.*

The Basis Theorem determines the additive structure of the cohomology ring $H^*(G^{\mathbb{P}}(d,n);\mathbb{Z})$. Since each odd cohomology group is zero we observe that the cup product is commutative and the ring $H^*(G^{\mathbb{P}}(d,n);\mathbb{Z})$ is a commutative ring.

To determine the multiplicative structure we need some combinatorics. Let b_i denote the i-th **Betti number** of $G^{\mathbb{P}}(d,n)$, i.e. $b_i = \text{rank}\left(H^i(G^{\mathbb{P}}(d,n);\mathbb{Z})\right)$. By the Basis Theorem, b_{2p} is equal to the number of solutions in integers a_i to the equation

$$[(d+1)(n-d) - \sum_{i=0}^{d}(a_i - i)] = p \quad \text{where} \quad 0 \le a_0 < a_1 < \cdots < a_d \le n.$$

We now calculate the cohomology groups of some Grassmannians and find their dimensions.

Example 1.37. *The projective space $\mathbb{P}^n = G^{\mathbb{P}}(0,n)$. The dimension of $\mathbb{P}^n$ is n. Using the Basis Theorem for $p = 0,1,\dots,n$, $H^{2p}(\mathbb{P}^n;\mathbb{Z})$ is one dimensional generated by the Schubert cycle $\Omega(a_0)$ such that $n - a_0 = p$. In fact, $\Omega(a_0)$ is a hyperplane of complex* *codimension* $n - a_0$*. The cohomology group* $H^r(\mathbb{P}^n;\mathbb{Z})$ *is* 0 *for* r *odd. So all the odd Betti numbers are zero and the even Betti numbers are equal to* 1*.*

Example 1.38. *The Grassmannian* $G(2,4) = G^{\mathbb{P}}(1,3)$*. The dimension of* $G(2,4)$ *is* 4*. For* $0 \leq p \leq 4$*,* $H^{2p}(G^{\mathbb{P}}(1,3);\mathbb{Z})$ *is generated by the Schubert cycle* $\Omega(a_0,a_1)$ *such that* $4 - [a_0 + (a_1 - 1)] = p$ *i.e.* $a_0 + a_1 = 5 - p$*. For* $p = 0$*, the only integer solution to* $a_0 + a_1 = 5$ *with* a_0 *and* a_1 *as in Schubert conditions is* $a_0 = 2$ *and* $a_1 = 3$*. Hence,* $H^0(G^{\mathbb{P}}(1,3);\mathbb{Z})$ *is generated by* $\Omega(2,3)$ *and has dimension 1. We summarize the calculations for the other cohomology groups in the following table.*

p	$\dim(H^{2p}(G^{\mathbb{P}}(1,3);\mathbb{Z}))$	generators
0	1	$\Omega(2,3)$
1	1	$\Omega(1,3)$
2	2	$\Omega(0,3),\Omega(1,2)$
3	1	$\Omega(0,2)$
4	1	$\Omega(0,1)$

Example 1.39. *The Grassmannian* $G(2,5) = G^{\mathbb{P}}(1,4)$*. The dimension of* $G(2,5)$ *is* 6*. For* $0 \leq p \leq 6$*,* $H^{2p}(G^{\mathbb{P}}(1,4);\mathbb{Z})$ *is generated by the cohomology class of* $\Omega(a_0,a_1)$ *with* $6 - [a_0 + (a_1 - 1)] = p$ *i.e.* $a_0 + a_1 = 7 - p$*. For* $p = 0$*, the only integer solution to* $a_0 + a_1 = 7$ *with* a_0 *and* a_1 *as in Schubert conditions is* $a_0 = 3$ *and* $a_1 = 4$*. We summarize the calculations for other cohomology groups in the following table.*

p	$\dim(H^{2p}(G^{\mathbb{P}}(1,4);\mathbb{Z}))$	*generators*
0	1	$\Omega(3,4)$
1	1	$\Omega(2,4)$
2	2	$\Omega(1,4),\Omega(2,3)$
3	2	$\Omega(0,4),\Omega(1,3)$
4	2	$\Omega(0,3),\Omega(1,2)$
5	1	$\Omega(0,2)$
6	1	$\Omega(0,1)$

Example 1.40. *The Grassmannian $G(3,6) = G^{\mathbb{P}}(2,5)$. The dimension of $G(3,6)$ is 9. For $0 \leq p \leq 9$, $H^{2p}(G^{\mathbb{P}}(1,4);\mathbb{Z})$ is generated by the cohomology classes of $\Omega(a_0,a_1,a_2)$ with $9 - [a_0 + (a_1 - 1) + (a_2 - 2)] = p$ i.e. $a_0 + a_1 + a_2 = 12 - p$. For $p = 0$, the only integer solution to $a_0 + a_1 + a_2 = 12$ with a_0, a_1 and a_2 as in Schubert conditions is $a_0 = 3, a_1 = 4$ and $a_2 = 5$. We summarize the calculations for other cohomology groups in the following table.*

p	$\dim(H^{2p}(G^{\mathbb{P}}(2,5);\mathbb{Z}))$	*generators*
0	1	$\Omega(3,4,5)$
1	1	$\Omega(2,4,5)$
2	2	$\Omega(1,4,5),\Omega(2,3,5)$
3	3	$\Omega(0,4,5),\Omega(1,3,5),\Omega(2,3,4)$
4	3	$\Omega(0,3,5),\Omega(1,2,5),\Omega(1,3,4)$
5	3	$\Omega(0,2,5),\Omega(0,3,4),\Omega(1,2,4)$
6	3	$\Omega(0,1,5),\Omega(0,2,4),\Omega(1,2,3)$
7	2	$\Omega(0,1,4),\Omega(0,2,3)$
8	1	$\Omega(0,1,3)$
9	1	$\Omega(0,1,2)$

1.3.3 Intersection theory of Schubert cycles

We now state without proof some important results for computing the products of Schubert cycles. The Basis Theorem says that the Schubert cycles, i.e. the cohomology classes of Schubert varieties, form an additive basis for the cohomology ring. Moreover the product of any two Schubert cycles can be uniquely expressed as a linear combination of Schubert cycles with **integer** coefficients.

Proposition 1.41. [12, p.1071] *Let* $m = \dim G^{\mathbb{P}}(d,n) = (d+1)(n-d)$. *The basis* $\{\Omega(a_0,\dots,a_d) \mid m - \dim\ \Omega(a_0,\dots,a_d) = p\ \}$ *of* $H^{2p}(G^{\mathbb{P}}(d,n);\mathbb{Z})$ *and the basis*

$$\{\Omega(n-a_d,\dots,n-a_0) \mid m - \dim\ \Omega(n-a_d,\dots,n-a_0) = n-p\}$$

of $H^{2(m-p)}(G^{\mathbb{P}}(d,n);\mathbb{Z})$ *are dual under the Poincaré duality pairing* $(v,w) \to \deg(v.w)$.

The Schubert cycles $\Omega(a_0,\cdots,a_d)$ and $\Omega(n-a_d,\cdots,n-a_0)$ are called dual cycles.

Corollary 1.42. [12, p.1071] *Let* $v \in H^{2p}(G^{\mathbb{P}}(d,n);\mathbb{Z})$. *Then* v *can be written uniquely as*

$$v = \sum \delta(n-a_d,\dots,n-a_0)\Omega(a_0,\dots,a_d),$$

where $\delta(n-a_d,\dots,n-a_0) = \deg(v.\Omega(n-a_d,\dots,n-a_0))$ *is an integer.*

Definition 1.43. The **special Schubert cycles** of the Grassmannian $G^{\mathbb{P}}(d,n)$ are defined by

$$\sigma(h) = \Omega(h, n-d+1,\dots,n), \quad \text{for} \quad h = 0,\dots,(n-d).$$

Example 1.44. *The special Schubert cycles of* $G^{\mathbb{P}}(1,3)$ *are*

$$\sigma(0) = \Omega(0,3),\ \sigma(1) = \Omega(1,3),\ \sigma(2) = \Omega(2,3).$$

Example 1.45. *The special Schubert cycles of* $G^{\mathbb{P}}(2,5)$ *are*

$$\sigma(0) = \Omega(0,4,5),\ \ \sigma(1) = \Omega(1,4,5),\ \ \sigma(2) = \Omega(2,4,5),\ \ \sigma(3) = \Omega(3,4,5).$$

Giambelli's Formula (Determinantal Formula) [12, p.1073] : Suppose that $0 \le a_0 < \cdots < a_d \le n$ is a sequence of integers. Then the following formula holds in $H^*(G^{\mathbb{P}}(d,n);\mathbb{Z})$:

$$\Omega(a_0,\ldots,a_d) = \begin{vmatrix} \sigma(a_0) & \ldots & \sigma(a_0-d) \\ \vdots & \vdots & \vdots \\ \sigma(a_d) & \ldots & \sigma(a_d-d) \end{vmatrix},$$

where $\sigma(h) = 0$ for $h \notin [0,(n-d)]$.

Example 1.46. *In* $G^{\mathbb{P}}(1,3)$ *consider the Schubert cycle* $\Omega(1,2)$. *Using Giambelli's Formula we have*

$$\Omega(1,2) = \begin{vmatrix} \sigma(1) & \sigma(0) \\ \sigma(2) & \sigma(1) \end{vmatrix} = \sigma(1)^2 - \sigma(2)\cdot\sigma(0).$$

Giambelli's Formula together with the Basis Theorem implies that every cohomology class is equal to a linear combination of products of special Schubert cycles i.e. the special Schubert cycles generate the cohomology ring $H^*(G^{\mathbb{P}}(d,n);\mathbb{Z})$ as a $\mathbb{Z}$-algebra. Moreover, Giambelli's Formula reduces the problem of determining the product of arbitrary Schubert cycles to finding the product of special Schubert cycles.

Pieri's Formula [12, p.1073] : Let $0 \le a_0 < \cdots < a_d \le n$ be any sequence of integers. Then for $h = 0,\ldots,(n-d)$ we have the following formula for the product in the cohomology ring $H^*(G^{\mathbb{P}}(d,n);\mathbb{Z})$:

$$\Omega(a_0,\ldots,a_d)\,.\,\sigma(h) = \sum \Omega(b_0,\ldots,b_d),$$

where the sum ranges over all sequences of integers $b_0 < \cdots < b_d$ satisfying conditions $0 \le b_0 \le a_0,\ \ a_0 < b_1 \le a_1, \cdots,\ a_{d-1} < b_d \le a_d$ and $\sum_{i=0}^{d} b_i = \sum_{i=0}^{d} a_i - (n-d-h)$.

Example 1.47. *We compute the product of Schubert cycles in* $G^{\mathbb{P}}(1,3)$ *and hence describe the cohomology ring* $H^*(G^{\mathbb{P}}(1,3);\mathbb{Z})$ *as a* $\mathbb{Z}$*- algebra. By Giambelli's Formula and the Basis Theorem we know that* $H^*(G^{\mathbb{P}}(1,3);\mathbb{Z})$ *is generated by special Schubert cycles* $\{\sigma(0),\sigma(1),\sigma(2)\}$ *as a* $\mathbb{Z}$*-algebra. One observes that* $\sigma(2)$ *acts as identity. So we can express* $H^*(G^{\mathbb{P}}(1,3);\mathbb{Z})$ *as* $\mathbb{Z}[\sigma(0),\sigma(1)]$ *with some relations. To find these relations we first compute the products of special Schubert cycles.*

1. $\sigma(0)^2 = \Omega(0,3)\cdot\sigma(0)$. *By Pieri's formula we have* $\sigma(0)^2 = \sum\Omega(b_0,b_1)$ *where* $b_0 < b_1$ *are distinct integers satisfying,* $0 \le b_0 \le 0,\ 0 < b_1 \le 3$ *and we have* $b_0 + b_1 = 3 - (2-0) = 1$. *Therefore* $\sigma(0)^2 = \Omega(0,1)$.

2. $\sigma(1)^2 = \Omega(1,3)\cdot\sigma(1)$. *By Pieri's formula we have* $\sigma(1)^2 = \sum\Omega(b_0,b_1)$ *where* $b_0 < b_1$ *are distinct integers satisfying,* $0 \le b_0 \le 1,\ 1 < b_1 \le 3$ *and we have* $b_0 + b_1 = 4 - (2-1) = 3$. *So* $\sigma(1)^2 = \Omega(0,3) + \Omega(1,2)$.

3. $\sigma(0)\cdot\sigma(1) = \Omega(0,3)\cdot\sigma(1) = \sum\Omega(b_0,b_1)$ *where* $b_0 < b_1$ *are distinct integers satisfying* $0 \le b_0 \le 0,\ 0 < b_1 \le 3,\ b_0 + b_1 = 3 - (2-1) = 2$. *Therefore the product* $\sigma(0)\cdot\sigma(1) = \Omega(0,2)$.

4. $\Omega(1,2)\cdot\sigma(1) = \sum\Omega(b_0,b_1)$ *where* $b_0 < b_1$ *are distinct integers satisfying the relations* $0 \le b_0 \le 1,\ 1 < b_1 \le 2,\ b_0 + b_1 = 3 - (2-1) = 2$. *Therefore the product* $\Omega(1,2).\sigma(1) = \Omega(0,2)$.

Computing in a similar way we can summarize the products of special Schubert cycles in $G^{\mathbb{P}}(1,3)$ *in the following table*

$\bullet$	$\sigma(0)$	$\sigma(1)$	$\sigma(2)$
$\sigma(0)$	$\Omega(0,1)$	$\Omega(0,2)$	$\Omega(0,3)$
$\sigma(1)$	$\Omega(0,2)$	$\Omega(0,3)+\Omega(1,2)$	$\Omega(1,3)$
$\sigma(2)$	$\Omega(0,3)$	$\Omega(1,3)$	$\Omega(2,3)$

We compute some more products

1. $\Omega(1,2)\cdot\sigma(0) = \sum\Omega(b_0,b_1)$ *where* $b_0 < b_1$ *are distinct integers satisfying the conditions* $0 \le b_0 \le 1,\ 1 < b_1 \le 2,\ b_0 + b_1 = 3-(2-0) = 1$. *We can't find integers* b_0, b_1 *satisfying these conditions. Hence the product* $\Omega(1,2)\cdot\sigma(0) = 0$.

2. $\Omega(1,2)\cdot\Omega(1,2)$. *By Giambelli's formula we have* $\Omega(1,2) = \sigma(1)^2 - \sigma(2)\cdot\sigma(0)$. *So* $\Omega(1,2)\cdot\Omega(1,2) = \Omega(1,2)[\sigma(1)^2 - \sigma(0)]$.
   Again using Pieri's formula we get

$$\Omega(1,2)\cdot\Omega(1,2) = \Omega(0,2)\cdot\sigma(1) - \Omega(1,2)\cdot\sigma(0) = \Omega(0,1) - 0 = \Omega(0,1).$$

We have now enough information to describe the relations between the generators. By the Basis Theorem we can write

$$\begin{aligned} H^*(G^{\mathbb{P}}(1,3);\mathbb{Z}) &= H^0 \oplus H^2 \oplus H^4 \oplus H^6 \oplus H^8 \\ &= \mathbb{Z}w_0 \oplus \mathbb{Z}w_2 \oplus (\mathbb{Z}w_4 \oplus \mathbb{Z}w_{4'}) \oplus \mathbb{Z}w_6 \oplus \mathbb{Z}w_8 \end{aligned}$$

where $w_0 = \Omega(2,3) = \sigma(2),\ w_2 = \Omega(1,3) = \sigma(1),\ w_4 = \Omega(0,3) = \sigma(0)$, $w_{4'} = \Omega(1,2) = \sigma(1)^2 - \sigma(0)\cdot\sigma(2),\ w_6 = \Omega(0,2) = \sigma(1)\sigma(0),\ w_8 = \Omega(0,1) = \sigma(0)^2$. *In this new notation each* w_i *is of weight* i *i.e. each* w_i *is a class in* H^i. *In order to find all possible relations we have to compute the products in all weights. So we have to compute the products* $w_2^2,\ w_2^3,\ w_2^4,\ w_2w_4,\ w_2w_{4'},\ w_2w_6,\ w_4^2,\ w_4w_{4'},\ w_4w_2^2,\ w_{4'}^2,\ w_{4'}w_2^2$. *Also we have relations* $w_iw_j = 0$ *for* $i + j > 8$. *Then referring to the computation done above, we have the following relations*

$$w_2^2 = w_4 + w_{4'}, \quad w_2w_4 = w_6, \quad w_2w_{4'} = w_6, \quad w_4^2 = w_8. \tag{1.2}$$

$$w_2w_6 = w_8, \quad w_4w_{4'} = w_8, \quad w_{4'}^2 = 0. \tag{1.3}$$

The relations $w_2^3 = 2w_6, \quad w_2^4 = 2w_8$ *can be obtained from the above set of relations. Also note that the relations* $w_4w_2^2 = w_8,\ w_{4'}w_2^2 = w_8$ *are redundant. They can be* *obtained by the relations above. The equations arising from $w_i w_j = 0$ for $i + j > 8$ are*

$$w_2 w_8, \quad w_4 w_6, \quad w_4 w_8 \quad w_{4'} w_6, \quad w_{4'} w_8, \quad w_6 w_8. \tag{1.4}$$

We have considered all possible weights. So these are enough relations. Hence as a $\mathbb{Z}$- algebra we can write

$$H^*(G^{\mathbb{P}}(1,3);\mathbb{Z}) = \mathbb{Z}[\sigma(0),\sigma(1)] = \mathbb{Z}[w_4, w_2]$$

with the relations given in equations 1.2,1.3 and 1.4.

Example 1.48. [12, p.1073] *Compute the number of lines $L \in \mathbb{P}^3$ which (simultaneously) intersect four given lines L_1, L_2, L_3 and L_4.*

As seen in Example 1.28, the lines which (simultaneously) intersect a given line A_0 in $\mathbb{P}^3$ are represented by the Schubert variety $\Omega(A_0, \mathbb{P}^3)$ defined by the flag

$$\underline{A} : A_0 \subset A_1 = \mathbb{P}^3.$$

Therefore the lines which intersect (simultaneously) four given lines are represented by the intersection of the Schubert varieties

$$Q = \bigcap_{i=1}^{4} \Omega(L_i, \mathbb{P}^3)$$

Assume that the set of lines intersecting four given lines is finite. Then this set has cardinality equal to the degree of the Schubert cycle $\Omega(1,3)^4$. Using the computations in Example 1.47, we have

$$\Omega(1,3)^4 = w_2^4 = 2w_8.$$

Now $w_8 = \Omega(0,1)$ is the class of a single point its degree is one. So the degree of the Schubert cycle $\Omega(1,3)^4$ is two. So the number of lines in $\mathbb{P}^3$ interesecting four given lines is either infinity or 2 or one (counted twice), with 2 being the "common case".

1.4 The Grassmannian as a scheme

Very interestingly Grassmannians exist in the category of schemes and can be considered as natural generalizations of the notion of classical Grassmannians over algebraically closed fields. For a detailed discussion on this refer to [6], III 2.7. Let S be any scheme and let $1 \leq d < n$ be integers. There exists a scheme $G_S(d,n)$, called the Grassmannian over S with the following properties.

1. If $T \to S$ is any morphism of schemes, then $G_T(d,n) = G_S(d,n) \times_S T$. In particular there exists a scheme $G_{\mathbb{Z}}(d,n)$ the Grassmannian over $\operatorname{Spec}\mathbb{Z}$ and any Grassmannian $G_S(d,n)$ can be realized as the fiber product $G_{\mathbb{Z}}(d,n) \times S$.

2. If $S = \operatorname{Spec}(k)$, k an algebraically closed field, then the scheme $G_S(d,n)$ is the classical Grassmann variety $G(d,n)$ over k.

To construct Grassmannians over a general scheme, we begin by constructing them over affine schemes. Then given any scheme S we can cover S by affine open schemes say $\{U_\alpha\}$ and glue together the Grassmannians $\{G_{U_\alpha}\}$.

To motivate our discussion we recall that in the classical setting Grassmannians are over an algebraically closed field k. There are at least two ways of constructing $G_k(d,n)$. We may consider $G_k(d,n)$ as a non-disjoint union of open sets each isomorphic to the affine space $\mathbb{A}_k^{d(n-d)}$. Alternatively we may consider it as a closed subvariety of $\mathbb{P}_k^N$ defined by the Plücker relations. Each of these constructions has an immediate extension to the category of schemes. Recall the following glueing construction of $G_k(d,n)$. The Grassmannian $G_k(d,n)$ over k, i.e. the set of d-dimensional subspaces of the vector space k^n, can be viewed as the set of $d \times n$ matrices M of rank d, modulo the multiplication on the left by invertible $d \times d$ matrices. For each subset $I \subset \{1,2,\ldots,n\}$ of cardinality d we can multiply any matrix M whose I-th minor is nonzero by the inverse of its I-th submatrix M_I, to obtain a matrix M'

whose I-th submatrix is the identity. Thus the set of all d-planes Λ complementary to the subspace of k^n spanned by the basis vectors $\{e_i\}_{i \notin I}$ can be identified with the affine space $\mathbb{A}^{d(n-d)}$ whose coordinates are the remaining entries of the matrix M'.

Now let $W \cong \mathbb{A}_k^{dn}$ be the space of $d \times n$ matrices. For each subset $I \subset \{1, 2, \dots, n\}$ of cardinality d consider the closed subset $W_I \subset W$ defined by the matrices with I-th submatrix equal to identity. For each I and $J \neq I$, let $W_{I,J} \subset W_I$ be the open subset of matrices whose J-th minor is nonzero. Define $\phi_{I,J} : W_{I,J} \to W_{J,I}$ by multiplication on the left by $M_I.M_J^{-1}$. Then ϕ is an isomorphism. Thus we can define the Grassmannian $G_k(d,n)$ as an abstract variety which is the union of affine spaces $W_I \cong \mathbb{A}_k^{d(n-d)}$ modulo the identifications of $W_{I,J}$ with $W_{J,I}$ given by $\phi_{I,J}$.

This construction has a natural extension to the Grassmannian over any affine scheme $G_S(d,n)$. Let $S = \operatorname{Spec} A$ be any affine scheme. Let

$$W = \operatorname{Spec} A[\dots, x_{i,j}, \dots] \cong \mathbb{A}_S^{dn}.$$

For each set $\{i_1, \dots, i_d\} \subset \{1, 2, \dots, n\}$ let $W_I \subset W$ be the closed subscheme corresponding to the matrices whose I-th $d \times d$ matrix is identity. This subscheme is the zero locus of the ideal $(\dots, x_{\alpha, i_\beta} - \delta_{\alpha,\beta}, \dots)$. For each I and $J \neq I$ define $W_{I,J}$ and $\phi_{I,J}$ as before. Then we glue all affine spaces $W_I \cong \mathbb{A}_S^{d(n-d)}$ along $\phi_{I,J}$ to get the scheme $G_S(d,n)$.

The other classical approach to Grassmannians is via the Plücker coordinates. Let $N = \binom{n}{d} - 1$. If $S = \operatorname{Spec} A$ is any affine scheme, we consider the polynomial ring $A[\dots, X_I, \dots]$ in $\binom{n}{d}$ variables over A where the variables are indexed by the subsets $I = (i_1 < i_2 < \cdots < i_d) \subset \{1, 2, \dots, n\}$. We can describe the Plücker ideal J as follows. Let φ be the map

$$A[\cdots, X_I, \cdots] \to A[x_{1,1}, \cdots, x_{d,n}]$$

$$X_I \mapsto \begin{vmatrix} x_{1,i_1} & \dots & x_{1,i_d} \\ \vdots & & \vdots \\ x_{d,i_1} & \dots & x_{d,i_d} \end{vmatrix}$$

sending each generator X_I of $A[\cdots, X_I, \cdots]$ to the corresponding minor of the matrix $(x_{i,j})$, and we let $J = \operatorname{Ker}\varphi$. Then the Grassmannian $G_S(d,n)$ is defined to be the projective scheme

$$G_S(d,n) = \text{Proj } A[\dots, X_I, \dots]/J \subset \text{Proj } A[\dots, X_I, \dots] = \mathbb{P}_S^{\binom{n}{d}-1}.$$

To see how the whole construction works we refer to [6, p.121 − 122].

1.4.1 Schemes and functors

One of the useful ways to describe schemes is through the notion of functor of points. The category of schemes can be embedded into the category of contravariant functors. The category of contravariant functors is a very large category and only some of these functors come from schemes. A scheme can be described via its functor of points. The functor of points of a scheme X is a functor

$$h_X : (\text{schemes})^{\text{o}} \to (\text{sets})$$

where (schemes)$^{\text{o}}$ and (sets) represent the category of schemes with arrows reversed and the category of sets. If Y is any scheme, we define

$$h_X(Y) = \text{Mor}(\text{Y}, \text{X}).$$

Also for every morphism $f : Y \to Z$ we define the map of sets $h_X(Z) \to h_X(Y)$ by sending $g \in h_X(Z) = \text{Mor}(\text{Z}, \text{X})$ to the composition $g \circ f \in \text{Mor}(\text{Y}, \text{X})$. A functor $F : (\text{schemes})^{\text{o}} \to (\text{sets})$ is said to be representable if it comes from a scheme, i.e. if $F = h_X$ for some scheme X. By Yoneda's Lemma below, such a scheme is unique if it exists.

For any scheme X the set $h_X(Y)$ is called the set of Y-valued points of X. If $Y = \operatorname{Spec} T$ is an affine scheme we write $h_X(T)$ instead of $h_X(\operatorname{Spec} T)$ and call it the set of T-valued points of X. Also we have the functor

$$h : (\text{schemes}) \to \operatorname{Fun}((\text{schemes})^{\text{o}}, (\text{sets}))$$

(where the morphisms in the category of functors are natural transformations) sending

$$X \to h_X$$

and associating a morphism $f : X \to X'$ the natural transformation $h_X \to h_{X'}$ that for any scheme Y sends $g \in h_X(Y)$ to the composition $f \circ g \in h_{X'}(Y)$.

Lemma 1.49. (Yoneda's Lemma)[6, p.252 – 253] *Let $\mathcal{C}$ be a category and let X and X' be objects of $\mathcal{C}$.*

1. *If F is any contravariant functor from $\mathcal{C}$ to the category of sets, the natural transformations from* $\operatorname{Mor}(-, X)$ *to F are in natural correspondence with the elements of $F(X)$.*

2. *If the functors* $\operatorname{Mor}(-, X)$ *and* $\operatorname{Mor}(-, X')$ *from $\mathcal{C}$ to the category of sets are isomorphic, then $X \cong X'$. More generally, the maps of functors from* $\operatorname{Mor}(-, X)$ *to* $\operatorname{Mor}(-, X')$ *are the same as the maps from X to X'; that is, the functor*

$$h : \mathcal{C} \to \operatorname{Fun}(\mathcal{C}^o, (\text{sets})),$$

sending X to h_X, is an equivalence of $\mathcal{C}$ with a full subcategory of the category of functors.

Viewing a scheme as its functor of points is often much easier than actually constructing a scheme. To show the existence of a certain scheme it is enough to define a functor from the category of schemes to the category of sets and then to prove an existence theorem asserting that there is a scheme of which it is the functor of points.

Definition 1.50. [6, p.259] A functor F : (rings) $\rightarrow$ (sets) is said to be a sheaf in the Zariski topology if for each ring R and each open covering of $X = \operatorname{Spec} R$ by distinguished open affines $U_i = \operatorname{Spec} R_{f_i}$, the functor F satisfies the sheaf axiom for the open covering $\{U_i\}$ of X. To be more precise, for every collection of elements $\alpha_i \in F(R_{f_i})$ such that α_i and α_j map to the same element in $F(R_{f_i f_j})$ there is a unique element $\alpha \in F(R)$ mapping to each of the α_i.

Theorem 1.51. [6, Theorem VI − 14] *A functor F : (rings) $\rightarrow$ (sets) is of the form h_Y for some scheme Y if :*

1. *F is a sheaf in the Zariski topology*

2. *There exist rings R_i and elements $\alpha_i \in F(R_i)$ - that is, by Lemma 1.49 the maps*

$$\alpha_i : h_{R_i} \rightarrow F,$$

   such that for each field K, $F(K)$ is the union of the images $h_{R_i}(K)$ under the maps α_i.

The goal of the next section is to show the representability of the Grassmann functor using above theorem. Before that we first see that the projective space $\mathbb{P}^n_{\mathbb{Z}}$ comes from the functor p : (rings) $\rightarrow$ (sets) given by

$$p(T) = \{T\text{-submodules } K \subset T^{n+1} \text{ that are locally rank } n \text{ direct summands of } T^{n+1}\}.$$

To see how it works we refer to the following theorems.

Theorem 1.52. [6, Proposition III-40] *If T is any ring, then*

$$\operatorname{Mor}(\operatorname{Spec} T, \mathbb{P}^n_{\mathbb{Z}}) = \{T\text{-submodules } K \subset T^{n+1} \text{ that are locally rank } n \text{ direct summands of } T^{n+1}\}$$

Theorem 1.53. [6, Theorem III-37] *For any scheme X, we have the natural bijections*

$\mathrm{Mor}(X, \mathbb{P}^n_{\mathbb{Z}}) = \{\mathcal{O}_X\text{-subsheaves } K \subset \mathcal{O}_X^{n+1} \text{ that are locally direct summands of rank } n\}$

Remark 1.54. *In general, we see that if K is a submodule of T^n which is locally a rank d direct summand of T^n then the quotient module T^n/K is a locally free module of rank $n-d$ and it is projective. We get that the following sequence splits:*

$$0 \to K \to T^n \to T^n/K \to 0$$

So K is a rank d direct summand of T^n. Indeed a submodule of a finitely generated free module that is locally a direct summand is a direct summand. Thus in the above theorems one can consider direct summands of T^{n+1} instead of locally direct summands.

1.4.2 Representability of the Grassmann functor

Let $0 < d < n$ be integers. The Grasmann scheme $G_{\mathbb{Z}}(d, n)$ is a closed subscheme of the projective space $\mathbb{P}^r_{\mathbb{Z}}$ where $r = \binom{n}{d} - 1$. Let g : (rings) $\to$ (sets) be the functor defined by

$$g(T) = \{\text{submodules } K \subset T^n \text{ that are rank } d \text{ direct summands of } T^n\}.$$

This functor is called the **Grasmann functor**. We now use Theorem 1.51 to show that the scheme $G_{\mathbb{Z}}(d, n)$ indeed represents the Grassmann functor.

We first show that g is a sheaf in the Zariski topology. Let R be a ring. Let $X = \operatorname{Spec} R$. Consider the open covering of X by distinguished open affine sets $U_i = \operatorname{Spec} R_{f_i}$. Suppose that for every collection of elements $W_i \in g(R_{f_i})$, W_i and W_j map to the same element in $g(R_{f_i f_j})$. So in $g(R_{f_i f_j})$ we have

$$W_i \otimes_{R_{f_i}} R_{f_i f_j} = W_j \otimes_{R_{f_j}} R_{f_i f_j}.$$

We wish to show that there exists a unique element $W \in g(R)$ that maps to each of the W_i. Let us first show the existence part. We want a rank d direct summand W of R^n such that

$$W \otimes_R R_{f_i} = W_i.$$

For each i we have an R_{f_i} module W_i which is a rank d direct summand of $R^n_{f_i}$. For every i let $\mathcal{F}_i = \widetilde{W_i}$ be the sheaf associated to W_i on SpecR_{f_i}. Now X is the union of distinguished open affine sets $U_i = \text{Spec}R_{f_i}$ and in $g(R_{f_i f_j})$ we have

$$W_i \otimes_{R_{f_i}} R_{f_i f_j} = W_j \otimes_{R_{f_j}} R_{f_i f_j}.$$

So for each i, j we get the isomorphisms $\varphi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \to \mathcal{F}_j|_{U_i \cap U_j}$. Then by using the glueing lemma of sheaves [8, p.69] there exists a unique sheaf $\mathcal{F}$ on X which is obtained by glueing the sheaves $\mathcal{F}_i$ and we have $\mathcal{F}|_{U_i} = \mathcal{F}_i$ for every i. Indeed the sheaf $\mathcal{F}$ on X is a coherent sheaf as X is covered by open affines U_i with the property that for each i we have a finitely generated R_{f_i} module W_i and $\mathcal{F}|_{U_i} \cong \widetilde{W_i}$. Then by using Lemma 5.3 and Proposition 5.4 of [8], we see that there exists a finitely generated R module W such that $\mathcal{F} = \widetilde{W}$. Going through the proof of [8, Proposition 5.4], it follows that this R module W is given by

$$W = \Gamma(X, \mathcal{F}) = \{w \in R^n \,|\, w \in R^n_{f_i}, w \in W_i \text{ for all } i\}.$$

Moreover, we have

$$\mathcal{F}(U_i) \cong W_i = W_{f_i} = W \otimes_R R_{f_i}.$$

Since each W_i is a rank d direct summand of $R^n_{f_i}$, by the above construction it follows that W is a rank d direct summand of R^n. To prove the uniqueness part let W_1 and W_2 be two elements of $g(R)$ such that

$$W_1 \otimes_R R_{f_i} = W_2 \otimes_R R_{f_i}.$$

So we get $(W_1)_{f_i} = (W_2)_{f_i}$ in $g(R_{f_i})$. We wish to show that $W_1 = W_2$. For this we use the following result.

Lemma 1.55. *Let A, B be R-modules. Let $\pi : A \to B$ be a morphism. Suppose that* $\operatorname{Spec} R = \bigcup \operatorname{Spec} R_{f_i}$. *If $\pi_{f_i} : A_{f_i} \to B_{f_i}$ is an isomorphism for all i then π is an isomorphism.*

Proof. The space $\operatorname{Spec} R = \bigcup \operatorname{Spec} R_{f_i}$ implies that f_i generate unit ideal. The map π_{f_i} is an isomorphism for all i. For each maximal ideal $\mathfrak{m}$ the map $A_{\mathfrak{m}} \to B_{\mathfrak{m}}$ is an isomorphism. Then we use [5, Corollary 2.9] to show that π is an isomorphism. □

Now we have the inclusion map $\pi : W_1 \cap W_2 \hookrightarrow W_1$ such that

$$(W_1 \cap W_2)_{f_i} = (W_1)_{f_i} \cap (W_2)_{f_i} = (W_1)_{f_i}, \quad \text{for all } i.$$

So π_{f_i} is an isomorphism for all i. Therefore by the above lemma π is an isomorphism i.e. $W_1 \cap W_2 = W_1$ which implies that $W_1 = W_2$.

Having verified g is a sheaf in the Zariski topology, we next have to show that there exist rings R_i and elements $W_i \in g(R_i)$ such that for every field F, $g(F)$ is the union of images $h_{R_i}(F)$ under the maps $W_i : h_{R_i} \to g$.

Let $\mathbb{P}^r_{\mathbb{Z}} = \operatorname{Proj}[\ldots, X_I, \ldots]$ be the projective space with homogeneous coordinates X_I corresponding to the subsets of cardinality d in $\{1, 2, \ldots, n\}$. Recall that the projective scheme $\mathbb{P}^r_{\mathbb{Z}}$ comes from the functor $h_{\mathbb{P}^r_{\mathbb{Z}}} : (\text{rings}) \to (\text{sets})$ given by

$$\begin{aligned} h_{\mathbb{P}^r_{\mathbb{Z}}}(T) &= \operatorname{Mor}(\operatorname{Spec} T, \mathbb{P}^r_{\mathbb{Z}}) \\ &= \{T\text{-submodules } K \subset T^{r+1} \text{ that are rank } r \text{ direct summands of } T^{r+1}\} \end{aligned}$$

We have a pairing

$$<,>: T^{r+1} \times T^{r+1} \to T$$

defined by

$$\langle (x_0, x_1, \ldots, x_r),\ (y_0, y_1, \ldots, y_r) \rangle = \sum_{i=0}^{r} x_i y_i.$$

In general, one can prove that if $M \subset T^{r+1}$ is a rank 1 direct summand of T^{r+1} then $M^{\perp} \subset T^{r+1}$ is a rank r direct summand of T^{r+1}. If $\{e_1, \ldots, e_n\}$ is the standard basis for T^n then $\{e_I \mid i_1 < \cdots < i_d\}$ is a basis to $T^{r+1} = \bigwedge^d T^n$. We would like to build a functor $\iota : g \to h_{\mathbb{P}^r_{\mathbb{Z}}}$ that sends

$$(K \subset T^n) \to ((\bigwedge^d K)^\perp \subset T^{r+1}).$$

Cover $\mathbb{P}^r_{\mathbb{Z}}$ by usual open affine subschemes $U_I \cong \mathbb{A}^r_{\mathbb{Z}}$. Referring to [6, p.261], these subschemes represent the subfunctors

$$U_I(T) = \left\{ \begin{array}{c} \text{rank } r \text{ summands of } T^{r+1} \text{ such that the } I\text{-th} \\ \text{basis vector } e_I \text{ of } T^{r+1} \text{ generates the cokernel} \end{array} \right\}.$$

The pairing $<,>$ allows us to associate a subspace to its annihilator. If K is a rank d direct summand of T^n spanned by $\{v_1, v_2, \ldots, v_d\}$, we have the corresponding perfect pairing

$$\bigwedge^d K \times \left(T^{r+1}/(\bigwedge^d K)^\perp \right) \to T.$$

We get

$$U_I(T) \bigcap \iota(g(T)) = \left\{ \begin{array}{c} \text{rank } d \text{ summands of } T^n \text{ such} \\ \text{that } e_I \text{ generates } T^{r+1}/(\bigwedge^d K)^\perp \end{array} \right\}.$$

Lemma 1.56. *Let K be a rank d direct summand of T^{r+1} spanned by the vectors $v_1, v_2, \cdots, v_d$. Then w generates $T^{r+1}/(\bigwedge^d K)^\perp$ if and only if $\langle v_1 \wedge \cdots \wedge v_d,\ w\rangle$ is a unit of T.*

Proof. As before have a pairing

$$<,>: T^{r+1} \times T^{r+1} \to T,$$

which gives rise to a perfect pairing

$$\bigwedge^d K \times \left(T^{r+1}/(\bigwedge^d K)^\perp \right) \to T.$$

Suppose first that w generates $T^{r+1}/(\bigwedge^d K)^{\perp}$. Now $\bigwedge^d K$ is a rank 1 direct summand of T^{r+1}. We can find $t_1, t_2 \in T$ such that

$$\langle t_1 v_1 \wedge \cdots \wedge v_d,\ t_2 w\rangle = 1.$$

So we have,

$$t_1 t_2 \langle v_1 \wedge \cdots \wedge v_d,\ w\rangle = 1.$$

Therefore $\langle v_1 \wedge \cdots \wedge v_d,\ w\rangle$ is a unit of T. Conversely let $\langle v_1 \wedge \cdots \wedge v_d,\ w\rangle$ be a unit of T. Without loss of generality let us assume that $\langle v_1 \wedge \cdots \wedge v_d,\ w\rangle = 1$. We need to show that w generates $T^{r+1}/(\bigwedge^d K)^{\perp}$. For any $v \in T^{r+1}/(\bigwedge^d K)^{\perp}$ let

$$\langle v_1 \wedge \cdots \wedge v_d,\ v\rangle = t.$$

Then for all s we have,

$$\langle s v_1 \wedge \cdots \wedge v_d,\ tw - v\rangle = st\,\langle v_1 \wedge \cdots \wedge v_d, w\rangle - s\,\langle v_1 \wedge \cdots \wedge v_d\rangle = 0,$$

and the pairing is perfect implies that $tw = v$. □

By the above discussion we see that

$$U_I(T) \cap \iota(g(T)) = \left\{ \text{rank } d \text{ summands } K \text{ of } T^n \text{ such that } e_I \text{ generates } \frac{T^{r+1}}{(\bigwedge^d K)^{\perp}} \right\}$$
$$= \{K \subset T^n \,|\, K = \mathrm{Sp}\{v_1, \ldots, v_d\} \text{with} \ \langle v_1 \wedge \cdots \wedge v_d,\ e_I\rangle \in T^*\}.$$

So it is enough to understand for a subspace $K = \text{Span} < e_{i_1}, \ldots, e_{i_d} >$, when is $\langle e_{i_1} \wedge \cdots \wedge e_{i_d}, e_I\rangle$ a unit of T.

Lemma 1.57. *Let $r = \binom{n}{d} - 1$. Let K be a rank d direct summand of T^n spanned by the set $\{v_1, \cdots, v_d\}$. Then in the above notations $\langle v_1 \wedge \cdots \wedge v_d, e_I\rangle \in T^*$ if and only if the image $P(K)$ of K via the Plücker map P belongs to the basic open set $U_I \cong \mathbb{A}^{d(n-d)}$ of $\mathbb{P}^r$ defined by $e_I \neq 0$.*

Proof. We first understand the classical case when T is a field k. Let V be a vector space over field k of dimension n. Let $I = (i_1 < \cdots < i_d)$. Let U be spanned by $\{e_{i_1}, \ldots, e_{i_{n-d}}\}$. Let $\lambda = P(U) = e_{i_1} \wedge \cdots \wedge e_{i_{n-d}}$ be the image of U via the Plücker map P. We can view λ as a linear form on $\mathbb{P}(\bigwedge^d V)$ as follows. For $v \in \bigwedge^d V$ define $\lambda(v) := v \wedge \lambda \in \bigwedge^n V \cong k$. Referring to the section 1.1.6, we see that U_I, the basic open set of $\mathbb{P}^r$ defined by $e_I \neq 0$ is given by

$$
\begin{aligned}
U_I &= \{\, \Lambda \in G(d,n) \,|\, \text{the } I\text{-th Plučker coordinate of } P(\Lambda) \text{ is nonzero (unit)} \,\} \\
&= [\mathbb{P}(\bigwedge^d V) - Z(\lambda)] \cap G(d,n) \\
&= \{P(K) \mid K \in G(d,n), P(K) \wedge \lambda \neq 0\} \\
&= \{P(K) \mid K \in G(d,n), V = K \oplus U\}.
\end{aligned}
$$

We now fix some splitting $V = K_0 \oplus U$ of V. We can then identify U_I with $\mathrm{Hom}(K_0, U)$. Therefore, the set of d-dimensional subspaces K spanned by $\{e_1, \ldots, e_d\}$ with the property that $< e_1 \wedge \cdots \wedge e_d,\ e_I > \neq 0$ is same as the set of all $K \in G(d,n)$ with $P(K) \in U_I$ and hence is isomorphic to the affine space $\mathbb{A}_k^{d(n-d)}$.

Let us now work in the the general case. Let $U \subset V = T^n$ be a free rank $n-d$ summand of V. Let $U = \mathrm{Span}\{v_{d+1}, \ldots, v_n\}$. Let $\lambda = P(U) = v_{d+1} \wedge \cdots \wedge v_n$ be the image of U via the Plücker map P. Then we can view λ as a linear form on $\mathbb{P}(\bigwedge^d V)$. As before let U_I be the basic open set of $\mathbb{P}^r$ defined by $e_I \neq 0$. Then

$$
\begin{aligned}
U_I &= [\mathbb{P}(\bigwedge^d V) - Z(\lambda)] \cap G(d,n) \\
&= \{P(K) \mid K \in G(d,n), P(K) \wedge \lambda \in T^*\}.
\end{aligned}
$$

Here $P(K) \wedge \lambda \in T^*$ means as follows. If $\{e_1, \ldots, e_n\}$ is a basis for V, $\bigwedge^n V$ is one dimensional with the canonical basis given by $e_J = e_1 \wedge \cdots \wedge e_n$ and $P(K) \wedge \lambda \in T^*$ means that if $P(K) \wedge \lambda = te_J$ then $t \in T^*$. Now we wish to identify the set U_I with

$$\{\, P(K) \,|\, K \in G(d,n), V = K \oplus U\}.$$

To see this identification let first $K \in G(d,n)$ such that $P(K) \wedge \lambda = te_J$ with $t \in T^*$. Let $K = \mathrm{Span}\{v_1, \dots, v_d\}$. Then

$$v_1 \wedge \cdots \wedge v_d \wedge v_{d+1} \wedge \cdots \wedge v_n = te_J, \quad t \in T^*.$$

But we have

$$v_1 \wedge \cdots \wedge v_d \wedge v_{d+1} \wedge \cdots \wedge v_n = \det A \cdot e_J, \quad t \in T^*,$$

where the matrix A has $v_1, \dots, v_n$ as its columns. Since $\det A = t \in T^*$ and $\{e_1, \dots, e_n\}$ is a basis for V the vectors $v_1, \dots, v_n$ form a basis for V. Therefore $V = K \oplus U$, the vectors $v_1 \dots, v_d$ span K and the vectors $v_{d+1}, \dots, v_n$ span U. Conversely suppose that $V = K \oplus U$ for some $K = \mathrm{Span}\{v_1, \dots, v_d\}$. Let

$$(v_1, \dots, v_n) = (e_1, \dots, e_n) \cdot P \quad \text{and} \quad (e_1, \dots, e_n) = (v_1, \dots, v_n) \cdot Q,$$

where $P, Q \in \mathrm{GL}(n, T)$. Then $PQ = \mathrm{Id}$ and $\det(P)$ is a unit of T. Therefore,

$$P(K) \wedge \lambda = \det(P) \cdot e_J = te_J,$$

and t is a unit of t. To summarize we have

$$U_I = \{\, P(K) \,|\, K \in G(d,n), V = K \oplus U\}.$$

Then fixing some splitting $K_0 \oplus U = V$ of V, we can identify the set $\mathrm{Hom}(K_0, U)$ with the set

$$\{P(K) \mid K \in G(d,n), V = K \oplus U\}$$

by associating to $\varphi : K_0 \to U$ its graph $\Gamma_\varphi = \{(w, \varphi(w)) \mid w \in K_0\}$. Note that

$$\Gamma_\varphi \cap U = \{(w, \varphi(w)) \mid w = 0\} = \{0\}$$

and given $v \in V, v = (w, u)$ for $w \in K_0, u \in U$ we can write

$$v = (w, \varphi(w)) + (0, u - \varphi(w)) \in \Gamma_\varphi \oplus U.$$

In this way we can identify U_I with $\mathrm{Hom}(K_0, U) \cong \mathbb{A}_T^{d(n-d)}$ and we have that for K spanned by $v_1, \dots, v_d$, $\langle v_1 \wedge \cdots \wedge v_d,\ e_I \rangle \in T^*$ if and only if $P(K) \in U_I \cong \mathbb{A}_T^{d(n-d)}$. $\square$

By this lemma, $U_I \bigcap \iota(g)$ is represented by affine scheme $\mathbb{A}^{d(n-d)} = \operatorname{Spec} \mathbb{Z}[(x_{ij})]$ where $1 \leq i \leq d,\ 1 \leq j \leq (n-d)$. Then taking the rings R_i in Theorem 1.51 as $\mathbb{Z}[x_{ij}]$, we have for any field F,

$$g(F) = \bigcup (U_I \cap i(g))(F).$$

Hence, the second condition in Theorem 1.51 is satisfied by the Grassmann functor. It follows that the Grassmann functor is represented by the Grassmann scheme.

1.4.3 Computation of the Zeta function of $G(d,n)$ using Schubert calculus

In section 1.2.3 we computed the Zeta function of $G(d,n)$ by simple combinatorics. It is noticed that by the knowledge of the cohomology groups of the Grassmannian in the characteristic p we can get the information of the cohomology in the characteristic zero. As an application of the Schubert Calculus we now compute the Zeta function with the Basis Theorem without actually going through the computations as in section 1.2.3 and we recover the information of the cohomology groups in characteristic p. First recall a few notions of the morphisms of schemes and Galois actions on étale cohomology groups. The best reference for this is [15]. If $X \to \operatorname{Spec} \mathbb{Z}_{(p)}$ is a smooth and proper morphism of schemes then the cohomology of $X \otimes \overline{\mathbb{Q}}$ with the Galois action gives the information of the cohomology of $X \otimes \overline{\mathbb{F}}_p$ with its Galois action. Let $\mathcal{O}$ be the ring of integers of $\overline{\mathbb{Q}}$. Suppose p is a prime and $\mathfrak{m}$ is a maximal ideal containing p. Then $\mathcal{O}_\mathfrak{m}$ is a local ring with unique maximal ideal $\mathfrak{m}\mathcal{O}_\mathfrak{m}$. The residue field $k = \mathcal{O}_\mathfrak{m}/\mathfrak{m}\mathcal{O}_\mathfrak{m} \cong \overline{\mathbb{F}}_p$. Let $\widetilde{X} = X \otimes \mathcal{O}_\mathfrak{m}$. If $\widetilde{X} \to \operatorname{Spec} \mathcal{O}_\mathfrak{m}$ is a smooth and proper morphism of schemes the cohomology of $\widetilde{X} \otimes \overline{\mathbb{Q}}$ with Galois action gives the cohomology of $\widetilde{X} \otimes k$ with its Galois action. Now let $X = G(d,n)$ be the Grassmannn variety. Let $m = \dim G(d,n) = d(n-d)$. The equations defining the Grassmannian i.e. the Plücker relations are relations with integer coefficients. So we can consider

$G(d,n)$ over fields of characteristic zero namely $\mathbb{Q}$, $\mathbb{C}$ and also over finite field $\mathbb{F}_q$. Let $G(d,n) \otimes_{\mathcal{O}_\mathfrak{m}} \overline{\mathbb{Q}}$ denote the Grassmann variety $G(d,n)$ over $\overline{\mathbb{Q}}$ and let $G(d,n) \otimes_{\mathcal{O}_\mathfrak{m}} k$ denote the Grassmann variety $G(d,n)$ over $\overline{\mathbb{F}}_p$. Since over any algebraically closed field L, $G(d,n)$ is smooth and proper, the morphism $G(d,n) \to \operatorname{Spec} \mathcal{O}_\mathfrak{m}$ is a smooth and proper morphism of schemes. Let l be a prime other than p. We have an isomorphism of étale cohomology groups [15, section 20.4] namely,

$$f: \ H^i_{\text{ét}}(G(d,n) \otimes \overline{\mathbb{Q}}; \mathbb{Q}_l) \to H^i_{\text{ét}}(G(d,n) \otimes k; \mathbb{Q}_l),$$

which is Galois equivariant. The Galois group $\operatorname{Gal}(\overline{\mathbb{Q}} \,/\, \mathbb{Q})$ contains the decomposition group $D_\mathfrak{m}$ and the inertia group $I_\mathfrak{m}$ as its subgroups. We have

$$I_\mathfrak{m} \subset D_\mathfrak{m} \subset \operatorname{Gal}(\overline{\mathbb{Q}} \,/\, \mathbb{Q}).$$

To say f is Galois equivariant means that, if $\tau \in D_\mathfrak{m}$ then, $\overline{\tau} \in \operatorname{Gal}(\overline{\mathbb{F}_p} \,/\, \mathbb{F}_p)$ and for a class $c \in H^{2i}_{\text{ét}}(G(d,n) \otimes \overline{\mathbb{Q}}; \mathbb{Q}_l)$ one has

$$f(\tau c) = \overline{\tau} \cdot f(c).$$

This implies that the inertia group $I_\mathfrak{m}$ acts trivially on $H^i_{\text{ét}}(G(d,n) \otimes \overline{\mathbb{Q}}; \mathbb{Q}_l)$. The Frobenius morphism $F : G(d,n) \otimes \mathbb{F}_p \to G(d,n) \otimes \mathbb{F}_p$ induces linear map F^* on cohomology. Let α in $\operatorname{Gal}(\overline{\mathbb{F}}_p \,/\, \mathbb{F}_p)$ be the geometric Frobenius morphism $x \mapsto x^{1/p}$. Let us also denote the induced linear map on cohomology by α. Then $\alpha = F^*$. Also there exists $\beta \in D_\mathfrak{m}$ such that $\overline{\beta} = \alpha$. We now use all this information to simplify the expression of the Zeta function of $G(d,n)$. Referring to [8, Appendix C], the Zeta function of $G(d,n)$ is given by

$$\begin{aligned} Z(G(d,n),t) &= \prod_{i=0}^{2m} \det[1 - tF^* \mid H^i_{\text{ét}}(G(d,n) \otimes \overline{\mathbb{F}}_p; \mathbb{Q}_l)]^{(-1)^{i+1}} \\ &= \prod_{i=0}^{2m} \det[1 - t\alpha \mid H^i_{\text{ét}}(G(d,n) \otimes \overline{\mathbb{F}}_p; \mathbb{Q}_l)]^{(-1)^{i+1}} \\ &= \prod_{i=0}^{2m} \det[1 - t\beta \mid H^i_{\text{ét}}(G(d,n) \otimes \overline{\mathbb{Q}}; \mathbb{Q}_l)]^{(-1)^{i+1}}. \end{aligned}$$

We use 3.6 and 3.7 of [8] Appendix C for $X = G(d,n)$ and get,

$$H^i_{\text{ét}}(X \otimes \overline{\mathbb{Q}}; \mathbb{Q}_l) \cong H^i_{\text{ét}}(X \otimes \mathbb{C}; \mathbb{Q}_l) \cong H^i_{\text{Betti}}(X \otimes \mathbb{C}; \mathbb{Q}_l) \cong H^i_{\text{Betti}}(X \otimes \mathbb{C}; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}_l.$$

By the Basis theorem of the Schubert calculus (Theorem 1.36), we know that the Schubert cycles generate $H^*(G(d,n) \otimes \mathbb{C}; \mathbb{Z})$. Now if Y is a subvariety of $G(d,n)$ of codimension i, it gives a class $[Y] \in H^{2i}_{\text{ét}}(G(d,n) \otimes \overline{\mathbb{Q}}; \mathbb{Q}_l)$ on which β acts by $\beta[Y] = p^i[\beta(Y)]$. Since the Schubert cycles are defined over $\mathbb{Q}$, we have a simpler formula for the Zeta function for $G(d,n)$ as

$$Z(G(d,n),t) = \frac{1}{\prod_{i=0}^{m}(1-p^i t)^{b_{2i}}},$$

where b_{2i} denotes the rank of $H^{2i}(G(d,n); \mathbb{Z})$ over $\mathbb{Z}$. Thus, the Zeta function of the Grassmann variety $G(d,n)$ of dimension m is given by

$$Z(G(d,n),t) = \frac{1}{(1-t)(1-pt)^{b_2}(1-p^2t)^{b_4}\dots(1-p^m t)^{b_{2m}}},$$

which agrees with the calculations done before in section 1.2.3. One observes that with the knowledge of the cohomology in characteristic p, we have the information of the cohomology in characteristic zero and vice versa.

1.4.4 The Zeta function of the Grassmann scheme

For an exposition of the Zeta function of schemes we refer to the paper by Serre [19]. Let X be a scheme of finite type over $\operatorname{Spec}\mathbb{Z}$. Such a scheme has a well defined dimension denoted by $\dim X$. Let $\max\,(X)$ denote the set of closed points of X. It can be shown that $\{x\}$ in X is closed in X if and only if the residue field $k(x)$ of x is finite. For $x \in \max(X)$, the norm $N(x)$ of x is defined as the number of elements of $k(x)$. Then the Zeta function of the scheme X is defined by a Eulerian product

$$\zeta(X,s) = \prod_{x \in \max\,(X)} \frac{1}{1-[N(x)]^{-s}}.$$

It can be seen that the product $\zeta(X,s)$ converges absolutely for $\mathrm{Re}(s) > \dim X$.

In the case when $X = \operatorname{Spec} A$, where A is the ring of integers of a number field K, $\zeta(X,s)$ coincides with the classical Dedekind Zeta function attached to K and thus is the same as the Riemann Zeta function when $A = \mathbb{Z}$. The Riemann Zeta function is defined by a Eulerian product

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}},$$

the product being taken over all positive integer primes p.

Now let X be a scheme of finite type over $\mathbb{F}_q$. If $x \in \max\,(X)$, the residue field $k(x)$ of x is a finite extension of $\mathbb{F}_q$; let $\deg(x)$ be its degree. Then we have $N(x) = q^{\deg(x)}$ and

$$\zeta(X,s) = \prod_{x\in\max\,(X)} \frac{1}{1-\left[q^{\deg(x)}\right]^{-s}}.$$

Recall the definition of the Zeta function of a smooth projective variety X over $k = \mathbb{F}_q$. The Zeta function is given by

$$Z(X,t) := \exp\left(\sum_{r=1}^{\infty} N_r.\frac{t^r}{r}\right) \in \mathbb{Q}[[t]],$$

where N_r is the number of points of X defined over $\mathbb{F}_{q^r}$. This can also be written as

$$Z(X,t) = \prod_{x\in\max\,(X)} \frac{1}{1-t^{\deg(x)}},$$

where the product is taken over the closed points of X. Therefore, we have

$$\zeta(X,s) = Z(X,q^{-s}).$$

If X is a disjoint union of subschemes X_i we have

$$\zeta(X,s) = \prod \zeta(X_i,s).$$

Example 1.58. *Consider the projective scheme* $\mathbb{P}^n_{\mathbb{Z}}$. *The Zeta function* $\zeta(\mathbb{P}^n_{\mathbb{Z}}, s)$ *is given by*

$$\zeta(\mathbb{P}^n_{\mathbb{Z}}, s) = \prod_p \zeta(\mathbb{P}^n_{\mathbb{F}_p}, s) = \prod_p Z(\mathbb{P}^n_{\mathbb{F}_p}, p^{-s})$$
$$= \prod_p \prod_{m=0}^{n} \frac{1}{[1 - p^{-(s-m)}]} = \prod_{m=0}^{n} \zeta(s-m).$$

Proposition 1.59. *The Zeta function of the Grassmann scheme* $G_{\mathbb{Z}}(d,n)$ *is a product of the Riemann zeta functions. If* b_i *denote the* i*-th Betti number of the Grassmannian* $G(d,n)$*, we have*

$$\zeta(G_{\mathbb{Z}}(d,n), s) = \prod_{i=0}^{d(n-d)} \zeta^{b_i}(s-i).$$

Proof. Let $m = \dim G_{\mathbb{Z}}(d,n) = d(n-d)$. Referring to the section 1.2.3, $Z(G_{\mathbb{F}_p}(d,n), t)$ is given by

$$Z(G_{\mathbb{Z}}(d,n) \otimes \mathbb{F}_p, t) = \frac{1}{(1-t)^{b_0}(1-pt)^{b_1}\dots(1-p^m t)^{b_m}},$$

where the Betti numbers b_i are given by the Gaussian binomial coefficients. Then the Zeta function $\zeta(G_{\mathbb{Z}}(d,n), s)$ is given by

$$\zeta(G_{\mathbb{Z}}(d,n), s) = \prod_p \zeta(G_{\mathbb{Z}}(d,n) \otimes \mathbb{F}_p, s) = \prod_p Z(G_{\mathbb{Z}}(d,n) \otimes \mathbb{F}_p, p^{-s})$$
$$= \prod_p \frac{1}{(1-t)^{b_0}(1-pt)^{b_1}\dots(1-p^m t)^{b_m}}, \quad \text{where} \quad t = p^{-s}$$
$$= \prod_{i=0}^{m} \zeta^{b_i}(s-i).$$

We conclude that $\zeta(G_{\mathbb{Z}}(d,n), s)$ can be expressed as a product of the Riemann Zeta functions. □

Chapter 2

Lagrangian Grassmannian

In this Chapter we discuss the Lagrangian Grassmannian $L(n, 2n)$, which parametrizes n-dimensional isotropic subspaces of a $2n$-dimensional vector space V endowed with a symplectic form $\langle\, ,\, \rangle$. In section 2.1 we discuss general notions of symplectic spaces and the Lagrangian Grassmannian. Section 2.2 discusses the Lagrangian Grassmannian as an algebraic variety and its covering by affine neighbourhoods. In section 2.3 we calculate the Zeta function and the Euler characteristic of the Lagrangian Grassmannian. In section 2.4 we discuss without proofs the Schubert calculus for the Lagrangian Grassmannian and using the Basis Theorem we compute the dimensions of the cohomology groups of the Lagrangian Grassmannian. Section 2.5 discusses the representability of the Lagrangian Grassmannn functor. Finally in section 2.6 we compute the Zeta function of the Lagrangian Grassmann scheme.

2.1 Lagrangian Grassmannian

First recall the notion of a symplectic space.

Definition 2.1. Let V be a vector space over field k. A **symplectic form**

$$\langle\, ,\, \rangle : V \times V \to k$$

is an antisymmetric and non-degenerate bilinear form i.e. it satisfies

$$\langle v, v \rangle = 0 \quad \text{for all } v \in V,$$

and if

$$\langle v, w \rangle = 0 \quad \text{for all} \quad v \in V,$$

then $w = 0$. A vector space V is called a **symplectic vector space** if it is equipped with a symplectic form.

One has the following theorem.

Theorem 2.2. [14, Theorem 2.3] or [2, p. $10 - 11$] *A symplectic vector space V is necessarily of even dimension and there exists a basis $u_1, \ldots, u_n, v_1, \ldots, v_n$ of V such that*

$$\langle u_i, u_j \rangle = \langle v_i, v_j \rangle = 0 \quad \textit{and} \quad \langle u_i, v_j \rangle = \delta_{ij}.$$

Such a basis for a symplectic vector space V is called a **standard basis**. With such choice a symplectic form can be described for $v = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ and $v' = (x'_1, \ldots, x'_n, y'_1, \ldots, y'_n)$ by

$$\langle v, v' \rangle = \sum_{i=1}^{n} (x_i y'_i - x'_i y_i).$$

It is easy to verify that the above pairing is a non-degenerate alternating pairing on the vector space V. The above form on V is called the **standard symplectic form**.

Definition 2.3. Let V be a symplectic vector space of dimension $2n$. Two vectors $v, w \in V$ are called **orthogonal** if $\langle v, w \rangle = 0$. This is denoted by $v \perp w$. If W is a m-dimensional subspace of V, we define the **orthogonal space** of W, $W^{\perp}$ by

$$W^{\perp} = \{ v \in V \mid \langle v, w \rangle = 0 \quad \text{for all} \quad w \in W \}.$$

Definition 2.4. Let V be a symplectic space of dimension $2n$.

1. A subspace $U \subset V$ is **isotropic** if $\langle u, u' \rangle = 0$ for all $u, u' \in U$.

2. A subspace $W \subset V$ is a **symplectic subspace** of V if the symplectic form on V when restricted to W remains symplectic.

3. A subspace $W \subset V$ is **coisotropic** if $W^{\perp}$ is isotropic.

4. A subspace $L \subset V$ is **Lagrangian** if it is both isotropic and coisotropic (thus $L = L^{\perp}$ and $\dim L = n$).

Definition 2.5. Let V be a symplectic vector space of dimension $2n$. Let $L(n, 2n)$ denote the collection of all Lagrangian subspaces of V. One can prove that $L(n, 2n)$ is a subvariety of the Grassmannian $G(n, 2n)$, called the **Lagrangian Grassmann variety** or the **Lagrangian Grassmannian**.

Definition 2.6. Let V_1, V_2 be two symplectic vector spaces. Let $\phi : V_1 \to V_2$ be a linear map. Then we call ϕ a **symplectic map** if for all $v, w \in V_1$

$$\langle \phi(v), \phi(w) \rangle = \langle v, w \rangle .$$

Suppose now $V_1 = V_2 = V$ with the same symplectic form. If $\phi : V \to V$ is any symplectic map then it is an automorphism of V. The collection of all symplectic automorphisms of V is a group under composition called the **symplectic group of** $(V, \langle\ ,\ \rangle)$ denoted by $\mathrm{Sp}(V)$. If $V = k^{2n}$ with the standard symplectic form, we write this group as $\mathrm{Sp}_{2n}(k) \subset \mathrm{GL}(2n, k)$. If we define

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

we can verify that [2, p.15] a matrix $A \in \mathrm{GL}(2n, k)$ leaves the standard form invariant if and only if $A^t J A = J$ where, A^t denotes the transpose of the matrix A. Thus,

$$\mathrm{Sp}_{2n}(k) = \{A \in \mathrm{GL}(2n, k) \mid A^t J A = J\}.$$

Lemma 2.7. *Let W be an n-dimensional k vector space. Let W^* be the dual space. Then $V = W \oplus W^*$ is a symplectic space with*

$$\langle\, ,\, \rangle : V \times V \to k$$

given by

$$\langle (w_1, f_1),\ (w_2, f_2)\rangle = f_1(w_2) - f_2(w_1)$$

for $w_1, w_2 \in W$ and $f_1, f_2 \in W^*$.

Proof. Clearly for $w \in W$ and $f \in W^*$ we have

$$\langle (w, f),\ (w, f)\rangle = f(w) - f(w) = 0.$$

Also, if

$$\langle (w_1, f_1), (w_2, f_2)\rangle = f_1(w_2) - f_2(w_1) = 0$$

for all $w_1 \in W$, $f_1 \in W^*$, then $f_1(w_2) = f_2(w_1)$, $\forall w_1 \in W$, $\forall f_1 \in W^*$. Hence, $w_2 = 0$ and $f_2 = 0$. □

2.2 The Lagrangian Grassmannian as an algebraic variety

Let V be a symplectic vector space of dimension $2n$. We now see that the Lagrangian Grassmannian over V, i.e. $L(n, 2n)$, is actually a closed subset of the Grassmannian $G(n, 2n)$. Let U be an n-dimensional isotropic subspace in $G(n, 2n)$. Since U is isotropic, by a result of linear algebra [3, section 1.5] there exists an isotropic linear subspace W such that

$$U \oplus W = V.$$

It follows that $W \cong U^*$ the isomorphism being given by $w \mapsto (\langle w, \cdot\rangle : U \to k)$. Fix a splitting $V = U \oplus U^*$ of V. Then we observe that the standard symplectic structure on V is same as the symplectic structure on V defined in Lemma 2.7. Recall that in case of the classical Grassmannian $G(d,n)$, an affine neighbourhood of $U \in G(d,n)$ is given by $\mathrm{Hom}(U,\Gamma)$ where Γ is a complementary subspace to U. To find the affine neighbourhood in the Lagrangian case we need the following Lemma.

Lemma 2.8. *Let $\varphi : U \to U^*$ be a linear map. Then $\varphi = \varphi^*$ if and only if the graph of φ, $\Gamma_\varphi \subset U \oplus U^*$ is a Lagrangian subspace with respect to the structure defined in the Lemma 2.7.*

Proof. Suppose that $\varphi = \varphi^*$. In general [4, p.414] the matrix of φ, $M(\varphi)$, is related to $M(\varphi^*)$ by $M(\varphi) = (M(\varphi^*))^t$. Consider two elements in the graph of φ namely, $(u_1, \varphi_{u_1}), (u_2, \varphi_{u_2}) \in U \oplus U^*$. We have

$$\begin{aligned}\langle (u_1, \varphi_{u_1}),\ (u_2, \varphi_{u_2})\rangle &= \varphi_{u_1}(u_2) - \varphi_{u_2}(u_1)\\ &= \langle \varphi(u_1),\ u_2\rangle - \langle \varphi(u_2),\ u_1\rangle\\ &= 0\end{aligned}$$

as the matrix of φ is symmetric. Conversely if Γ_φ is a Lagrangian subspace of $U \oplus U^*$, then for all $u_1, u_2 \in U$ we have

$$\langle \varphi(u_1), u_2\rangle = \langle \varphi(u_2), u_1\rangle .$$

It follows that $M(\varphi)$ is symmetric and $\varphi = \varphi^*$. □

By this lemma we have a neighbourhood of U namely,

$$\mathrm{Hom}(U,W)^{Sym} \subset \mathrm{Hom}(U,W),$$

where

$$\mathrm{Hom}(U,W)^{Sym} = \{f : U \to W \mid f = f^*\}.$$

We have $\mathrm{Hom}(U,W) \cong M_n(k) \cong \mathbb{A}^{n^2}$ and $\mathrm{Hom}\,(U,W)^{Sym} \cong \mathbb{A}^{\frac{n(n+1)}{2}}$. So if x_{ij} are the coordinates, the Lagrangian Grassmannian can be defined locally by $x_{ij} = x_{ji}$. Thus $L(n,2n)$ is closed in every open set and hence defines a closed subvariety of $G(n,2n)$ of dimension $\dfrac{n(n+1)}{2}$.

2.2.1 Examples

Example 2.9. *The Lagrangian Grassmannian* $L(2,4)$ *is the collection of all* 2 *dimensional isotropic subspaces of* $G(2,4)$. *Now* $U \in G(2,4)$ *is isotropic if and only if* $\langle u, v\rangle$ *is zero for all* $u, v \in U$. *We express this condition explicitly in terms of the Plücker coordinates. Let* $\{e_1, e_2, e_3, e_4\}$ *be the basis for* V. *Then the canonical basis for* $\bigwedge^2 V$ *is given by*

$$B = \{e_1 \wedge e_2,\ e_1 \wedge e_3,\ e_1 \wedge e_4,\ e_2 \wedge e_3,\ e_2 \wedge e_4,\ e_3 \wedge e_4\}.$$

Let $\{u_1, u_2\}$ *be a basis for* $U \in G(2,4)$ *with*

$$u_1 = a_{11}e_1 + a_{21}e_2 + a_{31}e_3 + a_{41}e_4, \quad u_2 = a_{12}e_1 + a_{22}e_2 + a_{32}e_3 + a_{42}e_4.$$

Then we get

$$\langle u_1, u_2\rangle = (a_{11}a_{32} - a_{31}a_{12}) + (a_{21}a_{42} - a_{41}a_{22}).$$

If $x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}$ *are the Plücker coordinates corresponding to the canonical basis, referring to Example 1.12 the Grassmannian* $G(2,4)$ *is defined by the relation*

$$x_{14}x_{23} - x_{24}x_{13} + x_{12}x_{34} = 0.$$

If further U *is isotropic then* $\langle u, u'\rangle = 0$ *for all* $u, u' \in U$. *This condition in the Plücker coordinates translates to*

$$x_{13} + x_{24} = 0.$$

Imposing this additional condition on the Grassmann condition we have that $L(2,4)$ *is a* 3*-dimensional variety of* $\mathbb{P}^4$ *defined by*

$${x_{13}}^2 = -(x_{14}x_{23} + x_{12}x_{34}).$$

2.3 The number of points in $L(n,2n)(\mathbb{F}_q)$

Lemma 2.10. *The symplectic group* $\mathrm{Sp}_{2n}(k)$ *acts transitively on the set of all isotropic subspaces of* $G(n,2n)(k)$*, i.e. on the Lagrangian Grassmannian. In other words given* L_1, L_2 *in* $L(n,2n)$ *there exists* $\phi \in Sp(V)$ *such that*

$$\phi(V_1) = V_2.$$

Proof. This is a consequence of a special case of a theorem of Witt. For the details refer to [2, Theorem 1.26 and Corollary 1.27]. □

By the above theorem we have

$$|L(n,2n)(\mathbb{F}_q)| = \frac{|\mathrm{Sp}_{2n}(\mathbb{F}_q)|}{|\text{Stabilizer of } X|} \quad \text{for} \quad X \in L(n,2n).$$

To find $|\mathrm{Sp}(2n)(\mathbb{F}_q)|$ we use the following result from linear algebra.

Lemma 2.11. [21, p. 373 – 374] *If* f *is a symplectic form on a* $2n$*-dimensional vector space* V *over a field of* q *elements then the number of pairs* $\{u,v\}$ *such that* $f(u,v) = \langle u,v\rangle = 1$ *is* $(q^{2n}-1)q^{2n-1}$.

Proposition 2.12. *The number of points of* $Sp_{2n}(\mathbb{F}_q)$ *is given by*

$$|Sp_{2n}(\mathbb{F}_q)| = q^{n^2}\prod_{i=1}^{n}(q^i-1)(q^i+1).$$

Proof. Given a symplectic form f on vector space V of dimension $2n$ by standard results there exists a symplectic basis $\{v_1, v_2, \cdots, v_{2n}\}$ for V such that

$$\langle v_i,\ v_{i+n}\rangle = 1 \text{ for } i = 1,\cdots,n \quad \text{and} \quad \langle v_i, v_j\rangle = 0 \quad \text{for} \quad |i-j| \neq n.$$

If $\{v_i\}$ is a symplectic basis of V then $\theta \in \mathrm{Sp}(V)$ if and only if $\{\theta v_i\}$ is also a symplectic basis for V [21, p.336]. Therefore we have

$$\langle \theta v_i,\ \theta v_{i+n}\rangle = 1 \text{ for } i = 1,\ldots,n \quad \text{and} \quad \langle \theta v_i,\ \theta v_j\rangle = 0 \quad \text{for} \quad |i-j| \neq n.$$

Using Lemma 2.11 the number of pairs $\{\theta v_1, \theta v_{1+n}\}$ such that $\langle \theta v_1, \theta v_{1+n}\rangle = 1$ is $(q^{2n}-1)q^{2n-1}$. Once we choose $\{\theta v_1, \theta v_{1+n}\}$ for $\{\theta v_i\}$ to be a symplectic basis the number of pairs $\{\theta v_2, \theta v_{2+n}\}$ such that $\langle \theta v_2, \theta v_{2+n}\rangle = 1$ is equal to $q^{(2n-2)-1}(q^{2n-2}-1)$, and so on. Finally, the number of pairs $\{\theta v_n, \theta v_{2n}\}$ such that $\langle \theta v_n, \theta v_{2n}\rangle = 1$ is $q(q^2-1)$. Thus we get

$$\begin{aligned}|\mathrm{Sp}(2n)(\mathbb{F}_q)| &= \prod_{i=1}^{n}(q^{2i}-1)q^{2i-1}\\ &= q^{n^2}\prod_{i=1}^{n}(q^{2i}-1)\\ &= q^{n^2}\prod_{i=1}^{n}(q^i-1)(q^i+1).\end{aligned}$$

□

Lemma 2.13. *Let V be a symplectic space of dimension $2n$. Let $u_1, \ldots, u_n, v_1, \ldots, v_n$ be the standard basis of V such that*

$$\langle u_i, u_j\rangle = \langle v_i, v_j\rangle = 0 \quad \text{and} \quad \langle u_i, v_j\rangle = \delta_{ij}.$$

The symplectic group $Sp_{2n}(\mathbb{F}_q)$ acts transitively on $L(n,2n)(\mathbb{F}_q)$ and the number of elements in the Stabilizer X for $X = Span\{u_1, u_2, \cdots, u_n\}$ is given by

$$|Stab\,(X)(\mathbb{F}_q)| = q^{\frac{n(n+1)}{2}}q^{\frac{n(n-1)}{2}}\prod_{i=1}^{n}(q^i-1).$$

Proof. First note that if $\begin{pmatrix} A & B \\ D & C \end{pmatrix} \in \mathrm{Stab}(X)$ then D has to be zero.

Let $M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in \mathrm{Stab}(X)$. If it has to be in $\mathrm{Sp}(2n)$ we must have

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}^t \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

that is

$$\begin{pmatrix} 0 & A^tC \\ -C^tA & B^tC - C^tB \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Thus we get, $C = (A^{-1})^t$ and $B^tC = C^tB$, i.e., C^tB is a symmetric matrix. So M is of the form

$$M = \begin{pmatrix} A & AS \\ 0 & (A^{-1})^t \end{pmatrix}$$

for some symmetric $n \times n$ matrix S. Let M_n^{Sym} be the group of symmetric $n \times n$ matrices. Consider the maps $\Phi_1 : \mathrm{GL}(n) \to \mathrm{Sp}(2n)$ and $\Phi_2 : M_n^{Sym} \to \mathrm{Sp}(2n)$ given by

$$\Phi_1(A) = \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^t \end{pmatrix} \quad \text{and} \quad \Phi_2(S) = \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}.$$

Then Φ_1 and Φ_2 are homomorphisms and

$$\Phi_1(A).\Phi_2(S) = \begin{pmatrix} A & 0 \\ 0 & (A^{-1})^t \end{pmatrix} \begin{pmatrix} I_n & S \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} A & AS \\ 0 & (A^{-1})^t \end{pmatrix}.$$

We have Im $(\phi_1) \cap$ Im $(\phi_2) = \{I\}$. It can be checked that Stab (X) is the semidirect product of $\mathrm{GL}(n)$, the general linear $n \times n$ group, and M_n^{sym}, the group of symmetric $n \times n$ matrices. Therefore, we get

$$\begin{aligned} |\mathrm{Stab}\,(X)(\mathbb{F}_q)| = |M_n^{sym}(\mathbb{F}_q)| \cdot |\mathrm{GL}(n)(\mathbb{F}_q)| &= q^{\frac{n(n+1)}{2}} \prod_{i=0}^{n-1} (q^n - q^i) \\ &= q^{\frac{n(n+1)}{2}} q^{\frac{n(n-1)}{2}} \prod_{i=1}^{n} (q^i - 1). \end{aligned}$$

□

Proposition 2.14. *The number of points in $L(n,2n)(\mathbb{F}_q)$ is given by*

$$|L(n,2n)(\mathbb{F}_q)| = \prod_{i=1}^{n}(1+q^i).$$

Proof. The proof follows immediately from Proposition 2.12 and Lemma 2.13. We have

$$|L(n,2n)(\mathbb{F}_q)| = \frac{|\mathrm{Sp}_{2n}(\mathbb{F}_q)|}{|\mathrm{GL}(n)(\mathbb{F}_q)|.|\mathrm{S}_n(\mathbb{F}_q)|} = \frac{q^{n^2}\prod_{i=1}^{n}(q^i-1)(q^i+1)}{q^{\frac{n(n+1)}{2}}q^{\frac{n(n-1)}{2}}\prod_{i=1}^{n}(q^i-1)}$$
$$= \prod_{i=1}^{n}(1+q^i).$$

□

2.3.1 The Zeta function of the Lagrangian Grassmannian

The Lagrangian Grassmannian $L(n,2n)$ is a smooth projective subvariety of the Grassmannian $G(n,2n)$ and we may consider it over any finite field $\mathbb{F}_q$. By using Proposition 2.14, the number of points in $L(n,2n)(\mathbb{F}_q)$ is given by

$$|L(n,2n)(\mathbb{F}_q)| = \prod_{i=1}^{n}(1+q^i).$$

As there are no terms in the denominator, N_r is a polynomial in powers of q^r and the Zeta function of such Grassmannians is easy to calculate.

Example 2.15. *The Lagrangian Grassmannian $L(2,4)\otimes\mathbb{F}_q$. By Proposition 2.14 we have $|L(2,4)(\mathbb{F}_q)| = (1+q)(1+q^2)$ and so, $N_r = 1+q^r+q^{2r}+q^{3r} = 1+q^r+q^{2r}+q^{3r}$. We get,*

$$Z(L(2,4)\otimes\mathbb{F}_q,t) = \frac{1}{(1-t)(1-qt)(1-q^2t)(1-q^3t)}.$$

Example 2.16. *The Lagrangian Grassmannian $L(3,6)\otimes\mathbb{F}_q$. We have*

$$|L(3,6)(\mathbb{F}_q)| = (1+q)(1+q^2)(1+q^3),$$

therefore we get, $N_r = 1 + q^r + q^{2r} + 2q^{3r} + q^{4r} + q^{5r} + q^{6r}$, *and*

$$Z(L(3,6)\otimes\mathbb{F}_q,t) = \frac{1}{(1-t)(1-qt)(1-q^2t)(1-q^3t)^2(1-q^4t)(1-q^5t)(1-q^6t)}.$$

Theorem 2.17. *The Zeta function of the Lagrangian Grassmannian $L(n,2n)$ is given by*

$$Z(L(n,2n)\otimes\mathbb{F}_q,t) = \frac{1}{(1-t)(1-qt)^{b_1}(1-q^2t)^{b_2}\dots(1-q^mt)^{b_m}},$$

where b_i is equal to the number of strict partitions of i whose parts do not exceed n and $m = \frac{n(n+1)}{2}$.

Proof. By Proposition 2.14 we get,

$$|L(n,2n)(\mathbb{F}_{q^r})| = \prod_{i=1}^{n}(1+q^{ir}).$$

For simplicity set $q^r = l$. Then

$$N_r = |L(n,2n)(\mathbb{F}_{q^r})| = \prod_{i=1}^{n}(1+l^i) = (1+l)(1+l^2)\dots(1+l^n) = \sum_{i=0}^{m} b_i l^i$$

where the coefficient b_i is equal to the number of strict partitions of i whose parts do not exceed n and $m = \frac{n(n+1)}{2}$. So the coefficients b_i can be calculated precisely and one observes that the Zeta function in the general case is given by

$$Z(L(n,2n)\otimes\mathbb{F}_q,t) = \frac{1}{(1-t)(1-qt)^{b_1}(1-q^2t)^{b_2}\dots(1-q^mt)^{b_m}},$$

where b_i and m are described as above. □

We observe that the odd Betti numbers of the Lagrangian Grassmannian are zero.

2.3.2 Euler characteristic of the Lagrangian Grassmannian

Consider the Lagrangian Grassmannian $L(n,2n)$. Referring to the last section the odd Betti numbers of the Lagrangian Grassmannian are zero and the even Betti numbers satisfy

$$\prod_{i=1}^{n}(1+l^i) = \sum_{i=0}^{m} b_i l^i$$

where $m = \frac{n(n+1)}{2}$ is the dimension of $L(n, 2n)$. Putting $l = 1$ in the above expression we immediately get the Euler characteristic of the Lagrangian Grassmannian as

$$E = \sum_{i=0}^{m} b_i = \prod_{i=1}^{m}(1 + 1^i) = 2^n.$$

2.4 Schubert calculus for Lagrangian Grassmannian

Definition 2.18. We denote a **partition** by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$. Thus λ_i are integers and we agree that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq 0$. Hence λ is **strict** if $\lambda_1 > \lambda_2 > \cdots > \lambda_l > 0$. We define the **length** of λ by $l(\lambda) = \text{Card}\,\{\, p \mid \lambda_p \neq 0\}$ and the **weight** of λ by $|\lambda| = \sum_{p=1}^{l} \lambda_p$. By $\rho(n)$ we mean the partition $(n, n-1, \cdots, 2, 1)$. We denote by D_n the set of all strict partitions λ with $\lambda_1 \leq n$.

Definition 2.19. Let $\mathcal{F} : 0 \subset F_1 \subset F_2 \subset \cdots \subset F_n \subset V$ be a fixed flag of isotropic subspaces of V such that dim $F_i = i$, $\forall i = 1, 2, \cdots, n$. Such a flag is called a **complete isotropic flag** of V.

Here F_n is an isotropic subspace of V of dimension n, so it is Lagrangian. In other words, a complete isotropic flag is nothing but a Lagrangian subspace F_n together with a complete flag of subspaces of F_n.

Note that any isotropic flag $\mathcal{F} : 0 \subset F_1 \subset F_2 \subset \cdots \subset F_n \subset V$ can be completed to a complete flag in V by setting $F_{n+i} = F_{n-i}^{\perp}$ for $1 \leq i \leq n$.

Definition 2.20. Let $\lambda \in D_n$, i.e., let $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_l > 0)$ be any strict partition with $\lambda_1 \leq n$. Let $\mathcal{F} : 0 \subset F_1 \subset F_2 \subset \cdots \subset F_n \subset V$ be a complete isotropic flag of V. With respect to this flag and partition we define the **Schubert variety in the Lagrangian case** as

$$X_\lambda = X_\lambda(\mathcal{F}) := \{L \in L(n, 2n) \mid \dim(L \cap F_{n+1-\lambda_i}) \geq i,\ 1 \leq i \leq l(\lambda)\}.$$

It can be shown that X_λ actually defines a complex projective variety of codimension $|\lambda|$ in $L(n, 2n)$. The variety X_λ determines a **Schubert class** $\Omega(\lambda) = [X_\lambda]$ in the cohomology group $H^{2|\lambda|}(L(n, 2n); \mathbb{Z})$. The Schubert classes $\Omega(i) = \sigma(i)$ for $i = 1, 2, \cdots, n$ (i.e. $l = 1$, $\lambda_1 = i$) are called special (they parametrize isotropic n-planes such that $\dim(L \bigcap F_{n+1-i}) \geq 1$. For the details of the subject our main reference is [22, p.$1 - 4$]. As before we define the cohomology ring $H^*(L(n, 2n), \mathbb{Z})$ of the Lagrangian Grassmannian as the direct sum $\bigoplus_{i \geq 0} H^i(L(n, 2n); \mathbb{Z})$.

Theorem 2.21. [22, p.2] *The cohomology group $H^*(L(n, 2n), \mathbb{Z})$ is a free abelian group and the Schubert classes $\Omega(\lambda), \lambda \in D_n$, form an additive basis for the cohomology ring $H^*(L(n, 2n), \mathbb{Z})$. We have an isomorphism of abelian groups*

$$H^*(L(n, 2n), \mathbb{Z}) = \bigoplus_{\lambda \in D_n} \mathbb{Z} \cdot \Omega(\lambda),$$

where the sum varies over all strict partitions $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_l > 0)$ with $\lambda_1 \leq n$.

2.4.1 Cohomology groups of the Lagrangian Grasmannian

We now calculate the dimensions of the cohomology groups of some Lagrangian Grassmannians. Using Theorem 2.21 and definition 2.20 we see that the codimension k classes of the Lagrangian Grassmannian are indexed by strict partitions of the form $\lambda : (\lambda_1 > \lambda_2 > \cdots \geq 0)$ with $\lambda_1 \leq n$ and weight of λ equal to k. So the k-th Betti number of $L(n, 2n)$ is equal to the number of such strict partitions.

Example 2.22. *The Lagrangian Grassmannian $L(2, 4)$. The dimension of the cohomology group $H^{2k}(L(2, 4); \mathbb{Z})$ equals the number of strict partitions λ such that $2 \geq \lambda_1$ and $|\lambda| = k$ for $k = 0, \cdots, 3$.*

Codimension k	*admissible partitions*	$\dim(H^{2k}(L(2,4);\mathbb{Z}))$
0	$\{0\}$	1
1	$\{1\}$	1
2	$\{2\}$	1
3	$\{(2,1)\}$	1

Example 2.23. *The Lagrangian Grassmannian* $L(3,6)$. *The dimension of the cohomology group* $H^{2k}(L(3,6);\mathbb{Z})$ *equals the number of strict partitions* λ *such that* $3 \geq \lambda_1$ *and* $|\lambda| = k$ *for* $k = 0, \cdots, 6$.

Codimension k	*admissible partitions*	$\dim(H^{2k}(L(3,6);\mathbb{Z}))$
0	$\{0\}$	1
1	$\{1\}$	1
2	$\{2\}$	1
3	$\{(2,1),3\}$	2
4	$\{(3,1)\}$	1
5	$\{(3,2)\}$	1
6	$\{(3,2,1)\}$	1

Example 2.24. *The Lagrangian Grassmannian* $L(4,8)$: *The dimension of the cohomology group* $H^{2k}(L(4,8);\mathbb{Z})$ *equals the number of strict partitions* λ *such that* $4 \geq \lambda_1$ *and* $|\lambda| = k$ *for* $k = 0, \cdots, 10$.

Codimension k	*admissible partitions*	$\dim(H^{2k}(L(4,8);\mathbb{Z}))$
0	$\{0\}$	1
1	$\{1\}$	1
2	$\{2\}$	1
3	$\{(2,1),3\}$	2
4	$\{4,(3,1)\}$	2
5	$\{(4,1),(3,2)\}$	2
6	$\{(4,2),(3,2,1)\}$	2
7	$\{(4,3),(4,2,1)\}$	2
8	$\{4,3,1\}$	1
9	$\{4,3,2\}$	1
10	$\{4,3,2,1\}$	1

By these examples we see that the dimensions of the cohomology groups match with the Betti numbers of the Lagrangian Grassmannians calculated before.

2.5 Representability of Lagrangian Grassmann functor

Let n be any positive integer. For a ring T consider the standard symplectic form on the T-module T^{2n}. We call a submodule K of T^{2n} isotropic if the standard symplectic form vanishes on it. The Lagrangian Grassmann functor is a functor $l : (\text{rings}) \to (\text{sets})$ is given by

$$l(T) = \{\text{isotropic } T\text{-submodules } K \subset T^{2n} \text{ that are rank } n \text{ direct summands of } T^{2n}\}$$

We will use theorem 1.51 to show that l is representable. We need to show that

1. l is a sheaf in the Zariski topology.

2. There exist rings R_i and elements $W_i \in l(R_i)$ such that for every field F, $l(F)$ is the union of images of $h_{R_i}(F)$ under the maps $W_i : h_{R_i} \to l$.

We first show that l is a sheaf in the Zariski topology. The proof proceeds as in the case of the classical Grasmann functor except for a few modifications. If $f : R_1 \to R_2$ is a ring homomorphism the corresponding morphism $\overline{f} : l(R_1) \to l(R_2)$ is given by

$$\overline{f}(V) = V \otimes_{R_1} R_2.$$

Note that if V is an isotropic rank n summand of R_1^{2n} then $V \otimes_{R_1} R_2$ is isotropic rank n summand of R_2^{2n}. Suppose that we have a nondegenerate alternate pairing

$$<,>: \; R_1^{2n} \times R_1^{2n} \to R_1.$$

Then f extends linearly and we get a pairing

$$<,>,: \; R_2^{2n} \times R_2^{2n} \to R_2.$$

If $V \subset R_1^{2n}$ is isotropic, $\langle v, v' \rangle = 0$ for all $v, v' \in V$. Then for $v_i, v_i' \in V$ and $r_i, r_i' \in R_2$ we have

$$\left\langle \sum_i v_1 \otimes r_i, \sum_j v_j' \otimes r_j' \right\rangle = \sum_{i,j} r_i r_j' \langle v_i, v_j' \rangle = 0.$$

Therefore, $V \otimes_{R_1} R_2$ is an isotropic summand of R_2^{2n}.

We now show that l is a sheaf in the Zariski topology. Let R be a ring. Let $X = \operatorname{Spec} R$. Consider the open covering of X by distinguished open affine sets $U_i = \operatorname{Spec} R_{f_i}$. Suppose that for every collection of elements $W_i \in l(R_{f_i})$, W_i and W_j map to the same element in $l(R_{f_i f_j})$. So in $l(R_{f_i f_j})$ let

$$W_i \otimes_{R_{f_i}} R_{f_i f_j} = W_j \otimes_{R_{f_j}} R_{f_i f_j}.$$

We wish to show that there exists a unique element $W \in l(R)$ that maps to each of the W_i. We can construct the required isotropic direct summand W exactly as in the classical case (for details refer to section 1.4.1). Such a W exists uniquely and is defined by

$$W = \{w \in R^n \mid w \in R^n_{f_i}, w \in W_i \text{ for all } i\}.$$

Since $W_i \in l(R_{f_i})$ are isotropic for all i and W is contained in W_i for all i we see that the summand W is also isotropic. Thus, l is a sheaf in the Zariski topology. Now we have to show that there exist rings R_i and elements $W_i \in l(R_i)$ such that for every field F, $l(F)$ is the union of images of $h_{R_i}(F)$ under the maps $W_i : h_{R_i} \to l$. Let $r = \binom{2n}{n} - 1$. Let $g(n, 2n) := g : \text{(rings)} \to \text{(sets)}$ be the Grassmann functor given by

$$g(T) = \{T\text{-submodules } K \subset T^{2n} \text{ that are rank } n \text{ direct summands of } T^{2n}\}.$$

Let $\mathbb{P}^r_{\mathbb{Z}} = \text{Proj}\,[\ldots, X_I, \ldots]$ be the projective space with homogeneous coordinates X_I corresponding to the subsets of cardinality n in $\{1, 2, \ldots, 2n\}$. Recall that the projective scheme $\mathbb{P}^r_{\mathbb{Z}}$ comes from the functor

$$\begin{aligned} h_{\mathbb{P}^r_{\mathbb{Z}}}(T) &= \text{Mor}\,(\text{Spec}\,T, \mathbb{P}^r_{\mathbb{Z}}) \\ &= \{T\text{-submodules } K \subset T^{r+1} \text{ that are rank } r \text{ direct summands of } T^{r+1}\} \end{aligned}$$

We now refer to the section 1.4.2. With the same notations of section 1.4.2, replacing d by n and n by $2n$ we get,

$$\begin{aligned} U_I(T) \cap \iota(g(T)) &= \left\{ \text{rank } n \text{ summands } K \subset T^{2n} \text{ such that } e_I \text{ generates } \frac{T^{r+1}}{(\bigwedge^n K)^\perp} \right\} \\ &= \{K \subset T^{2n} \mid K = \text{Sp}\{v_1, \ldots, v_n\} \text{with} \, \langle v_1 \wedge \cdots \wedge v_n, e_I \rangle \in T^*\}. \end{aligned}$$

Thus we see that $U_I(T) \cap \iota(l(T))$ is given by

$$\left\{ \text{isotropic rank } n \text{ summands } K \subset T^{2n} \text{ such that } e_I \text{ generates } \frac{T^{r+1}}{(\bigwedge^n K)^\perp} \right\},$$

which is equal to

$$\{K \subset T^{2n} \mid \ K \text{ isotropic spanned by } v_1, \ldots, v_n,\ \langle v_1 \wedge \cdots \wedge v_n, e_I \rangle \in T^*\}.$$

By Lemma 1.57 we see that for a direct summand $K \subset T^{2n}$ spanned by $v_1, \dots, v_n$, $\langle v_1 \wedge \cdots \wedge v_n, e_I \rangle \in T^*$ if and only if the image of K via the Plücker map i.e. $P(K)$ is in U_I. Let U be the complemetary subspace of K. If further K is isotropic then we see from section 2.2 that the affine neighbourhood U_I of K is isomorphic to

$$\{\varphi : K \to U \mid \varphi = \varphi^*\} = \mathrm{Hom}^{Sym}(K, U) \cong \mathbb{A}_T^{\frac{n(n+1)}{2}}.$$

Therefore $U_I \bigcap \iota(l)$ is represented by affine scheme $\mathbb{A}^{\frac{n(n+1)}{2}} = \mathrm{Spec}\,\mathbb{Z}[(x_i), i = 1, \cdots, \frac{n(n+1)}{2}]$. Then taking R_i as $\mathbb{Z}[x_i]$ we see that for any field F

$$g(F) = \bigcup (U_I \cap i(l))(F).$$

Therefore the second condition in Theorem 1.51 is satisfied by the Lagrangian Grassmann functor. Thus the Lagrangian Grassmann functor is representable.

Computation of the Zeta function of $L(n, 2n)$ using Schubert calculus

Note that the Zeta function of $L(n, 2n)$ can also be computed using Schubert calculus. The Lagrangian Grassmannian can be considered over fields of characteristic zero namely $\mathbb{Q}$, $\mathbb{C}$ and also over finite field $\mathbb{F}_q$. Exactly following section 1.4.3 we have for $X = L(n, 2n)$ the following isomorphisms

$$H^i_{\text{ét}}(X \otimes \overline{\mathbb{Q}}; \mathbb{Q}_l) \cong H^i_{\text{ét}}(X \otimes \mathbb{C}; \mathbb{Q}_l) \cong H^i_{\text{Betti}}(X \otimes \mathbb{C}; \mathbb{Q}_l) \cong H^i_{\text{Betti}}(X \otimes \mathbb{C}; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}_l.$$

Then with the Basis Theorem of the Schubert calculus in Lagrangian case (Theorem 2.21), the Zeta function of $L(n, 2n)$ can be computed as in section 1.4.3 which comes out to be

$$Z(L(n, 2n), t) = \frac{1}{\prod_{i=0}^{m} (1 - p^i t)^{b_{2i}}},$$

where m is the dimension of the Lagrangian Grassmannian and b_{2i} denotes the rank of $H^{2i}(L(n, 2n); \mathbb{Z})$ over $\mathbb{Z}$. This agrees with the calculations done before in section 2.3.1.

2.5.1 The Zeta function of the Lagrangian Grassmann Scheme

Theorem 2.25. *The Zeta function of the Lagrangian Grassmann scheme $L_{\mathbb{Z}}(d,n)$ is a product of Riemann Zeta functions given by*

$$\zeta(L_{\mathbb{Z}}(n,2n),s) = \prod_{i=0}^{m} \zeta^{b_i}(s-i),$$

where m is the dimension of $L(n,2n)$.

Proof. Consider the Lagrangian Grassmann scheme $L_{\mathbb{Z}}(d,n)$. Let us assume that $m = \dim L(n,2n) = \frac{n(n+1)}{2}$. The $Z(L_{\mathbb{Z}}(n,2n)\otimes \mathbb{F}_p, t)$ is given by

$$Z(L_{\mathbb{Z}}(n,2n)\otimes \mathbb{F}_p, t) = \frac{1}{(1-t)^{b_0}(1-pt)^{b_1}\dots(1-p^m t)^{b_m}},$$

where the Betti number b_i equals the number of strict partitions of i whose parts do not exceed n. Then referring to section 1.4.4, $\zeta(L_{\mathbb{Z}}(n,2n),s)$ is given by

$$\begin{aligned}
\zeta(L_{\mathbb{Z}}(n,2n),s) &= \prod_p \zeta(L_{\mathbb{Z}}(n,2n)\otimes \mathbb{F}_p, s) = \prod_p Z(L_{\mathbb{Z}}(n,2n)\otimes \mathbb{F}_p, p^{-s}) \\
&= \prod_p \frac{1}{(1-t)^{b_0}(1-pt)^{b_1}\dots(1-p^m t)^{b_m}} \quad \text{where} \quad t = p^{-s} \\
&= \prod_{i=0}^{m} \zeta^{b_i}(s-i).
\end{aligned}$$

Thus, we see that $\zeta(L_{\mathbb{Z}}(d,n),s)$ can be expressed as a product of the Riemann Zeta functions. □

Chapter 3

Other subschemes of the Grassmannian

In Chapter 1 we saw that the usual Grassmannian $G(d, n)$ can be considered as a scheme over $\operatorname{Spec} \mathbb{Z}$. The Grassmann scheme $G = G_{\mathbb{Z}}(d, n)$ represents the Grassmann functor g : (rings) $\to$ (sets) given by

$$g(T) = \{T\text{-submodules } K \subset T^n \text{ that are rank } d \text{ direct summands of } T^n\}.$$

By the results of Chapter 1 the Grassmann scheme $G = G_{\mathbb{Z}}(d, n)$ is a closed subscheme of the projective space $\mathbb{P}^r_{\mathbb{Z}}$ where $r = \binom{n}{d} - 1$. It is a smooth projective variety over $\operatorname{Spec} \mathbb{Z}$ of relative dimension $d(n-d)$. Moreover the Grassmannian $G(d, n)$ has a covering by spaces each isomorphic to affine space $\mathbb{A}^{d(n-d)}$.

In Chapter 2 we saw that the Lagrangian Grassmann scheme $L = L_{\mathbb{Z}}(n, 2n)$ represents the Lagrangian Grassmann functor l : (rings) $\to$ (sets) given by

$$l(T) = \{\text{isotropic } T\text{-submodules } K \subset T^{2n} \text{ that are rank } n \text{ direct summands of } T^{2n}\}.$$

The Lagrangian Grassmann scheme L is a closed subscheme of $\mathbb{P}^r_{\mathbb{Z}}$ where r is given by $r = \binom{2n}{n} - 1$. It is a smooth projective variety over $\operatorname{Spec} \mathbb{Z}$ of relative dimension

$\frac{n(n+1)}{2}$. Moreover the Lagrangian Grassmannian has a covering by spaces each isomorphic to affine space $\mathbb{A}^{\frac{n(n+1)}{2}}$.

Now fix a ring $R \subset M_n(\mathbb{Z})$ and an integer, $0 < d < n$. Let f_R : (rings) $\to$ (sets) be the functor such that $f_R(T)$ is

$$\{T\text{-submodules } K \subset T^n \text{ that are } R\text{-invariant rank } d \text{ direct summands of } T^n\},$$

where we say that a summand $K \subset T^n$ is R-invariant if $r.a \in K$ for all r in R and for all $a \in K$. In this Chapter we prove that f_R is represented by a closed scheme $F_R \subset G_{\mathbb{Z}}(d,n)$. Moreover we provide an explicit affine covering of F_R.

We focus on the following case. Let R be the ring of integers of a number field Q. Let $[Q : \mathbb{Q}] = d$ and $n = 2d$. The embedding $R \subset M_{2d}(\mathbb{Z})$ can be chosen as follows. The ring R acts diagonally on $R \oplus R$ by $r(r_1, r_2) = (rr_1, rr_2)$. If $R = \bigoplus_{i=1}^d \mathbb{Z}v_i$ as groups then $R \oplus R \cong \mathbb{Z}^{2d}$. We study below the local structure of F_R and its Zeta function in some particular cases. Our motivation is the following.

Suppose k is an algebraically closed field and A is an abelian variety over $\operatorname{Spec} k$ of dimension d. Let $\mathcal{C}_k$ denote the category of local Artinian rings $(B, \mathfrak{m})$ such that $B/\mathfrak{m} = k$. Let $\delta : \mathcal{C}_k \to$ (sets) be the deformation functor given by

$$\delta(B) = \{(\mathcal{A}, \phi) \,|\, \mathcal{A}/\operatorname{Spec} B \text{ an abelian scheme}, \phi : \mathcal{A} \otimes_B (B/\mathfrak{m}) \cong A \;\}/\cong$$

Theorem 3.1. (Grothendieck, de Jong, Deligne - Pappas, others) *There exists a complete local noetherian ring D with residue field k that pro-represents δ; for any ring B in $\mathcal{C}_k$, $\delta(B) = \operatorname{Hom}_{cont}(D, B)$. Moreover there is a k- rational point, say x, of the Grassmannian $G(n, 2n)$ such that*

$$D \cong \widehat{\mathcal{O}}_{G,x} \cong \operatorname{Spec} W(k)[[x_1, x_2, \ldots, x_{d(n-d)}]]$$

(if k has characteristic 0 we can replace $W(k)$ by k).

In general, deformations of abelian varieties possibly provided with extra data such as an embedding $R \subset \mathrm{End}(A)$ or a principal polarization are related to completed local rings on the Grassmannian F_R or L. In particular we have the following theorem.

Theorem 3.1.(continued) *Let R be a ring of integers. Let (A, i) be an abelian variety over k with an embedding $i : R \to \mathrm{End}_k(A)$. Let $\delta_R : \mathcal{C}_k \to (\mathrm{sets})$ be the deformation functor given by*

$$\delta_R(B) = \left\{ \begin{array}{l} (\mathcal{A}, I, \phi) \mid \mathcal{A}/\operatorname{Spec} B \text{ is an abelian scheme,} \\ I : R \to \mathrm{End}_B(A),\ \phi : \mathcal{A} \otimes_B (B/\mathfrak{m}) \cong A,\ I \textit{ induces } i \end{array} \right\}.$$

Then δ_R is represented by a complete local noetherian ring D_R with residue field k and moreover $D_R \cong \widehat{\mathcal{O}}_{F_R,x}$, the completion of the local ring of the Grassmannian F_R at a suitable point x.

3.1 Representability of the functor f_R

Recall that for a fixed ring $R \subset M_n(\mathbb{Z})$ and an integer $0 < d < n$, $f_R(T)$ is given by

$$\{T\text{-submodules } K \subset T^n \text{ that are } R\text{-invariant rank } d \text{ direct summands of } T^n\}.$$

Let W be a free rank d submodule of T^n which is R-invariant. We first show that f_R is a sheaf in the Zariski topology. Let T be a ring. Let $X = \operatorname{Spec} T$. Consider the open covering of X by distinguished open affine sets $U_i = \operatorname{Spec} T_{f_i}$. Suppose for every collection of elements $W_i \in f_R(T_{f_i})$, W_i and W_j map to the same element in $f_R(T_{f_i f_j})$. So in $f_R(T_{f_i f_j})$ we have

$$W_i \otimes_{T_{f_i}} T_{f_i f_j} = W_j \otimes_{T_{f_j}} T_{f_i f_j}.$$

We wish to show that there exists a unique element $W \in f_R(T)$ that maps to each of the W_i. The existence and uniqueness part follows from the case of the Grassmann functor. We have to verify that if W_i are R-invariant for all i then the direct summand W as obtained in the section 1.4.2 is also R-invariant. Referring to section 1.4.2 we get a rank d direct summand W of T^n as

$$W = \Gamma(X, \mathcal{F}) = \{w \in T^n \mid w \in T^n_{f_i}, w \in W_i \text{ for all } i\}.$$

Now if $r \in R$ and $w \in W$ we see that $w \in W_i$ for all i and as each W_i is R-invariant, $r \cdot w \in W_i$. It follows that $r \cdot w \in W$. Thus W is R-invariant.

Now let W be a rank d summand of T^n. Suppose we have a splitting of T^n as $T^n = W \oplus U$ where U is a rank $(n-d)$ free submodule of T^n. Then we have an affine neighbourhood of W namely, $\mathrm{Hom}^{[R]}(W, U) \subset \mathrm{Hom}(W, U)$ given by

$$\mathrm{Hom}^{[R]}(W, U) = \{\varphi : W \to U \mid \Gamma_\varphi \text{ is } R\text{- invariant}\}.$$

Lemma 3.2. *The subset* $\mathrm{Hom}^{[R]}(W, U) \subset \mathrm{Hom}\ (W, U) \cong \mathbb{A}^{d(n-d)}$ *is closed.*

Proof. We have $\mathrm{Hom}^{[R]}(W, U) = \bigcap_{r \in R} \mathrm{Hom}^{[r]}(W, U)$ where by definition

$$\mathrm{Hom}^{[r]}(W, U) = \{\varphi : W \to U \mid \Gamma_\varphi \text{ is } r\text{-invariant}\}.$$

Let $\{e_1, \ldots, e_d\}$ be a basis for W. Then a basis for the graph of φ, Γ_φ is given by $\{(e_i, \varphi(e_i)) \mid 1 \leq i \leq d\}$. Extend the basis $\{e_1, \ldots, e_d\}$ of W to a basis of T^n. We have

$$\begin{aligned} \mathrm{Hom}^{[R]}(W, U) &= \bigcap_{r \in R} \bigcap_{i=1}^{d} \mathrm{Hom}^{[r],i}(W, U) \\ &= \bigcap_{r \in R} \bigcap_{i=1}^{d} \{\varphi : W \to U \mid r(e_i, \varphi(e_i)) \in \Gamma_\varphi\}. \end{aligned}$$

Let M_i be a $n \times (d+1)$ matrix with the first d colums given by $e_i + \varphi(e_i)$ for $i = 1, \cdots, d$ and the $(d+1)$-th column given by $r(e_i + \varphi(e_i))$. Then $r(e_i, \varphi(e_i)) \in \Gamma_\varphi$ if and only if all the $(d+1) \times (d+1)$ determinants of the matrix M_i vanish. This defines a closed condition. Thus each $\mathrm{Hom}^{[r],i}(W, U)$ is a closed set in $\mathrm{Hom}(W, U)$ and $\mathrm{Hom}^{[R]}(W, U)$ is a closed subset of $\mathrm{Hom}(W, U)$ being a intersection of closed sets. $\square$

Corollary 3.3. *The functor f_R is representable.*

Proof. By the above discussion we see that the functor f_R is a sheaf in the Zariski topology. Also by Lemma 3.2, given W a rank d summand which is R-invariant and complementary submodule U, we have a neighbourhood of W, namely

$$\mathrm{Hom}^{[R]}(W,U) \subset \mathrm{Hom}\ (W,U) \cong \mathbb{A}^{d(n-d)}.$$

Then referring to Theorem 1.51 it follows that the functor f_R is representable. □

Now, let V be a vector space of dimension n over an algebraically closed field k. Suppose a ring R acts on V. Let W be a rank d direct summand of V that is R-invariant. Suppose we have a splitting $V = W \oplus U$. Then $U = V/W$ has a R-module structure. Note that U as a subspace of V may not be R-invariant. In general, the set

$$\{\Gamma_\varphi \mid \varphi : W \to U,\ \varphi \text{ is a } k\text{-linear map with } \Gamma_\varphi\ R\text{-invariant}\}$$

is not same as the set $\mathrm{Hom}_{R\otimes k}(W,U)$. However we will see that $\mathrm{Hom}_{R\otimes k}(W,U)$ is in natural bijection with the tangent space to the scheme F_R at W.

3.2 Properties of F_R

In this section we will give a description of the Zariski tangent space to the Grassmannian $G(d,n)$ at a k-valued point. Hence we will describe the tangent space to the scheme F_R at a k-valued point W. Recall that if X is any scheme, then for any k-rational point $x \in X$ the Zariski tangent space T_x to X at x is $\mathrm{Hom}(\mathfrak{m}/\mathfrak{m}^2, k)$, where $\mathfrak{m} = \mathfrak{m}_{X,x}$ is the maximal ideal in the local ring of X at x and k is the residue field of X at x. Now suppose that X is a scheme over field k. Then to give a k morphism of $\mathrm{Spec}\, k[\epsilon]/(\epsilon^2)$ to X is equivalent to giving a point $x \in X$, rational over k (i.e. $k(x) = k$), and an element of T_x. For the details refer to [6, p.256-257].

Proposition 3.4. *The Zariski tangent space to the Grassmannian $G(d,n)$ at a k-valued point W is isomorphic to* $\mathrm{Hom}_k(W, k^n/W)$.

Proof. If W is a k-valued point of the Grassmannian $G(d,n)$ it is a rank d summand of k^n. Let $k[\epsilon] = k(\epsilon)/(\epsilon^2)$. By the above discussion, to give a tangent vector to $G(d,n)(k)$ at W is to giving a k-morphism $k[\epsilon] \to G(d,n)(k)$, i.e. by section 1.4.2 of Chapter 1, giving a rank d direct summand of $k[\epsilon]^n$ which reduces to W mod ϵ. Thus, the Zariski tangent space to $G(d,n)(k)$ at W is the set of all rank d summands M of $k[\epsilon]^n$ which reduce to W modulo ϵ. We now show that the collection of all such M can be identified with $\mathrm{Hom}_k(W, k^n/W)$. Let first M be a rank d summand of $k[\epsilon]^n$ that is W modulo ϵ. Then $M \otimes_{k[\epsilon]} k = W$, and we get an exact sequence

$$0 \to \epsilon M \to M \to W \to 0$$

We have $\epsilon M = \epsilon W$, and for all $w \in W$ there is a $\varphi(w) \in k^n$, such that $w + \varphi(w)\epsilon \in M$. Using this and the fact that the above sequence is exact, the module M has the form

$$M = \{w + \varphi(w)\epsilon \mid w \in W\} + \epsilon W.$$

Then φ gives a map $W \to k^n/W$. Moreover, if $w_1, w_2 \in W$, we have

$$([w_1 + \varphi(w_1)\epsilon] + [w_2 + \varphi(w_2)\epsilon] - [w_1 + w_2 + \varphi(w_1 + w_2)\epsilon]) \in \epsilon W.$$

Therefore, we have

$$\varphi(w_1) + \varphi(w_2) - \varphi(w_1 + w_2) \in W,$$

which implies $\varphi(w_1) + \varphi(w_2) - \varphi(w_1 + w_2)$ is zero in k^n/W. Similarly, one finds that $\varphi(aw) - a\varphi(w) \in W$ for all $w \in W$ and $a \in k$. Thus, the map φ defines a k-linear map $W \to k^n/W$. Also, if $\psi : W \to k^n/W$ is another function such that for all $w \in W$, $(w + \psi(w)\epsilon) \in M$ then

$$\varphi(w)\epsilon - \psi(w)\epsilon \in M \Rightarrow \varphi(w) - \psi(w) \in W.$$

Thus $\varphi(w) - \psi(w)$ is zero in k^n/W and $\varphi = \psi$. So, given a rank d summand M of $k[\epsilon]^n$ that is W modulo ϵ we get a well defined map in $\mathrm{Hom}_k(W, k^n/W)$. Now let $\varphi : W \to k^n/W$ be a k-linear map. We want to define a rank d summand M of $k[\epsilon]^n$ that is W modulo ϵ. Lift φ to a map $\widetilde{\varphi} : W \to k^n$. Define M by

$$M = \{w + \widetilde{\varphi}(w)\epsilon \mid w \in W\} + \epsilon W.$$

Then M is closed under addition, closed under multiplication by elements in $k[\epsilon]$ and is independent of the choice of $\widetilde{\varphi}$. So M is a $k[\epsilon]$ module that is W modulo ϵ and has rank d as W has rank d, and M is the required summand. □

Corollary 3.5. *The Zariski tangent space to the scheme F_R at a k-valued point W is isomorphic to* $\mathrm{Hom}_{R\otimes k}(W, k^n/W)$.

Proof. If W is a k-valued point of F_R it is an R- invariant rank d summand of k^n. By the above proposition the Zariski tangent space to F_R at W is the set of all R-invariant rank d summands M of $k[\epsilon]^n$ that reduce to W modulo ϵ. Suppose first that M is R-invariant. So we get a k linear map $\varphi \in \mathrm{Hom}_k(W, k^n/W)$. Then for $r \in R$, $w \in W$ we have

$$r(w + \varphi(w)\epsilon) \in M \quad \text{and} \quad rw + \varphi(rw)\epsilon \in M.$$

Therefore,

$$(r\varphi(w) - \varphi(rw))\epsilon \in M \Rightarrow r\varphi(w) - \varphi(rw) \in W.$$

Thus $r\varphi(w) - \varphi(rw)$ is zero in k^n/W and we get that φ is R-linear. Conversely if we have a R-linear map $\varphi : W \to k^n/W$ then for $r \in R$, $\varphi(rw) = r(\varphi(w))$. So for the corresponding lift $\widetilde{\varphi} : W \to k^n$, $\widetilde{\varphi}(rw) = r(\widetilde{\varphi}(w))$ modulo W. Therefore the module

$$M = \{w + \tilde{\varphi}(w)\epsilon \mid w \in W\} + \epsilon W,$$

as defined in the above theorem, is R-invariant. □

Example 3.6. *Suppose k is an algebraically closed field. Let $R \otimes k = k[\epsilon]$. Suppose $V = k[\epsilon]^2$, $W = k[\epsilon]$, $U = k[\epsilon]$ and $V = W \oplus U$. Then*

$$\mathrm{Hom}_{R\otimes k}(W, U) = \mathrm{Hom}_{k[\epsilon]}(k[\epsilon],\ k[\epsilon]) \cong k[\epsilon],$$

which is 2-dimensional over k. However, if $W = (\epsilon) \oplus (\epsilon)$, then $V/W \cong (\epsilon) \oplus (\epsilon)$ as $k[\epsilon]$ modules and the tangent space to W as a point on F_R is given by

$$\mathrm{Hom}_{k[\epsilon]}(W, U) = \mathrm{Hom}_k(W, U),$$

which is four dimensional. Thus W, as a point on F_R, is singular.

Proposition 3.7. *Let V be a vector space over an algebraically closed field k. Suppose a ring R acts on V. Assume that $V = W \oplus U$ where W and U are R-invariant. Let $\varphi : W \to U$ be a k-linear homomorphism. Then the graph of φ, Γ_φ is R-invariant if and only if φ is R-equivariant. Thus an affine neighbourhood of F_R is given by $\mathrm{Hom}_{R\otimes k}(W, U)$.*

Proof. Let Γ_φ be R-invariant. Let $r \in R$, $w \in W$. Then $(w, \varphi(w)) \in \Gamma_\varphi$ implies $(rw, r\varphi(w))$ and $(rw, \varphi(rw))$ belong to Γ_φ. Thus $r(\varphi(w)) = \varphi(rw)$ and so φ is R-equivariant. Conversely suppose that φ is R-equivariant. Let $(w, \varphi(w)) \in \Gamma_\varphi$. Then for $r \in R$

$$r(w, \varphi(w)) = (rw, r(\varphi(w))) = (rw, \varphi(rw)) \in \Gamma_\varphi.$$

Hence, Γ_φ is R-invariant. □

Let T be any algebraically closed field. Let R be the ring of integers of a number field $Q(R)$ with $[Q(R) : \mathbb{Q}] = d$. Assume that $R \otimes T$ is a sum of fields, say

$$R \otimes_{\mathbb{Z}} T = \bigoplus_{\alpha} T \cong T^d,$$

where the sum varies over all ring homomorphisms $\alpha : R \to T$. To give a T-submodule $M \subset T^{2d}$ of rank d that is R-invariant is same as giving a $R \otimes T$ invariant submodule of $(R \otimes_{\mathbb{Z}} T)^2$ of rank d over T.

Lemma 3.8. *Let R and T be as above. Let $M \subset (R \otimes T)^2$ be a submodule of rank d over T. Then M is $R \otimes T$ invariant if and only if $M = \bigoplus_{i=1}^d M_i$, where M_i is a T-submodule of T^2 of rank r_i and $\sum_{i=1}^d r_i = d$.*

Proof. Note that R and T are commutative rings with unity. Every submodule M of $(R \otimes T)^2$ can be written as

$$M = \bigoplus_{i=1}^{d} e_i M = \bigoplus_{i=1}^{d} M_i,$$

where e_i are the idempotents corresponding to the decomposition $R \otimes T = \bigoplus_\alpha T$. Each M_i is a T module of T^2. If M has rank d and M_i has rank r_i then $\sum r_i = d$. □

The invariants $(r_1, \cdots, r_d)$ are discrete invariants of the module M. We see that the Grassmann variety $F_{R\otimes T}$ is a disjoint union $\coprod_{r=(r_1,\cdots,r_d)} V_r$, where the union is over vectors $(r_1, \cdots, r_d)$ such that each $0 \leq r_i \leq 2$ and $\sum r_i = d$. The component V_r parametrizes modules M with invariant r.

Now let $M \subset (R \otimes T)^2$ be an R-invariant rank d summand of T^{2d} as a T-module. Suppose $T^{2d} = M \oplus U$ where U is also R-invariant. Write $M = \oplus M_i$ and $U = \oplus U_i$ where each M_i and U_i is a T-module. An affine neighbourhood of M in $F_{R\otimes T}$ is given by

$$\mathrm{Hom}_{R\otimes T}(M, U) = \{\Gamma_\varphi \mid \varphi : M \to U \text{ such that } \Gamma_\varphi \text{ is } R \otimes T\text{-invariant}\}.$$

But, we have

$$\mathrm{Hom}_{R\otimes T}(M, U) = \bigoplus_i \mathrm{Hom}_T(M_i, U_i).$$

Thus, for $\varphi : W \to U$, the graph of φ, Γ_φ, has the form

$$\Gamma_\varphi = \bigoplus_{i=1}^{d} \Gamma_{\varphi_i},$$

where each Γ_{φ_i} has rank r_i. We see that V_r equals j copies of $\mathbb{P}^1$ where j is number of i such that $r_i = 1$.

In the case when $d = 2$ we have following possibilities :

1. M_i is 0 and U_i has rank 2.

2. U_i is 0 and M_i has rank 2.

3. Both M_i and U_i have rank 1

3.3 Example : Quadratic field

We now consider the following example of a quadratic field. Suppose $L = \mathbb{Q}(\sqrt{D})$ is a quadratic extension of $\mathbb{Q}$ where D is a squarefree integer. We have $[L : \mathbb{Q}] = 2$. Let R be the ring of integers in L. Then

$$R = \begin{cases} \mathbb{Z}[\sqrt{D}] & \text{if } D \equiv 2,3 \pmod 4; \\ \mathbb{Z}[\frac{1+\sqrt{D}}{2}] & \text{if } D \equiv 1 \pmod 4. \end{cases}$$

The discriminant d_L of R is

$$d_L = \begin{cases} 4D & \text{if } D \equiv 2,3 \pmod 4; \\ D & \text{if } D \equiv 1 \pmod 4. \end{cases}$$

Let $k = \overline{\mathbb{F}}_p$ and $R_1 = R \otimes k$. Consider $R_1^2 = R_1 \oplus R_1$. It has two structures, namely:

1. R_1^2 is an R_1- module. For $r_1, r_2, r \in R_1$ define $r(r_1, r_2) = (rr_1, rr_2)$.

2. R_1^2 is a k-vector space of dimension 4 i.e. $R_1^2 \cong k^4$.

We are interested in a subscheme $F_{R_1}(2,4)(k) \subset G(2,4)(k)$ which is the collection of all 2-dimensional subspaces of R_1^2 that are R_1- invariant. We concentrate on the case when $D \equiv 2,3 \pmod 4$. We have

$$R_1 = R \otimes k = \begin{cases} \overline{\mathbb{F}}_p \oplus \overline{\mathbb{F}}_p, & \overline{\mathbb{F}}_p \text{ characteristic } p, (p) \text{ inert in } L; \\ \overline{\mathbb{F}}_p \oplus \overline{\mathbb{F}}_p, & \overline{\mathbb{F}}_p \text{ characteristic } p, (p) \text{ split in } L\ ; \\ \overline{\mathbb{F}}_p[t]/t^2, & \overline{\mathbb{F}}_p \text{ characteristic } p,\ (p) \text{ ramified in } L. \end{cases}$$

First two cases: If (p) is inert or spilt in L, $R_1 = \overline{\mathbb{F}}_p \oplus \overline{\mathbb{F}}_p$, $R_1^2 = (\overline{\mathbb{F}}_p \oplus \overline{\mathbb{F}}_p) \oplus (\overline{\mathbb{F}}_p \oplus \overline{\mathbb{F}}_p)$. We are interested in 2- dimensional subspaces of R_1^2 that are R_1-invariant. Note that U is such a subspace if and only if U is preserved under $(1, 0)$ and $(0, 1)$. But

$$(0, 1) = (1, 1) - (1, 0).$$

So it is enough to consider the subspaces of R_1^2 that are invariant under $(1, 0)$. Let $e_1 = (1, 0, 0, 0)$, $e_2 = (0, 1, 0, 0)$, $e_3 = (0, 0, 1, 0)$, $e_4 = (0, 0, 0, 1)$ be the standard basis for k^4. Let $t : R_1^2 \to R_1^2$ be the linear transformation given by

$$t = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}$$

Then we are interested in 2-dimensional subspaces of R_1^2 that are preserved under t. The basis $\{e_1, e_2, e_3, e_4\}$ of R_1^2 has the property that t acts as identity on e_1, e_3 and acts as zero on e_2, e_4. Let $M_1 = \text{span}\{e_2, e_4\}$, $M_2 = \text{span}\{e_1, e_3\}$. Then $R_1^2 = M_1 \oplus M_2$. If N is a submodule of R_1^2 of dimension 2 by Lemma 3.8 we can write it as $N = N_1 \oplus N_2$ where N_1, N_2 are submodules of M_1, M_2 respectively. If N has dimension 2 we have the following three possibilities.

1. $N = M_1 = \text{span}\{e_2, e_4\}$;
2. $N = M_2 = \text{span}\{e_1, e_3\}$;
3. $N=$ span $\{a_2e_2 + a_4e_4, a_1e_1 + a_3e_3 \mid a_2 \neq 0$ or $a_4 \neq 0$, and $a_1 \neq 0$ or $a_3 \neq 0\}$.

We now analyse these three cases. Let $x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}$ be the Plücker coordinates corresponding to the canonical basis of $\bigwedge^2 k^4$. Then, referring to section 1.12 in Chapter 1, we see that the Grassmann relation satisfied by $G(2, 4)$ is given by

$$x_{14}x_{23} - x_{24}x_{13} + x_{12}x_{34} = 0.$$

1. When $N = \text{span}\{e_2, e_4\}$ its image under the Plücker map is the point in $\mathbb{P}^5$ given by $(0 : 0 : 0 : 0 : 1 : 0)$. This is a closed subset of $\mathbb{P}^5$ defined by

$$Z_1 = V\{x_{12}, x_{13}, x_{14}, x_{23}, x_{34}\}.$$

2. When $N = \text{span}\{e_1, e_3\}$ its image in $\mathbb{P}^5$ is $(0:1:0:0:0:0)$. This is a closed subset of $\mathbb{P}^5$ defined by

$$Z_2 = V\{x_{12}, x_{14}, x_{23}, x_{24}, x_{34}\}.$$

3. Assume we are in the third case. Let $v_1 = a_2e_2 + a_4e_4,\ v_2 = a_1e_1 + a_3e_3$. Then

$$v_1 \wedge v_2 = (-a_1a_2)e_1 \wedge e_2 + (0)e_1 \wedge e_3 + (-a_1a_4)e_1 \wedge e_4 \\ + (a_2a_3)e_2 \wedge e_3 + (0)e_2 \wedge e_4 + (-a_3a_4)e_3 \wedge e_4.$$

So the Plücker coordinates of such N are

$$(-a_1a_2 : 0 : -a_1a_4 : a_2a_3 : 0 : -a_3a_4).$$

In addition to the Plücker relations these points in $\mathbb{P}^5$ satisfy the following relations in the coordinates

$$x_{12}x_{34} + x_{14}x_{23} = 0, \quad x_{13} = 0, \quad x_{24} = 0.$$

Moreover, one checks that every point on the closed set of $G(2,4)$ defined by those relations comes from some N. In fact the collection of all such points is isomorphic to a quadratic surface in $\mathbb{P}^3$ defined by

$$Z_3 = V\{x_{12}x_{34} + x_{14}x_{23}\}$$

and so is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ over $\overline{\mathbb{F}}_p$. This agrees with the discussion in the previous section. However, since the isomorphism $Z_3 \cong \mathbb{P}^1 \times \mathbb{P}^1$ is only over $\overline{\mathbb{F}}_p$, we will use explicit equations below to compute the Zeta function.

We now compute the Zeta function of this Grassmannian $F_{R_1}(2,4)(k)$ contained in $G(2,4)(k)$. Let $\sigma : \mathbb{A}^4_k \to \mathbb{A}^4_k$ be the Frobenius morphism. The Galois group $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ acts on $R_1 = (R \otimes \overline{\mathbb{F}}_p)$ by

$$\sigma(l \otimes \lambda) = l \otimes \sigma(\lambda).$$

This action induces the action of the Galois group on R_1^2. We want to see the action of the Galois group on the Plücker coordinates.

3.3.1 The Zeta function : (p) inert in L

Assume that (p) is inert in L. Then

$$\mathbb{Z}[\sqrt{D}]/(p) \cong \mathbb{F}_p[x]/(x^2 - D) =: \mathbb{F},$$

which is a field with p^2 elements. There exist two embeddings $\tau_1, \tau_2 : \mathbb{F} \to \overline{\mathbb{F}}_p$ and $R \otimes \overline{\mathbb{F}}_p \cong \overline{\mathbb{F}}_p^2$ by the map $f : l \otimes \lambda \mapsto (\tau_1(l)\lambda,\ \tau_2(l)\lambda)$. Note that $R = \mathbb{Z} \cdot 1 \oplus \mathbb{Z}[\sqrt{D}]$ as a group and

$$f(1 \otimes 1) = (1, 1), \quad f(\sqrt{D} \otimes 1) = (\sqrt{D}, -\sqrt{D}).$$

The elements $(1, 1), (\sqrt{D}, \sqrt{-D})$ form a basis to $\overline{\mathbb{F}}_p^2$ and the Frobenius morphism σ acts on $(1, 1)$ and $(\sqrt{D}, 1)$ as the identity.
Let $M = \begin{pmatrix} 1 & \sqrt{D} \\ 1 & -\sqrt{D} \end{pmatrix}$. Then $M^{-1} = \frac{-1}{2\sqrt{D}} \begin{pmatrix} -\sqrt{D} & -\sqrt{D} \\ -1 & 1 \end{pmatrix}$. The matrix M^{-1} changes coordinates from the standard basis of $\overline{\mathbb{F}}_p^2$ to the basis $\{(1, 1), (\sqrt{D}, \sqrt{-D})\}$. Let $\alpha = (\alpha_1, \alpha_2)$ be any general element of $\overline{\mathbb{F}}_p^2$. Suppose

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = M^{-1} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

Then $\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{\alpha_1+\alpha_2}{2} \\ \frac{\alpha_1-\alpha_2}{2\sqrt{D}} \end{pmatrix}$ are the coordiantes of α in the basis $\{(1, 1), (\sqrt{d}, \sqrt{-d})\}$.
Therefore, we get

$$\sigma \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{\sigma(\alpha_1)+\sigma(\alpha_2)}{2} \\ \frac{\sigma(\alpha_1)-\sigma(\alpha_2)}{2\sigma(\sqrt{D})} \end{pmatrix} = \begin{pmatrix} \frac{\sigma(\alpha_1)+\sigma(\alpha_2)}{2} \\ \frac{\sigma(\alpha_1)-\sigma(\alpha_2)}{-2\sqrt{D}} \end{pmatrix}.$$

Now,

$$\sigma \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = M\sigma \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{D} \\ 1 & -\sqrt{D} \end{pmatrix} \begin{pmatrix} \sigma(\alpha_1) + \sigma(\alpha_2) \\ \frac{\sigma(\alpha_1)-\sigma(\alpha_2)}{-\sqrt{D}} \end{pmatrix} = \begin{pmatrix} \sigma(\alpha_2) \\ \sigma(\alpha_1) \end{pmatrix}.$$

So σ acts on $(\alpha_1, \alpha_2) \in k^2$ by

$$\sigma \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \sigma(\alpha_2) \\ \sigma(\alpha_1) \end{pmatrix}.$$

Thus, σ acts on k^4 by

$$\sigma(a_1, a_2, a_3, a_4) = (\sigma(a_2), \sigma(a_1), \sigma(a_4), \sigma(a_3)),$$

and it acts on the Plücker coordinates by

$$\sigma(x_{12}) = -x_{12}, \quad \sigma(x_{13}) = x_{24}, \quad \sigma(x_{14}) = x_{23},$$

$$\sigma(x_{23}) = x_{14}, \quad \sigma(x_{24}) = x_{13}, \quad \sigma(x_{34}) = -x_{34}.$$

Now we come back to the subvariety of $G(2, 4)$ defined by

$$x_{12}x_{34} + x_{14}x_{23} = 0, \quad x_{13} = x_{24} = 0.$$

The points on this subvariety have Plücker coordinates $(c_{12} : 0 : c_{14} : c_{23} : 0 : c_{34})$ satisfying the relation $c_{12}c_{34} + c_{14}c_{23} = 0$. The Frobenius morphism σ acts on $(c_{12} : 0 : c_{14} : c_{23} : 0 : c_{34})$ by

$$\sigma(c_{12} : 0 : c_{14} : c_{23} : 0 : c_{34}) = (-\sigma(c_{12}) : 0 : \sigma(c_{23}) : \sigma(c_{14}) : 0 : -\sigma(c_{34})).$$

One has

$$\sigma^2(c_{12} : 0 : c_{14} : c_{23} : 0 : c_{34}) = (\sigma^2(c_{12}) : 0 : \sigma^2(c_{14}) : \sigma^2(c_{23}) : 0 : \sigma^2(c_{34})).$$

In fact we see that if r is any even positive integer then σ^r acts by

$$\sigma^r(c_{12} : 0 : c_{14} : c_{23} : 0 : c_{34}) = (\sigma^r(c_{12}) : 0 : \sigma^r(c_{14}) : \sigma^r(c_{23}) : 0 : \sigma^r(c_{34})).$$

This action of σ^r is usual componentwise action as in case of projective spaces. So the number of subspaces invariant under the action of σ^r is simply equal to the number of solutions to $x_{12}x_{34} + x_{14}x_{23} = 0$ over $\mathbb{F}_q$ with $q = p^r$. To count those we distinguish two cases.

1. Suppose $x_{12} \neq 0$. Each of x_{14} and x_{23} can be chosen arbitrarily. Hence each of them has q choices and x_{34} is determined by $x_{34} = -\dfrac{x_{14}x_{23}}{x_{12}}$. So the total number of solutions in this case is q^2.

2. Suppose $x_{12} = 0$. Then either $x_{14} = 0$ or $x_{23} = 0$ or both of them are zero. The total number of solutions in this case is $(q+1)+(q+1)-1 = 2q+1$.

Adding the number of solutions in both the cases we get the number of subspaces invariant under the action of σ^r, for r even, is $p^{2r} + 2p^r + 1$. Also if r is even, the points $(0:1:0:0:0:0)$ and $(0:0:0:0:1:0)$ are preserved under σ^r.

To see how σ^r acts if r is odd, first note that the set of points in $\mathbb{P}^5$ given by $(c_{12}:0:c_{14}:c_{23}:0:c_{34})$ satisfying $c_{12}c_{34} + c_{14}c_{23} = 0$ can be identified with the set of points $(a:b:c:d)$ in $\mathbb{P}^3$ satisfying $ad+bc = 0$. First consider the case $r = 1$. We consider 2 cases

1. Assume $a \neq 0$. Let $a = 1$. So we have

$$\sigma(1:b:c:d) = (-\sigma(1):\sigma(c):\sigma(b):-\sigma(d)) = (-1:\sigma(c):\sigma(b):-\sigma(d)).$$

   These are proportional if and only if the multiplication factor is -1. Therefore we get, $b = -\sigma(c)$ and $c = -\sigma(b)$ which implies that $b, c \in \mathbb{F}_{p^2}$. Also $d \in \mathbb{F}_p$ as $\sigma(d) = d$. The relation $b \cdot \sigma(b) = -d$ gives $b \cdot b^p = -d$. So we have to find the number of $b \in \mathbb{F}_{p^2}$ satisfying $b^{p+1} \in \mathbb{F}_p$. But $b^{p+1} = \mathrm{Norm}_{\mathbb{F}_{p^2}/\mathbb{F}_p}(b)$ and so any $b \in \mathbb{F}_{p^2}$ satisfies $b^{p+1} \in \mathbb{F}_p$. Thus the number of such b is p^2.

2. Now let $a = 0$, $b = 0$.

$$\sigma(0:0:c:d) = (0:0:\sigma(0):-\sigma(d)) = (0:0:0:-\sigma(d)).$$

   There is only one solution in this case. We note that one gets the same solution when $a = 0, c = 0$.

Adding the total number of solutions in both the cases we get that the number of points invariant under the action of σ is p^2+1. We can do similar calculations for all odd powers of σ and get that the number of points invariant under the action of σ^{2r-1} is equal to $p^{2(2r-1)}+1$.

The Grassmannian $F_{R_1}(\overline{\mathbb{F}}_p)$ that we have is the union of two sets X_1 and X_2,

$$X_1 = \{(0:1:0:0:0:0),(0:0:0:0:1:0)\},$$

$$X_2 = \{(c_{12}:0:c_{14}:c_{23}:0:c_{34}) \mid c_{12}c_{34}+c_{14}c_{23}=0\}.$$

The Zeta function of $F_{R_1}(\overline{\mathbb{F}}_p)$ is given by

$$Z(F_{R_1}(\overline{\mathbb{F}}_p),t) = Z(X_1,t)\cdot Z(X_2,t).$$

We have

$$Z(X_1,t) = \exp\left(\sum_{r=1}^{\infty} 2\frac{t^{2r}}{2r}\right) = \exp\left(\sum_{r=1}^{\infty}\frac{t^{2r}}{r}\right) = \frac{1}{1-t^2}.$$

The Zeta function of X_2 is given by

$$\begin{aligned} Z(X_2,t) &= \exp\left(\sum_{r=1}^{\infty}(p^{2r}+1)^2\frac{t^{2r}}{2r}+\sum_{r=1}^{\infty}(p^{2(2r-1)}+1)\frac{t^{2r-1}}{2r-1}\right) \\ &= \exp\left(\sum_{r=1}^{\infty}(p^{4r}+2p^{2r}+1)\frac{t^{2r}}{2r}+\sum_{r=1}^{\infty}(p^{2(2r-1)}+1)\frac{t^{2r-1}}{2r-1}\right). \end{aligned}$$

Combining proper terms together we get

$$\begin{aligned} Z(X_2,t) &= \exp\left(\sum_{r=1}^{\infty}p^{2r}\frac{t^r}{r}+\sum_{r=1}^{\infty}2p^{2r}\frac{t^{2r}}{2r}+\sum_{r=1}^{\infty}\frac{t^r}{r}\right) \\ &= \frac{1}{(1-t)(1-p^2t)(1-p^2t^2)}. \end{aligned}$$

Therefore we get

$$Z(F_{R_1}(\overline{\mathbb{F}}_p),t) = \frac{1}{(1-t)(1-t^2)(1-p^2t)(1-p^2t^2)}.$$

3.3.2 The Zeta function : (p) split in L

Let (p) be split in L. Then $R \otimes \mathbb{F}_p = \mathbb{F}_p \oplus \mathbb{F}_p$ and it follows that the Galois action on $R \otimes \overline{\mathbb{F}}_p = \overline{\mathbb{F}}_p \oplus \overline{\mathbb{F}}_p$ and hence on the Plücker coordinates, is componentwise in this case. The Grassmann subvariety $F_{R_1}(\overline{\mathbb{F}}_p)$ is the union of two sets X_1 and X_2,

$$X_1 = \{(0:1:0:0:0:0),(0:0:0:0:1:0)\},$$

$$X_2 = \{(c_{12}:0:c_{14}:c_{23}:0:c_{34}) \mid c_{12}c_{34}+c_{14}c_{23}=0\}.$$

The Zeta function of $F_{R_1}(\overline{\mathbb{F}}_p)$ is given by

$$Z(F_{R_1}(\overline{\mathbb{F}}_p),t) = Z(X_1,t)\cdot Z(X_2,t).$$

The Zeta function of X_1 is given by

$$Z(X_1,t) = \exp\left(\sum_{r=1}^{\infty} 2\frac{t^r}{r}\right) = \frac{1}{(1-t)^2}.$$

Since the Galois action is natural, for a even positive integer r, σ^r acts by

$$\sigma^r(c_{12}:0:c_{14}:c_{23}:0:c_{34}) = (\sigma^r(c_{12}):0:\sigma^r(c_{14}):\sigma^r(c_{23}):0:\sigma^r(c_{34})).$$

This action of σ^r is the usual componentwise action as in case of projective spaces. So the number of subspaces invariant under the action of σ^r is simply equal to the number of solutions to $x_{12}x_{34}+x_{14}x_{23}=0$ over $\mathbb{F}_p$. We have the following cases.

1. Suppose $x_{12} \neq 0$. Assume that $x_{12}=1$. Each of x_{14} and x_{23} can be chosen arbitrarily. Hence each of them has p choices and x_{34} is determined by the relation $x_{34} = -\dfrac{x_{14}x_{23}}{x_{12}}$. So the total number of solutions in this case is p^2.

2. Suppose $x_{12}=0$. Then either $x_{14}=0$ or $x_{23}=0$ or both of them are zero. The total number of solutions in this case is $(p+1)+(p+1)-1=2p+1$.

Adding the solutions in both the cases we get the number of $\mathbb{F}_{p^r}$ rational points as $p^{2r}+2p^r+1$ and the Zeta function of X_2 is given by

$$\begin{aligned} Z(X_2,t) &= \exp\left(\sum_{r=1}^{\infty}(p^{2r}+2p^r+1)\frac{t^r}{r}\right) \\ &= \frac{1}{(1-t)(1-pt)^2(1-p^2t)}. \end{aligned}$$

It follows that

$$Z(F_{R_1}(\overline{\mathbb{F}}_p),t) = \frac{1}{(1-t)^3(1-pt)^2(1-p^2t)}.$$

3.3.3 The Zeta function : (p) ramified in L

The case when (p) is ramified in L. As before $R_1 = R\otimes k = k[t]/t^2$. Consider the nilpotent linear transformation $t : R_1^2 \to R_1^2$. We are interested in the set $F_{R_1}(\overline{\mathbb{F}}_p)$ which is the set of 2-dimensional subspaces of R_1^2 that are R_1-invariant i.e. the subspaces that are preserved by the action of t. The vectors $w_1 = e_1 = (1,0)$, $w_2 = te_1 = (t,0), w_3 = e_2 = (0,1),\ w_4 = te_2 = (0,t)$ form a basis for R_1^2. We see that $\ker(t)$ is the space spanned by te_1 and te_2 as $t^2 = 0$. Now let U be any two dimensional subspace of V. Then tU is a subspace of V of dimension less than 2. If $U = \ker(t) = \mathrm{span}\{te_1, te_2\}$, it is zero dimensional, otherwise tU is one dimensional. And in that case if $\{v_1, v_2\}$ is a basis of U, either $tv_1 \neq 0$ or $tv_2 \neq 0$. If $tv_1 \neq 0$ since v_1, tv_1 are linearly independent, they form a basis for U. Conversely if $v_1 \in V$ such that $tv_1 \neq 0$ then $\{v_1, tv_1\}$ span a 2 dimensional subspace of R_1^2 which is t invariant. Thus every $U \in F_R(\overline{\mathbb{F}}_p)$ with $U \neq \ker(t)$ can be written as the span of $\{v_1, tv_1\}$ where $v_1 \in U$ is not in the $\ker(t)$.

$$v_1 = a_1e_1 + a_2te_1 + a_3e_2 + a_4te_2 = a_1w_1 + a_2w_2 + a_3w_3 + a_4w_4.$$

Then we get

$$tv_1 = a_1te_1 + a_3te_2 = a_1w_2 + a_3w_4.$$

The set $F_{R_1}(2,4)(\overline{\mathbb{F}}_p) \subset G(2,4)(\overline{\mathbb{F}}_p)$ can be embedded in $\mathbb{P}^5$ via the Plücker map. If $\{v_1, tv_1\}$ is a basis for U its image in $\mathbb{P}^5$ via the Plücker map is the element in $\mathbb{P}^5$ determined by $v_1 \wedge tv_1$. Now

$$\begin{aligned} v_1 \wedge tv_1 = (a_1^2)\, w_1 \wedge w_2 + (0)\, w_1 \wedge w_3 + (a_1 a_3)\, w_1 \wedge w_4 \\ + (-a_1 a_3)\, w_2 \wedge w_3 + (a_2 a_3 - a_1 a_4)\, w_2 \wedge w_4 + (a_3^2)\, w_3 \wedge w_4. \end{aligned}$$

If we set x_{ij} as the Plücker coordinates as before then we have the relations

$$x_{13} = 0,\ x_{14} + x_{23} = 0,\ x_{14}^2 - x_{12}x_{34} = 0$$

in addition to the Grassmann relation

$$x_{14}x_{23} - x_{24}x_{13} + x_{12}x_{34} = 0.$$

Thus $F_{R_1}(\overline{\mathbb{F}}_p)$ can be described as the zero set of $\{x_{13},\ x_{14} + x_{23},\ x_{14}^2 - x_{12}x_{34}\}$ in $\mathbb{P}^5$. Note that

$$F_{R_1}(\overline{\mathbb{F}}_p) \cong V\{(a:b:c:d) \in \mathbb{P}^3 \mid b^2 = ad\},$$

with the Galois action on V being the usual one. We now find number of solutions over $\mathbb{F}_{p^r}$.

1. Let $a = 0$. Therefore $b = 0$. The number of solutions in this case is $p^r + 1$.
2. Let $a \neq 0$. Suppose $a = 1$. So, $b^2 = d$. The number of solutions in this case is p^{2r}.

Adding the number of solutions in both the cases we get the number of $\mathbb{F}_{p^r}$ rational points is $p^{2r} + p^r + 1$. Therefore the Zeta function of $F_{R_1}(\overline{\mathbb{F}}_p)$ in this case is given by

$$\begin{aligned} Z(F_{R_1}(\overline{\mathbb{F}}_p), t) &= \exp\left(\sum_{r=1}^{\infty}(p^{2r} + p^r + 1)\frac{t^r}{r}\right) \\ &= \frac{1}{(1-t)(1-pt)(1-p^2t)}. \end{aligned}$$

3.4 Zeta function of F_R as a scheme over $\mathbb{Z}$

In the above example of quadratic field, the scheme F_R is of finite type over the ring of integers $\mathbb{Z}$. The global Zeta function in this case is obtained by combining the Zeta functions in (p) inert, split and ramified cases.

Proposition 3.9. *The Zeta function of F_R as a scheme over $\mathbb{Z}$ is given by*

$$\zeta(F_R, s) = \zeta_L(s) \cdot \zeta_L(s-1) \cdot \zeta(s-2) \cdot \zeta_{d_L}(s),$$

where $\zeta_L(s)$ is the Dedekind Zeta function, $\zeta(s-2)$ is Riemann Zeta function and $\zeta_{d_L}(s) = \prod_{p \nmid d_L} \frac{1}{(1-t)}$ is the Euler factor at d_L.

Proof. We recall from the last section, the computation of the Zeta function in all three cases.

1. The Zeta function when (p) is inert in L is

$$Z(F_{R_1}(\overline{\mathbb{F}}_p), t) = \frac{1}{(1-t)(1-t^2)(1-p^2t)(1-p^2t^2)},$$

2. The Zeta function when (p) is split in L is

$$Z(F_{R_1}(\overline{\mathbb{F}}_p), t) = \frac{1}{(1-t)^3(1-pt)^2(1-p^2t)},$$

3. The Zeta function when (p) is ramified in L is

$$Z(F_{R_1}(\overline{\mathbb{F}}_p), t) = \frac{1}{(1-t)(1-pt)(1-p^2t)}.$$

The global Zeta function $\zeta(F_R, s)$ is given by the product of the following three functions (where $t = p^{-s}$)

$$\prod_{\text{all } p} \frac{1}{(1-t)} \cdot \frac{1}{(1-p^2t)} \cdot \frac{1}{(1-pt)}, \tag{3.1}$$

$$\left(\prod_{p \text{ spilt}} \frac{1}{(1+t)} \prod_{p \text{ inert}} \frac{1}{(1-t)}\right) \cdot \left(\prod_{p \text{ spilt}} \frac{1}{(1+pt)} \prod_{p \text{ inert}} \frac{1}{(1-pt)}\right), \tag{3.2}$$

$$\prod_{p \text{ spilt}} \frac{1}{(1-t)} \prod_{p \text{ inert}} \frac{1}{(1-t)}. \tag{3.3}$$

Let χ be the quadratic Dirichlet character $(\mathbb{Z}/d_L\mathbb{Z})^* \to \{\pm 1\}$. We conclude that

$$\begin{aligned}\zeta(F_R, s) &= \zeta(s) \cdot \zeta(s-2) \cdot \zeta(s-1) \cdot L(\chi, s) \cdot L(\chi, s-1) \cdot \prod_{p \nmid d_L} \frac{1}{(1-t)} \\ &= \zeta_L(s) \cdot \zeta_L(s-1) \cdot \zeta(s-2) \cdot \zeta_{d_L}(s),\end{aligned}$$

where $\zeta_L(s)$ is the Dedekind Zeta function, $\zeta(s-2)$ is Riemann Zeta function and $\zeta_{d_L}(s) = \prod_{p \nmid d_L} \frac{1}{(1-t)}$ is the Euler factor at d_L. □

Conclusion

In this text we studied in detail the classical Grassmannian and the Lagrangian Grassmannian. After studying the local properties and the Zeta function of these varieties we introduced Schubert calculus. As one of its applications we noticed that the information of the cohomology groups of the Grassmannian in characteristic zero gives the information of the cohomology groups in characteristic p and vice versa. It remains an interesting problem to determine the ring structure of the cohomology using point counting in characteristic p.

Schubert calculus, essentially founded by H. Schubert in 1874, helps understand questions in enumerative geometry i.e. to find number of points, lines, planes etc satisfying certain geometric conditions. The subject was then connected with the branch of combinatorics which deals with symmetric functions, Young tableaux, plane partitions etc. An excellent account of this subject can be found in a survey article by Kleiman and Laksov [12]. In Chapter 1 we computed the products of Schubert cycles using the Basis Theorem, Giambelli's Formula and Pieri's Formula. A further step in this direction could be to understand the literature dealing with the connection between the multiplication of Schur-S polynomials (defined by Schur in his 1901 thesis) and the cohomology ring of the Grassmannian. This connection was first observed by Lesieur [13]. In the classical case, the multiplication of Schubert cycles agrees with the corresponding product of Schur-S polynomials.

The Schubert calculus for Lagrangian Grassmannian is also quite interesting and only a few pieces are introduced in this thesis. Further step in this direction could be to understand the literature dealing with the multiplicative structure of the cohomology ring of the Lagrangian Grassmannian. In [16], Pragacz showed that the product of Schubert cycles in Lagrangian case agrees with the corresponding product of Schur-Q polynomials.

In Chapter 1, following [6] we showed the representability of the Grassmann functor. In the next Chapter, we modified the same definition and used isotropic summands to define the Lagrangian Grassmann functor. The Grassmannians can also be viewed as Hilbert schemes. The Grassmannian $G_S(d, n)$ parametrizes subschemes X of degree one and dimension d in the projective space $\mathbb{P}^n_S$. We computed the Euler characteristic and the Zeta functions of Grassmannians. It would be interesting to compute their Hilbert polynomials and realise them as Hilbert schemes.

We then remark that the above Grassmann varieties come under a large class of generalized Grassmann varieties. For a field k, integers $0 < d < n$, a generalized Grassmann variety $G(R; d, n)$ is the set of all d-dimensional subspaces of k^n that are preserved under a subring $R \subseteq \mathrm{End}(k^n)$. The case $R = \{0\}$ leads to the classical Grassmannian $G(d, n)(k)$. If further we define an alternating pairing on k^{2n} and concentrate only on isotropic subspaces of dimension n, we get Lagrangian Grassmannian $L(n, 2n)(k)$. We agree to denote the generalized Grassmann variety $G(R; d, n)$ simply by $G(d, n)$ when $R = \{0\}$. In the last Chapter we fixed a ring $R \subset M_n(\mathbb{Z})$ and an integer, $0 < d < n$. We defined the functor f_R : (rings) $\to$ (sets) which sends ring T to

$$\{T\text{-submodules } K \subset T^n \text{ that are } R\text{-invariant rank } d \text{ direct summands of } T^n\},$$

and showed that f_R is represented by the generalized Grassmann scheme denoted by $F_R = G(R; d, n)$. We studied in detail the Zariski tangent space to the scheme F_R at a k-valued point W. In the case of the classical Grassmannian $G(d, n)(k)$, an affine neighbourhood of W is in natural bijection with $\mathrm{Hom}_k(W, k^n/W)$. However it is noticed that this is not true in the case of the scheme F_R. Instead we saw that the tangent space to W as a point on F_R is in bijection with $\mathrm{Hom}_{R\otimes k}(W, k^n/W)$. It still remains to understand the local structure of the scheme F_R more closely and compute the dimension of the scheme F_R. We analysed the scheme F_R when R is the ring of integers of a number field $L = \mathbb{Q}[\sqrt{D}]$ when D is a squarefree integer with $D \equiv 2, 3 \pmod 4$. We computed the Zeta function of F_R in this case as a product of some Dedekind Zeta functions and Riemann Zeta functions with some Euler factor.

It is an interesting problem to develop Schubert calculus in this setting and calculate the Zeta function in characteristic p using cohomology, base change and vanishing cycles.

Bibliography

[1] G. Berman and K. Fryer. *Introduction to Combinatorics*. Academic Press, 1972.

[2] R. Berndt. *An Introduction to Symplectic Geometry*, volume 26. Graduate Studies in Mathematics, American Mathematical Society, 1998.

[3] J. Duistermaat. Symplectic geometry. *http://www.math.uu.nl/people/duistermaat/sympl.pdf*, 2004.

[4] D. Dummit and R. Foote. *Abstract Algebra*. John Wiley and Sons, Inc., 1999.

[5] D. Eisenbud. *Commutative Algebra with a view toward Algebraic Geometry*, volume 150. Springer Verlag, 1995.

[6] D. Eisenbud and J. Harris. *The Geometry of Schemes*. Graduate Texts in Mathematics, Springer Verlag, 2000.

[7] P. Griffiths and J. Harris. *Principles of Algebraic Geometry*. Wiely Classics Library Edition, 1994.

[8] R. Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics, Springer Verlag, 1977.

[9] B. Hassett. Lecture notes in computational algebraic geometry. *http://www.math.rice.edu/ hassett/CAGbook/CAGtoc.html*, 2004.

[10] A. Hatcher. Algebraic topology. *http://www.math.cornell.edu/hatcher/* , 2001.

[11] W. Hodge and D. Pedoe. *Methods of Algebraic Geometry*, volume 2. Cambridge University Press, 1952.

[12] S. Kleiman and D.Laksov. Schubert Calculus. *The American Mathematical Monthly*, 79:1061–1082, 1972.

[13] L.Lesieur. Les problèmes d'intersections sur une variété de Grassmann. *C. R. Acad. Sc. Paris* 225, pages 916–917, 1947.

[14] D. McDuff and D. Salamon. *Introduction to Symplectic Topology*. Academic Press, 1972.

[15] J. S. Milne. Lecture notes on Étale cohomology. *http://www.jmilne.org/math/CourseNotes/math732.html*, 1998.

[16] P. Pragacz. *Algebro-geometric applications of Schur-S and Q- polynomials* . *Séminare d'Algèbre Dubreil-Malliavin*, pages 130–191, 1991.

[17] W. F. P. Pragacz. Schubert Varieties and Degeneracy Loci. *Lecture Notes in Mathematics 1689, Springer Verlag*, 1998.

[18] J.-P. Serre. *Corps Locaux*. Paris: Hermann, 1962.

[19] J.-P. Serre. Zeta and L - functions. *Proc. conf. Purdue Univ*, pages 249–259, 1963.

[20] R. Stanley. Some Combinatorial Aspects of the Schubert Calculus. *Lecture Notes in Mathematics, Combinatoire et Représentation du Groupe Symétrique, Strasbourg*, 579:217–251, 1976.

[21] M. Suzuki. *Group Theory I*. Springer Verlag, 1982.

[22] H. Tamvakis. Quantum Cohomology of Lagrangian and Orthogonal Grassmannians. *http://people.brandeis.edu/harryt/* , June 16, 2001.