THE INFLATION-RESTRICTION SEQUENCE : AN INTRODUCTION TO SPECTRAL SEQUENCES

TOM WESTON

1. Example

We begin with abelian groups $E_0^{p,q}$ for every $p, q \geq 0$ and maps

$$d_0^{p,q} : E_0^{p,q} \to E_0^{p,q+1}$$

(here, as in all of homological algebra, all maps are indexed by the group where they originate) making the columns into complexes; that is,

$$d_0^{p,q+1} d_0^{p,q} = 0$$

for all p and q. For convenience, if either p or q is negative we define $E_0^{p,q} = 0$, and we let $d_0^{p,q}$ be the zero map.

$$\begin{tikzcd}
{} & {} & {} & {} \\
E_0^{0,2} \arrow[u] & E_0^{1,2} \arrow[u] & E_0^{2,2} \arrow[u] & E_0^{3,2} \arrow[u] \\
E_0^{0,1} \arrow[u] & E_0^{1,1} \arrow[u] & E_0^{2,1} \arrow[u] & E_0^{3,1} \arrow[u] \\
E_0^{0,0} \arrow[u] & E_0^{1,0} \arrow[u] & E_0^{2,0} \arrow[u] & E_0^{3,0} \arrow[u]
\end{tikzcd}$$

Now, since the columns are complexes, we can take their cohomology. Specifically, for all p and q we define

$$E_1^{p,q} = \ker d_0^{p,q} / \operatorname{im} d_0^{p,q-1}.$$

(By our conventions before we automatically have $E_1^{p,q} = 0$ if either p or q is negative.) Now, suppose that we somehow magically obtain horizontal maps

$$d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$$

which make the rows into complexes; that is,

$$d_1^{p+1,q} d_1^{p,q} = 0$$

for all p and q.

$$\begin{tikzcd}
E_1^{0,2} \arrow[r] & E_1^{1,2} \arrow[r] & E_1^{2,2} \arrow[r] & E_1^{3,2} \arrow[r] & {} \\
E_1^{0,1} \arrow[r] & E_1^{1,1} \arrow[r] & E_1^{2,1} \arrow[r] & E_1^{3,1} \arrow[r] & {} \\
E_1^{0,0} \arrow[r] & E_1^{1,0} \arrow[r] & E_1^{2,0} \arrow[r] & E_1^{3,0} \arrow[r] & {}
\end{tikzcd}$$

Again, let us take the cohomology; that is, define

$$E_2^{p,q} = \ker d_1^{p,q} / \operatorname{im} d_1^{p-1,q}$$

for all p and q. Further, let us suppose that we somehow obtain maps

$$d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$$

satisfying $d_2^{p+2,q-1} d_2^{p,q} = 0$. Here the terminal point of each map has shifted one to the right and one down from the terminal point at the previous step.

$$\begin{tikzcd}
E_2^{0,2} \arrow[drr] & E_2^{1,2} \arrow[drr] & E_2^{2,2} \arrow[drr] & E_2^{3,2} \arrow[drr] & & \\
E_2^{0,1} \arrow[drr] & E_2^{1,1} \arrow[drr] & E_2^{2,1} \arrow[drr] & E_2^{3,1} \arrow[drr] & {} & {} \\
\boxed{E_2^{0,0}} & \boxed{E_2^{1,0}} & E_2^{2,0} & E_2^{3,0} & {} & {}
\end{tikzcd}$$

Note that the maps leaving the bottow row $E_2^{p,0}$ all must be 0, since they land in $E_2^{p+2,-1} = 0$. Similarly, the maps entering the first two columns $E_2^{0,q}$ and $E_2^{1,q}$ all originiate in 0 groups, and thus are 0. In particular, for both groups $E_2^{0,0}$ and $E_2^{1,0}$ *both* the maps entering and leaving are 0; we will return to this observation in a moment.

Now, continue our procedure another step. That is, define

$$E_3^{p,q} = \ker d_2^{p,q} / \operatorname{im} d_3^{p-2,q+1}.$$

Again, let us suppose that we somehow obtain maps

$$d_3^{p,q} : E_3^{p,q} \to E_3^{p+3,q-2},$$

shifted one to the right and one down of the previous maps, such that $d_3^{p+3,q-2} d_3^{p,q} = 0$.

$$\begin{tikzcd}
E_3^{0,2} \arrow[ddrrr] & E_3^{1,2} \arrow[ddrrr] & E_3^{2,2} \arrow[ddrrr] & E_3^{3,2} \arrow[ddrrr] & & & \\
\boxed{E_3^{0,1}} & \boxed{E_3^{1,1}} & \boxed{E_3^{2,1}} & E_3^{3,1} & & & \\
\boxed{E_3^{0,0}} & \boxed{E_3^{1,0}} & \boxed{E_3^{2,0}} & E_3^{3,0} & {} & {} & {}
\end{tikzcd}$$

Returning to our previous observation, we see that since both the maps $d_2^{0,0}$ and $d_2^{-2,1}$ entering and leaving $E_2^{0,0}$ are zero,

$$E_3^{0,0} = E_2^{0,0}.$$

Similarly,

$$E_3^{1,0} = E_2^{1,0}.$$

Thus these two entries have stabilized at the E_2 step. We will denote these (now and forever) constant values by $E_\infty^{0,0}$ and $E_\infty^{1,0}$ respectively.

We get more entries stabilizing now at the E_3 step. This time, all of the maps leaving the bottom two rows and entering the first three columns are the zero map. Thus all of the entries $E_3^{0,0}$, $E_3^{1,0}$, $E_3^{2,0}$, $E_3^{0,1}$, $E_3^{1,1}$ and $E_3^{1,2}$ have stabilized. As before, we denote these constant values by $E_\infty^{p,q}$ for the appropriate p and q; this is of course consistent with our earlier definitions of $E_\infty^{0,0}$ and $E_\infty^{1,0}$.

Roughly speaking, a spectral sequence is all of the data in the above construction. The general idea is that one starts with interesting groups $E_r^{p,q}$ at some early stage r and that the stable values $E_\infty^{p,q}$ are also related to other interesting groups. Thus the spectral sequence would somehow codify a relationship between these two families of groups. Specifically, the content of a spectral sequence is in three pieces of information : interesting interpretations of $E_r^{p,q}$ for some small r (very often 2); the construction of the new maps $d_r^{p,q}$ at each stage; and interesting interpretations of the stable values $E_\infty^{p,q}$. Of course, it is not clear how precisely the spectral sequence relates all of these groups; in general the relationship is so complicated as to be unusable. There are several important cases where precise information can be obtained, however; we will give two examples below.

2. Formal Definitions

Definition 1. A a^{th}*-stage (first quadrant cohomological) spectral sequence* is a collection of abelian groups $E_r^{p,q}$ for all $p, q \geq 0$ and for all $r \geq a$ for some positive integer a, together with maps

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

such that

$$d_r^{p+r,q-r+1} d_r^{p,q} = 0$$

and

$$E_{r+1}^{p,q} \cong \ker d_r^{p,q} / \operatorname{im} d_r^{p-r,q+r-1}$$

for all p, q and r as above.

As we have seen above, in any spectral sequence the (p, q) spot eventually stabilizes; we denote this stable value of $E_r^{p,q}$ by $E_\infty^{p,q}$. As can be seen easily by continuing the construction of the first section, this will be achieved at least by the $p + q + 2$ step, and often earlier.

The definition above makes no mention of these stable values, however. To relate these to the spectral sequence we need to introduce the notion of *convergence*.

Definition 2. An a^{th}-stage spectral sequence $E_r^{p,q}$ is said to *converge* to groups H^n, written

$$E_r^{p,q} \Rightarrow H^{p+q},$$

if there is a filtration

$$0 = H_{n+1}^n \subseteq H_n^n \subseteq H_{n-1}^n \subseteq \cdots \subseteq H_2^n \subseteq H_1^n \subseteq H_0^n = H^n$$

such that

$$E_\infty^{p,n-p} \cong H_p^n / H_{p+1}^n$$

for all p.

$$0 \underbrace{\hookrightarrow}_{E_\infty^{n,0}} H_n^n \hookrightarrow \cdots \hookrightarrow H_{p+1}^n \underbrace{\hookrightarrow}_{E_\infty^{p,n-p}} H_p^n \hookrightarrow \cdots \hookrightarrow H_1^n \underbrace{\hookrightarrow}_{E_\infty^{0,n}} H^n$$

Thus the stable values $E_\infty^{p,q}$ on the line $p+q=n$ are the succesive quotients in a filtration of H^n, so that knowledge of them gives a lot of information about H^n. The simplest such information are the *edge maps*

$$E_a^{n,0} \twoheadrightarrow E_\infty^{n,0} \cong H_n^n \hookrightarrow H^n$$

and

$$H^n \twoheadrightarrow H^n/H_1^n \cong E_\infty^{0,n} \hookrightarrow E_a^{0,n}.$$

($E_\infty^{n,0}$ is a quotient of $E_a^{n,0}$ since all maps leaving the $(n,0)$-entry are 0, so that at each stage the kernel is the whole entry. $E_\infty^{0,n}$ is a subgroup of $E_a^{0,n}$ for similar reasons.) There is even a very important special case where knowledge of the $E_\infty^{p,q}$ actually gives complete information about H^n.

Definition 3. An a^{th}-stage spectral sequence $E_r^{p,q}$ is said to *collapse* at the b^{th}-stage if there is only one non-zero row or column in $E_b^{p,q}$.

Let

$$E_a^{p,q} \Rightarrow H^{p+q}$$

be a convergent spectral sequence collapsing at the b^{th}-stage, and assume that $b \geq 2$ To fix ideas let us suppose that only the q_0 row is non-zero.

$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ \\ E_b^{0,q_0} & E_b^{1,q_0} & E_b^{2,q_0} & E_b^{3,q_0} \\ \\ 0 & 0 & 0 & 0 \end{array}$$

Two important things now happen. First, every $d_b^{p,q}$ map must be the zero-map. Thus we must have

$$E_{b+1}^{p,q} = E_b^{p,q}$$

for all p and q, and continuing in this way we see that

$$E_\infty^{p,q} = E_b^{p,q}$$

for all p and q. Second, the $p+q=n$ diagonal only has the one non-zero term $E_b^{n-q_0,q_0}$, so our above filtration yields

$$H^n \cong E_b^{n-q_0,q_0}.$$

Thus for a collapsing spectral sequence there is no ambiguity about the H^n.

3. The Hochschild-Serre Spectral Sequence

It is well past time for an example. We will give the most important general family of examples. For details see [1, Chapter 5]; the construction is fairly involved.

So, suppose that we have two abelian categories $\mathcal{A}$ and $\mathcal{B}$. Further suppose that we have functors $G : \mathcal{A} \to \mathcal{B}$ and $F : \mathcal{B} \to \mathbf{Ab}$, where $\mathbf{Ab}$ is the category of abelian groups. (There isn't really any need to require that G takes values in $\mathbf{Ab}$; it could just as well be any abelian category. One merely has to replace the words "abelian group" with "object in an abelian category" throughout the previous two sections.)

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "G"] \arrow[dr, "FG"'] & & \mathcal{B} \arrow[dl, "F"] \\ & \mathbf{Ab} & \end{tikzcd}$$

We suppose further that both $\mathcal{A}$ and $\mathcal{B}$ have enough injectives, and that F and G are *left exact*, so that they have right derived functions R^rF and R^rG for all $r \geq 0$. The *Grothendieck spectral sequence* compares the composition of the derived functors with the derived functors of the composition.

Theorem 3.1 (Grothendieck Spectral Sequence)**.** *Given the above setup, suppose further that for any injective object I of $\mathcal{A}$, $R^rF(G(I)) = 0$ for all $r > 0$. Then for any object A of $\mathcal{A}$, there exists a second stage (first-quadrant cohomological) spectral sequence*

$$E_2^{p,q} = (R^pF)(R^qG)(A) \Rightarrow R^{p+q}(FG)(A).$$

The edge maps

$$(R^pF)(GA) \to R^p(FG)(A)$$

and

$$R^q(FG)(A) \to F(R^qG(A))$$

are the natural maps.

As a particular example, we have the Hochschild-Serre spectral sequence for the cohomology of groups.

Theorem 3.2 (Hochschild-Serre Spectral Sequence)**.** *Let G be a group, H a normal subgroup and A a G-module. Then there is a second stage (first-quadrant, cohomological) spectral sequence*

$$E_2^{p,q} = H^p(G/H; H^q(H; A)) \Rightarrow H^{p+q}(G; A).$$

The edge maps

$$H^n(G/H; A^H) \to H^n(G; A)$$

and

$$H^n(G; A) \to H^n(H; A)^{G/H}$$

are induced from inflation and restriction respectively.

Proof. This is a special case of the Grothendieck spectral sequence; for details, see [1, Chapter 6, Section 8]. ∎

$$\begin{array}{ccc}
H^0(G/H; H^2(H;M)) & H^1(G/H; H^2(H;M)) & H^2(G/H; H^2(H;M)) \\
\\
H^0(G/H; H^1(H;M)) & H^1(G/H; H^1(H;M)) & H^2(G/H; H^1(H;M)) \\
\\
H^0(G/H; H^0(H;M)) & H^1(G/H; H^0(H;M)) & H^2(G/H; H^0(H;M))
\end{array}$$

4. The Inflation-Restriction Sequence

Even when a spectral sequence does not collapse, one can often still obtain information about some of the low-degree terms. In this section we will construct an exact sequence of fundamental importance in group cohomology.

Let

$$E_2^{p,q} \Rightarrow H^{p+q}$$

be a convergent second stage spectral sequence.

$$\begin{tikzcd}
E_2^{0,2} \arrow[drr] & E_2^{1,2} \arrow[drr] & E_2^{2,2} \arrow[drr] & & \\
E_2^{0,1} \arrow[drr] & E_2^{1,1} \arrow[drr] & E_2^{2,1} \arrow[drr] & & \\
E_2^{0,0} & E_2^{1,0} & E_2^{2,0} & &
\end{tikzcd}$$

Let us follow through the computation of $E_3^{p,q}$ a bit. Straight from the definitions, we get the following picture for the third stage, in which the boxed entries have already stabilized to their final values.

$$\begin{array}{ccc}
\ker E_2^{0,2} \to E_2^{2,1} & \ker E_2^{1,2} \to E_2^{3,1} & \text{mess} \\
\\
\boxed{\ker E_2^{0,1} \to E_2^{2,0}} & \boxed{\ker E_2^{1,1} \to E_2^{3,0}} & \boxed{\text{mess}} \\
\\
\boxed{E_2^{0,0}} & \boxed{E_2^{1,0}} & \boxed{E_2^{2,0} / \operatorname{im} E_2^{0,1}}
\end{array}$$

Now, let us combine this with our knowledge of convergence. First, directly from the filtration of H^1, we get an exact sequence

$$0 \to E_2^{1,0} \to H^1 \to (\ker E_2^{0,1} \to E_2^{2,0}) \to 0.$$

Next, we know that we have an edge map

$$0 \to E_2^{2,0} / \operatorname{im} E_2^{0,1} \to H^2.$$

Thus these two exact sequences splice together to give a five term exact sequence

$$0 \to E_2^{1,0} \to H^1 \to E_2^{0,1} \to E_2^{2,0} \to H^2.$$

Further, every map except for $E_2^{0,1} \to E_2^{2,0}$ is just an edge map. We state the special case of this for the Hochschild-Serre spectral sequence as a theorem.

Theorem 4.1 (Inflation-Restriction)**.** *If H is a normal subgroup of G and M is a G-module, then there is an exact sequence*

$$\begin{aligned} 0 \longrightarrow H^1(G/H; M^H) &\xrightarrow{\text{inf}} H^1(G; M) \xrightarrow{\text{res}} H^1(H; M)^{G/H} \\ \longrightarrow H^2(G/H; M^H) &\xrightarrow{\text{inf}} H^2(G; M) \end{aligned}$$

where inf *and* res *are the inflation and restriction maps.*

Now, suppose further that the $q = 1$ row of our spectral sequence vanishes. That is, suppose that $E_2^{p,1} = 0$ for all p. Since every $E_r^{p,1}$ is a subquotient of $E_2^{p,1}$, it follows that this row is zero for all r. (In general, any entry in the spectral sequence which is ever zero is always and forever zero.) In particular, our above exact sequence now becomes an isomorphism

$$E_2^{1,0} \xrightarrow{\cong} H^1$$

and an injection

$$E_2^{2,0} \hookrightarrow H^2.$$

In this case we can actually extend this second exact sequence farther to the right.

So, we begin with a second stage spectral sequence with zero $q = 1$ row.

$$\begin{tikzcd}
E_2^{0,2} \arrow[drr] & E_2^{1,2} \arrow[drr] & E_2^{2,2} \arrow[drr] & E_2^{3,2} \arrow[drr] & & \\
0 \arrow[drr] & 0 \arrow[drr] & 0 \arrow[drr] & 0 \arrow[drr] & {} & {} \\
E_2^{0,0} & E_2^{1,0} & E_2^{2,0} & E_2^{3,0} & {} & {}
\end{tikzcd}$$

We again follow along to the third stage. This time we get

$$\begin{tikzcd}
E_2^{0,2} \arrow[ddrrr] & E_2^{1,2} \arrow[ddrrr] & \text{mess} & \text{mess} & \\
0 & 0 & 0 & 0 & \\
E_2^{0,0} & E_2^{1,0} & E_2^{2,0} & E_2^{3,0} & {}
\end{tikzcd}$$

Now, let us go to the fourth stage. This time we get

$$\begin{array}{cccc}
\ker E_2^{0,2} \to E_2^{3,0} & \ker E_2^{1,2} \to E_2^{3,1} & \text{mess} & \text{mess} \\
 & & & \\
0 & 0 & 0 & 0 \\
 & & & \\
E_2^{0,0} & E_2^{1,0} & E_2^{2,0} & E_2^{3,0}/\operatorname{im} E_2^{0,2}
\end{array}$$

where all of the entires pictured above have stabilized to their ∞ values. As before, the filtration of H^2 is reflected by the exact sequence

$$0 \to E_2^{2,0} \to H^2 \to (\ker E_2^{0,2} \to E_2^{3,0}) \to 0.$$

We also again get an edge map injection

$$0 \to E_2^{3,0}/\operatorname{im} E_2^{0,2} \to H^3.$$

These splice together to give the five term exact sequence

$$0 \to E_2^{2,0} \to H^2 \to E_2^{0,2} \to E_2^{3,0} \to H^3 \to 0,$$

in which all but one of the maps is simply an edge map.

It is easy to generalize the above construction to the case where each of the rows $q = 1$ up to $q = q_0 - 1$ vanishes. We state the group cohomology case as a theorem.

Theorem 4.2 (Higher Inflation-Restriction)**.** *Let H be a normal subgroup of G and M a G-module. Suppose that $H^q(H;M) = 0$ for $1 \le q < q_0$. Then for $1 \le q < q_0$, inflation induces isomorphisms*

$$H^q(G/H; M^H) \xrightarrow{\cong} H^q(G;M)$$

and there is an exact sequence

$$\begin{aligned}
0 \longrightarrow H^{q_0}(G/H; M^H) &\xrightarrow{\text{inf}} H^{q_0}(G;M) \xrightarrow{\text{res}} H^{q_0}(H;M)^{G/H} \\
\longrightarrow H^{q_0+1}(G/H; M^H) &\xrightarrow{\text{inf}} H^{q_0+1}(G;M)
\end{aligned}$$

where inf *and* res *are the inflation and restriction maps.*

References

[1] Charles Weibel, *An introduction to homological algebra.* Cambridge; Cambridge University Press, 1994.