ETALE COHOMOLOGY - PART 2

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Draft Version as of March 15, 2004

Contents

1. Grothendieck Topologies

We recall the definition of a Grothendieck topology (or site) from [AG03]:

Definition 1.1. *A* site *is a category T with a notion of covering, that is, a collection of sets $\{U_i \to U\}$ in T (called coverings) such that*

1. *If $\{U_i \to U\}$ is a covering and $V \to U \in T$, then the fibred product $U_i \times_U V$ exists for all i, and $\{U_i \times_U V \to V\}$ is a covering.*
2. *If $\{U_i \to U$ for $i \in A\}$ is a covering, and for each i we have a covering $\{V_{ij} \to U_i$ for $j \in B_i\}$, then we have a covering*
$$\{V_{ij} \to U \text{ for } i \in A \text{ and for } j \in B_i\}.$$
3. *If $U' \to U$ is an isomorphism, then $\{U' \to U\}$ is a covering.*

Remark 1.2. We will often call the category by the same name as the site, even though the extra data of the coverings is an essential part of the definition of a site.

Definition 1.3. *A* morphism of sites $f\colon T \to T'$ *(sometimes called a "continuous function" in the literature) is a functor* $f^{-1}\colon T' \to T$ *such that*

(1) *if* $\{U_i' \to U'\}$ *is a covering in* T', *then* $\left\{f^{-1}(U_i' \to U')\right\}$ *is a covering in* T, *and*
(2) *if* $\{W_i' \to U'\}$ *is a covering in* T' *and* $V' \to U'$ *is in* T', *then*
$$f^{-1}(V' \times_{U'} W_i') \cong f^{-1}(V') \times_{f^{-1}(U')} f^{-1}(W_i')$$
for every i.

Example 1.4. If E is a class of morphisms of schemes such that

(1) all isomorphisms are in E,
(2) composites of morphisms in E are again in E, and
(3) the base change of a morphism in E by any morphism is again in E,

then the *small* E *site over* X, denoted E/X, is the full subcategory of X-schemes with structure morphism in E, and where $\left\{U_i \xrightarrow{g_i} U\right\}$ is a covering if and only if each g_i is in E and $\coprod U_i \xrightarrow{g_i} U$ is universally surjective. Note that we exclude families of morphisms in which the coverings are not sets.

Important examples of this are

(1) the *Zariski site* Zar/X or X_{Zar}, where we take E to be open immersions, and
(2) the *small étale site* $\text{ét}/X$ or $X_{\text{ét}}$, where we take E to be the class of étale morphisms of finite type.

Observe that when the morphisms in E are open maps as in these two cases, a collection $\coprod U_i \xrightarrow{g_i} U$ is universally surjective exactly when $U = \bigcup g_i(U_i)$, and "universally surjective" simply reduces to the more obvious "surjective".

Example 1.5. Morphisms of sites:

(1) There is a morphism of sites $\text{ét}/X \to \mathrm{Zar}/X$ given by the inclusion map: an open immersion is an étale map.
(2) If $X \to Y$ is any morphism, we get a map $\text{ét}/X \to \text{ét}/Y$ by base change:
$$\begin{array}{ccc} \text{ét}/X & \to & \text{ét}/Y \\ X \times_Y U' & \leftarrow\!\shortmid & U' \end{array}$$
(3) If $X \to Y$ is an étale morphism, we get a map $\text{ét}/Y \to \text{ét}/X$ by composition:
$$\begin{array}{ccc} \text{ét}/Y & \to & \text{ét}/X \\ Z \to X \to Y & \leftarrow\!\shortmid & Z \to X \end{array}$$

Example 1.6. The *big* E *site* is the full subcategory of Sch/X of schemes of locally finite type over X and in which $\left\{U_i \xrightarrow{g_i} U\right\}$ is a covering exactly when $U = \bigcup g_i(U_i)$ and each g_i is in E. The flat site X_{Fl} is the big site for E the class of flat morphisms locally of finite type.

Example 1.7. Let G be a (profinite) group, and let T_G be the category of (continuous) G-sets. Coverings are the obvious choice, namely $\left\{U_i \xrightarrow{g_i} U\right\}$ is a covering if $U = \bigcup g_i(U_i)$.

2. The Category of Sheaves on a Site

Definition 2.1. *A* presheaf *on a site T is a contravariant functor $F\colon T \to \boldsymbol{Ab}$.*

The category of presheaves P_T forms an abelian category in which $F' \to F \to F''$ is exact if and only if $F'(U) \to F(U) \to F''(U)$ is for all $U \in T$.

Proposition 2.2. *The category P_T is an abelian category satisfying AB5 and AB4*.*

Recall that AB3 is the axiom that asserts that any family has a direct sum. The axiom AB4 asserts AB3 and that a direct sum of exact sequences is exact. The axiom AB5 asserts AB3 and that a filtered direct limit of exact sequences is exact. Finally, for any n, the axiom ABn^* asserts that axiom ABn holds in the dual category. In other words, one replaces the word "sum" with "product" and "direct" with "inverse", so for example AB3* asserts that any family has a product.

Proof. By definition $P_T = \mathrm{Hom}(T, \mathbf{Ab})$, and $\mathbf{Ab}$ satisfies AB5 and AB4*, so P_T does. ☐

Example 2.3. Some simple presheaves:

(1) If M is an abelian group, set $F_M(U) = \begin{cases} M & \text{if } U \neq \emptyset \\ 0 & \text{if } U = \emptyset \end{cases}$ on E/X. This is the *constant presheaf* associated to M on E/X.
(2) If $\mathcal{G}$ is a commutative group scheme, set $\mathcal{G}(U) = \mathrm{Hom}(U, \mathcal{G})$ on E/X.

Definition 2.4. *A presheaf $\mathcal{F}$ is a* sheaf *if it satisfies the additional criterion*

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$$

is exact[1] *for all coverings $\{U_i \to U\}$.*

Remark 2.5. On ét$/X$ and X_{Fl} the sheaf condition is equivalent to: the sheaf condition is satisfied for all Zariski covers $\{U_i \to U\}$ and all singleton covers $\{U' \to U\}$.

Example 2.6. Sheaves on various sites:

(1) If G is a finite group, sheaves on T_G correspond to left G-modules by
$$\begin{aligned} \mathcal{F} &\mapsto \mathcal{F}(G) \\ \mathrm{Hom}_G(-, A) &\leftarrow\!\shortmid A \end{aligned}$$
(2) If G is a profinite group, sheaves on T_G correspond to continuous left G-modules by
$$\begin{aligned} \mathcal{F} &\mapsto \varinjlim_H \mathcal{F}(G/H) \\ \mathrm{Hom}_{G,\mathrm{cont}}(-, A) &\leftarrow\!\shortmid A \end{aligned}$$

[1] Some care should be taken when interpreting the sheaf condition, as "exactness" means something unusual in this context. Suppose that we have a cover $\left\{U_i \xrightarrow{r_i} U\right\}$ of U and a collection of sections s_i such that on $U_i \times_U U_j$ we have $\mathcal{F}(p_1)(s_i) = \mathcal{F}(p_2)(s_j)$. Then the sheaf axiom tells us that there exists a unique $s \in \mathcal{F}(U)$ such that $\mathcal{F}(r_i)(s) = s_i$. In particular, this diagram should in a sense be exact not only in the middle but also on the left.

(3) If K is a field, and $G = \mathrm{Gal}(K^{\mathrm{sep}}/K)$, then we have an equivalence of categories

$$\begin{aligned} T_G &\to \text{ét}/(\operatorname{Spec} K) \\ X(K^{\mathrm{sep}}) &\leftarrow\!\shortmid X \end{aligned}$$

So an étale sheaf on $\operatorname{Spec} K$ corresponds to a sheaf on T_G, which corresponds to a continuous G-module. If $\mathcal{F}$ is an étale sheaf on $\operatorname{Spec} K$, the corresponding G-module is

$$\varinjlim_{H} \mathcal{F}'(G/H) = \varinjlim_{K'/K \text{ finite}} \mathcal{F}(\operatorname{Spec} K').$$

(4) If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules (that is, a sheaf in the usual algebraic geometry sense, which is just a sheaf on Zar/X), we can set $\mathcal{F}_E(U) = (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_U)(U)$, giving a presheaf on E/X or X_E. If $\mathcal{F}$ is quasi-coherent and the site is ét$/X$ or X_{Fl}, then $\mathcal{F}$ is a sheaf by Remark 2.5 combined with the fact that if $f\colon A \to B$ is faithfully flat and M is an A-module, then the complex

$$0 \to M \to M \otimes_A B \to M \otimes_A B \otimes_A B \to \cdots$$

is exact. For example, if $\mathcal{F} = \mathcal{O}_X$, we see that the presheaf $\mathbb{G}_a$ with $\mathbb{G}_a(U) = \mathcal{O}_U(U)$ (obtained from the group scheme $\mathbb{G}_a$) is a sheaf.

(5) On ét$/X$, if X is an S-scheme, then $\left(\Omega^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{O}_U\right)(U) = \Omega^1_{U/S}(U)$, so $U \mapsto \Omega^1_{U/S}(U)$ is an étale sheaf.

(6) If $\mathcal{G}$ is a group scheme on X, the presheaf associated to $\mathcal{G}$ on ét$/X$, Zar/X or Fl/X is a sheaf. This follows from Remark 2.5 and fpqc descent (see [Alo03]). For example, taking the group scheme $\mathbb{G}_m$ we obtain a sheaf for which $\mathbb{G}_m(U) = \mathcal{O}_U(U)^*$. We also denote this sheaf $\mathcal{O}_X^*$.

Remark 2.7. On the fppf site (the E-site for E the class of faithfully flat of finite presentation) sheaves are representable by schemes. That is, given any sheaf $\mathcal{F}$ on the fppf site, there is some abelian group scheme $\mathcal{G}$ such that $\mathcal{F}$ is the sheaf associated to $\mathcal{G}$.

Let P_T, S_T be the categories of presheaves and sheaves on T respectively. There is clearly an inclusion $\iota\colon S_T \hookrightarrow P_T$.

Theorem 2.8. *The functor ι has a left adjoint $\sharp\colon P_T \to S_T$ (that is, $\mathrm{Hom}_{S_T}(G^\sharp, \mathcal{F}) \cong \mathrm{Hom}_{P_T}(G, \iota\mathcal{F})$), which we call the "sheafification" functor.*

Sketch of proof. If G is a presheaf, set

$$G^\dagger(U) = \varinjlim_{\{U_i \to U\}} \ker\left(\prod_{U_i} G(U_i) \rightrightarrows \prod_{U_i, U_j} G(U_i \times U_j)\right),$$

the limit taken over the category of all covers with refinement maps (a refinement map $\{U_j \to U\} \to \{U_i \to U\}$ is in fact a family of morphisms $U_j \to U_{f(j)}$ making the appropriate diagrams commute).

Then $G^\dagger$ is a presheaf with the following properties:

(1) $G^\dagger$ is separated (that is, if $\{U_i \to U\}$ is a covering and $s \in G^\dagger(U)$ has $s|_{U_i} = 0$ for every U_i then $s = 0$).
(2) If $G^\dagger$ is separated, the $G^\dagger$ is a sheaf.

So set $G^\sharp = (G^\dagger)^\dagger$. □

Corollary 2.9. *We have*

(1) ι *is left-exact and preserves inverse limits,*
(2) $\sharp$ *is* exact *and preserves direct limits,*
(3) ι *sends injectives to injectives, and*
(4) *if* $\mathcal{F}$ *is a sheaf,* $(\iota\mathcal{F})^\sharp = \mathcal{F}$.

Proof. Statement (1) and the right exactness of (2) are automatic from the existence of a left adjoint. If we can prove that $\dagger$ is left-exact, then the left-exactness of $\sharp$ will follow. The functors $H^0_{\{U_i \to U\}}$, defined to be $G \mapsto \ker\left(\prod G(U_i) \rightrightarrows G(U_i \times U_j)\right)$ are left exact (kernels are left exact). *Filtered* limits of abelian groups are always exact, but this fact does not apply directly here, as two refinements may not have a common refinement (if $\left\{U'_j \to U\right\}$ is a refinement of $\{U_i \to U\}$, one refinement map may send U'_j to U_i and another may send U'_j to $U_{i'}$). Instead, one proves that if $\left\{U'_j \to U\right\}$ is a refinement of $\{U_i \to U\}$ then the induced map $H^0_{U_i \to U} \to H^0_{U'j \to U}$ is independent of the refinement map one uses. This is a result of the functoriality of H^0, which is a result of the functoriality of G. Thus the limit factors through the ordered set of all covers, and is exact.

Statement (3) is automatic from statement (2) (a general fact about right adjoints to an exact functor). Finally, (4) is clear from the construction. □

In turn, this implies the following:

Theorem 2.10. S_T *is an abelian category satisfying AB5 and AB3*. Moreover:*

(1) *The sequence* $0 \to F' \to F \to F''$ *is exact in* S_T *if and only if it is exact in* P_T*, which is true if and only if* $0 \to F'(U) \to F(U) \to F''(U)$ *is exact for every* U.
(2) $F \xrightarrow{\phi} F'' \to 0$ *is exact in* S_T *if and only if for every* $s \in F''(U)$ *there exists a covering* $\{U_i \to U\}$ *and* $s_i \in F(U_i)$ *such that* $\phi(s_i) = s|_{U_i}$.
(3) *We can form inverse limits in* S_T *(for example kernels and products) by forming limits in* P_T*; the result is a sheaf.*
(4) *We can form direct limits in* S_T *(for example cokernels and sums) by forming limits in* P_T *and sheafifying.*

Proof. The proof is formal from Proposition 2.2 and Corollary 2.9. For example, if $F \to G$, we show $\operatorname{coker}(F \to G) = (\operatorname{coker}(\iota F \to \iota G))^\sharp$. Indeed,

$$\iota F \to \iota G \to C \to 0$$

implies

$$0 \to \operatorname{Hom}(C, \iota H) \to \operatorname{Hom}(\iota G, \iota H) \to \operatorname{Hom}(\iota F, \iota H) \quad \text{for all } H,$$

which is true if and only if

$$0 \to \operatorname{Hom}(C^\sharp, H) \to \operatorname{Hom}((\iota G)^\sharp, H) \to \operatorname{Hom}((\iota F)^\sharp, H) \quad \text{for all } H.$$

Since $(\iota G)^\sharp = G$ and $(\iota F)^\sharp = F$, the result follows. □

Theorem 2.11. *The categories* P_T *and* S_T *have enough injectives.*

Remark 2.12. This is not, strictly speaking, true in this generality, because of foundational (set-theoretic) reasons. If the isomorphism classes of objects of T do not form a set — for example, if T is the crystalline site — one must fix a suitable universe $\mathscr{U}$ so that T is a $\mathscr{U}$-site, and restrict to the category of $\mathscr{U}$-valued sheaves. See the appendix (section A) for an overview or [SGA72a, the appendix to Exposé 1] for details.

Proof. Recall that an abelian category is said to "have generators" if there is a *set* A of objects such that for each monomorphism $G \xhookrightarrow{i} F$ that is not an isomorphism, there exists $F' \in A$ and $F' \xrightarrow{\phi} F$ such that ϕ does not factor through i. For example, the set $\{\mathbb{Z}\}$ is a set of generators for $\mathbf{Ab}$, since if $H \hookrightarrow G$ is not an isomorphism, we can construct a map taking 1 to some element x not in the image of H. A criterion of Grothendieck ([Gro57]) guarantees that an abelian category satisfying AB5, AB3*, and having generators has enough injectives.

One can prove, generally, that if C and C' are abelian categories and C' satisfies AB3 and has generators, then $\mathrm{Hom}(C, C')$ has both these properties provided C is small. Applying this result with $C = T^{\mathrm{op}}$ and $C' = \mathbf{Ab}$, we find that P_T has generators when T is small. Sheafifying gives a set of generators in S_T.

When T is not small, one gives a similar proof using $\mathscr{U}$-versions of these results; see [SGA72b, 6.7] and [SGA72b, 6.9]. $\square$

We can be more explicit when T is small. If for each $U \in T$ there is a sheaf $\mathbb{Z}_U$ so that $\mathrm{Hom}(\mathbb{Z}_U, \mathcal{F}) = \mathcal{F}(U)$, then the collection of $\mathbb{Z}_U$ are generators: indeed if $\mathcal{G} \xhookrightarrow{i} \mathcal{F}$ is not an isomorphism, choose U so that $\mathcal{G}(U) \subsetneq \mathcal{F}(U)$ and $\sigma \in \mathcal{F}(U) - \mathcal{G}(U)$. Then the map $\phi\colon \mathbb{Z}_U \to \mathcal{F}$ corresponding to σ does not factor through i. We shall see shortly how to construct such sheaves $\mathbb{Z}_U$.

3. Operations on presheaves and sheaves

Suppose $f\colon T \to T'$ is a morphism of sites.

If F is a presheaf on T, define $f_p F(U) = F(f^{-1}(U))$ on T'. Clearly f_p is exact and takes sheaves to sheaves.

Theorem 3.1. *If T, T' are small, then f_p has a left adjoint $f^p\colon P_{T'} \to P_T$ (so $\mathrm{Hom}_A(f^p G, F) = \mathrm{Hom}_{A'}(G, f_p F)$). If T' has a final object and finite inverse limits then f^p is exact (and not just right exact). This occurs in, for example, ét$/X$ and Zar$/X$.*

Proof. The existence of f^p is formal, by the following result from [HS97] (or [Mil80, Ch. I 2.2]).

Theorem 3.2. *If C, C' are small categories, $p\colon C \to C'$ a functor, A a category with direct limits, then the functor $\mathrm{Hom}(C', A) \to \mathrm{Hom}(C, A)$ induced by p has a left adjoint.*

Apply this with $C = T$, $C' = T'$, $A = \mathbf{Ab}$. $\square$

Explicitly, we set
$$(f^p G)(U) = \varinjlim_{V} G(f^{-1}(V)),$$
the limit taken over diagrams
$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & f^{-1}(V) \arrow[dl] \\ & X & \end{tikzcd} .$$

When T has a final object and finite inverse limits, the dual category to the category of these diagrams is pseudofiltered[2], and so the limit defining f^p is exact.

Remark 3.3. When f^p is exact, f_p maps injectives to injectives.

Remark 3.4. If T, T' are not small, my impression is that one often considers only morphisms such that f^p exists? (Or is there a $\mathscr{U}$-version here?)

Remark 3.5. If T is small, define $Z_U(V) = \oplus_{\mathrm{Hom}(V,U)}\mathbb{Z}$. Then

$$\mathrm{Hom}_{P_T}(Z_U, F) = \mathrm{Hom}(\mathbb{Z}, F(U) = F(U),$$

so we may take $\mathbb{Z}_U = Z_U^{\sharp}$ as promised earlier.

If $T = E/X$, we may construct $\mathbb{Z}_U$ as follows. Construct the morphism of sites $j\colon E/U \to E/X$ induced by $j\colon U \to X$. Then $j^p\colon P_X \to P_U$ is simply the map $\mathrm{Hom}(E/X^{\mathrm{op}}, \mathbf{Ab}) \to \mathrm{Hom}(E/U^{\mathrm{op}}, \mathbf{Ab})$ induced by j. By Theorem 3.2, j^p has a left adjoint $j_!$, and we set $\mathbb{Z}_U = j_!\mathbb{Z}$, where $\mathbb{Z}$ denotes the constant sheaf on U. Then

$$\mathrm{Hom}_{P_X}(j_!\mathbb{Z}, F) = \mathrm{Hom}_{P_U}(\mathbb{Z}, j^pF) = F(U).$$

Next, we set

$$\begin{aligned} f_s &= \sharp \circ f_p \circ i\colon S_T \to S_{T'} \\ f^s &= \sharp \circ f^p \circ i\colon S_T \to S_{T'}. \end{aligned}$$

Since $f_p \circ i$ is already a sheaf, f_s is just the restriction of f_p from P_T to S_T,

We immediately have:

Theorem 3.6. *The functor f^s is left adjoint to f_s. In particular, f_s is left exact and f^s is right exact. If f^p is actually exact (for example, if T has a final object and finite inverse limits) then so is f^s; if f^s is exact, then f_s maps injectives to injectives.*

Proof. Everything follows from the adjointness properties for f_p and f^P and for i and $\sharp$. For example, if f^p is exact, then since i is left exact, f^s is left exact as well. $\square$

Example 3.7. Let $U \xrightarrow{j} X$ be an E-morphism and $j\colon E/U \to E/X$ the corresponding map of small E-sites. If $\mathcal{F} \in S_{E/X}$ then $j^s(\mathcal{F}) = \mathcal{F}|_{E/U}$. Indeed, as we have seen, $j^p(i\mathcal{F}) = (i\mathcal{F})|_{E/U}$. But $(i\mathcal{F})|_{E/U}$ is already a sheaf.

4. Stalks of étale sheaves; the mapping cylinder

4.1. **Stalks.** If F is a presheaf on ét$/X$ and $P = \operatorname{Spec}\Omega \xrightarrow{u} X$ is a geometric point, set $F_P = (u^pF)(P)$.

[2]A category is pseudofiltered if every diagram of the form $\begin{tikzcd}[row sep=tiny] & j \\ i \arrow[ur] \arrow[dr] & \\ & j' \end{tikzcd}$ can be completed to a diagram of the form $\begin{tikzcd}[row sep=tiny] & j \arrow[dr] & \\ i \arrow[ur] \arrow[dr] & & k \\ & j' \arrow[ur] & \end{tikzcd}$, and if every diagram of the form $i \underset{v}{\overset{u}{\rightrightarrows}} j$ can be completed to a diagram $i \underset{v}{\overset{u}{\rightrightarrows}} j \xrightarrow{w} k$ such that $wu = wv$. The category is connected if for any two objects i and j there is a (finite) sequence $i \to j_1 \leftarrow i_1 \to \cdots \leftarrow j$ of morphisms. A category is filtered if it is pseudofiltered and connected.

Remark 4.1. If F is in fact a sheaf, then $F_P = (u^s F)(P)$ as well. This follows from: if G is a presheaf on $\text{ét}/\operatorname{Spec}\Omega$, for Ω separably closed, then $G(\operatorname{Spec}\Omega) = G^\sharp(\operatorname{Spec}\Omega)$. This last fact follows from: for any cover $\{U_i \to \operatorname{Spec}\Omega\}$, there is a refinement $\{\operatorname{Spec}\Omega \to \operatorname{Spec}\Omega\} \to \{U_i \to \operatorname{Spec}\Omega\}$ since Ω is a separably closed field, and its étale covers are therefore simply disjoint unions of copies of $\operatorname{Spec}\Omega$.

If $x \in X$, let $\overline{x}$ be $\operatorname{Spec} k(x)^{\text{sep}} \to X$.

Theorem 4.2. *Exactness properties of maps of étale sheaves may be detected on stalks:*

1. *The functor $\mathcal{F} \to \mathcal{F}_P$ is exact on $S_{\text{ét}/X}$.*
2. *A section $s \in \mathcal{F}$ of an étale sheaf is 0 if and only if $s_{\overline{x}} = 0$ for all $x \in U$.*
3. *A map $\mathcal{F} \to \mathcal{F}'$ of étale sheaves is an isomorphism (respectively, injection, surjection, zero) if and only if, for all $x \in X$, $\mathcal{F}_{\overline{x}} \to \mathcal{F}'_{\overline{x}}$ is.*
4. *A sequence $\mathcal{F}'' \to \mathcal{F} \to \mathcal{F}'$ is exact if and only if for all $x \in X$, $\mathcal{F}''_{\overline{x}} \to \mathcal{F}_{\bar{x}} \to \mathcal{F}'_{\overline{x}}$ is.*

Proof. (1) $\mathcal{F} \to \mathcal{F}_P$ is the composite of u^s and

$$\begin{aligned} \Gamma \colon S_{\text{ét}/\operatorname{Spec}\Omega} &\to \mathbf{Ab} \\ \mathcal{G} &\mapsto \mathcal{G}(\operatorname{Spec}\Omega), \end{aligned}$$

with Ω separably closed (see Example 2.6 part 3), so both functors are exact.

(2) If $s_{\overline{x}} = 0$, take an étale neighborhood

$$\begin{tikzcd} & \overline{x} \arrow[dl] \arrow[dr] & \\ U_x \arrow[rr] & & X \end{tikzcd}$$

of $\overline{x}$ such that $s|_{U_{\overline{x}}} = 0$. Then $\{U_x \to X\}$ is a cover of x, and by the sheaf condition $s = 0$.

(3) and (4) follow with little difficulty from (2) and Theorem 2.10. $\square$

The following result is useful for computing the stalks of étale sheaves.

Proposition 4.3. *If G is a presheaf on ét/X, then*

$$(G^\sharp)_P = \varinjlim_{X'} G(X'),$$

the limit taken over étale neighborhoods X' of P.

We first remark:

Lemma 4.4. *If $f \colon T' \to T$ is a morphism of sites and G is a presheaf on T, then the canonical morphism $(f^p G)^\sharp \to f^s(G^\sharp)$ is an isomorphism.*

Proof. Let $\mathcal{F}$ be any sheaf on T'. Then

$$\begin{aligned} \operatorname{Hom}((f^p G)^\sharp, \mathcal{F}) &= \operatorname{Hom}(f^p G, i\mathcal{F}) \\ &= \operatorname{Hom}(G, f_p(i\mathcal{F})) \\ &= \operatorname{Hom}(G, i(f_s\mathcal{F})) \\ &= \operatorname{Hom}(G^\sharp, f_s\mathcal{F}) \\ &= \operatorname{Hom}(f^s(G^\sharp), \mathcal{F}) \end{aligned}$$

so $(f^p G)^\sharp \xrightarrow{\sim} f^s(G^\sharp)$. $\square$

Proof of proposition 4.3. Certainly

$$\varinjlim_{X'} G(X') = (u^p G)(P).$$

But we already know $(u^p G)(P) = (u^p G)^\sharp(P)$ and by the lemma this is isomorphic to $u^s(G^\sharp)(P)$. The proof follows. $\square$

Example 4.5. If $X = x = \operatorname{Spec} K$ and F is any presheaf, then

$$F_{\overline{x}} = \varinjlim_{K'/K \text{ finite sep.}} F(\operatorname{Spec} K').$$

Definition 4.6. *Let $\mathcal{O}_{X,P}$ be the stalk of the étale sheaf $\mathcal{O}_X$ (defined in Example 4) at P. Then we see that*

$$\mathcal{O}_{X,P} = \varinjlim_{U} \mathcal{O}_U(U) = \varinjlim_{U} \mathcal{O}_{U,u},$$

where the limit is over étale neighborhoods (U, u) of x with u mapping to x, and where $\mathcal{O}_{U,u}$ denotes the usual (Zariski) stalk of the usual structure sheaf on U at u.

magic occurs

Example 4.7. If Y is a group scheme, locally of finite type over X, then Y defines an étale sheaf on X, and

$$Y_{\overline{x}} = \varinjlim_{U} Y(U).$$

But if Y is locally of finite type over X and (U_i) is a filtered inverse system of X-schemes then

$$\varinjlim_{U_i} Y(U_i) = Y(\varprojlim U_i).$$

Hence $Y_{\overline{x}} = Y(\mathcal{O}_{X,\overline{x}})$. For example, $(\mathbb{G}_m)_{\overline{x}} = \mathcal{O}_{X,\overline{x}}$.

Recall that if A is a local ring with residue field k and k' is an extension of k, then the limit $\widetilde{A} = \varinjlim B$ over all diagrams

$$\begin{tikzcd} A \arrow[rr, "\text{local, étale}"] \arrow[dr] & & B \arrow[dl] \\ & k' & \end{tikzcd}$$

is henselian with residue field k'.

If $k' = k^{\text{sep}}$, then $\widetilde{A} = A^{\text{sh}}$ is the strict henselization of A.

Example 4.8. If $\mathcal{O}_K$ is the ring of integers in a local field K, then $\mathcal{O}_K^{\text{sh}}$ is the ring of integers $\mathcal{O}_{K^{\text{un}}}$ of the maximal unramified extension of K.

Example 4.9.

$$\mathcal{O}_{X,\overline{x}} = (\mathcal{O}_{X,x})^{\text{sh}}$$

Proposition 4.10. *Suppose $X \xrightarrow{f} X'$ and $f(x) = x'$.*

(1) *If F' is a sheaf on X', then $(f^s F')_{\overline{x}} = F'_{\overline{x}}$.*

(2) *If f is quasi-compact and quasi-separated, let $\widetilde{f}$ be such that this diagram is cartesian:*

$$\begin{tikzcd} X \arrow[d] & X \times_{X'} \operatorname{Spec} \mathcal{O}_{X',\overline{x'}} \arrow[l, "\widetilde{f}"'] \arrow[d] \\ X' & \operatorname{Spec} \mathcal{O}_{X',\overline{x'}} \arrow[l] \end{tikzcd}$$

Then

$$(f_s F)_{\overline{x'}} = \left(\widetilde{f}^s F\right)\left(X \times_{X'} \operatorname{Spec} \mathcal{O}_{X',\overline{x'}}\right).$$

Proof. (1) is clear.

To prove (2), look at

$$(f_s F)_{\overline{x'}} = \varinjlim_{U} F(X \times_{X'} U) \to \widetilde{f}^s F\left(X \times_{X'} \operatorname{Spec} \mathcal{O}_{X',\overline{x'}}\right),$$

which is an isomorphism if f is quasi-compact and quasi-separated as each étale neighborhood $X \times_{X'} \operatorname{Spec} \mathcal{O}_{X',\overline{x'}} \to U'$ factors through some $X \times_{X'} U$. $\square$

Lemma 4.11. *Let $\mathcal{F}$ be a sheaf on ét$/X$, and let $f\colon \operatorname{Spec} \mathcal{O}_{X,P} \to X$ for a geometric point P. Then*

$$\mathcal{F}_P = (f^s \mathcal{F})(\mathcal{O}_{X,P}).$$

Proof. By definition we have $P \to \operatorname{Spec} \mathcal{O}_{X,P} \xrightarrow{f} X$, so by (1) we need only show that $\mathcal{G}_P = \mathcal{G}(\mathcal{O}_{X,P})$ for any sheaf $\mathcal{G}$ on $\mathcal{O}_{X,P}$. But this is true because any étale map to a local ring has a section. $\square$

Corollary 4.12. *Let $X \to X'$ be a finite morphism and let $\mathcal{F}$ be an étale sheaf on X. Then*

$$(f_* \mathcal{F})_{\overline{x'}} = \prod_{\overline{x} \in X_{\overline{x'}}} \mathcal{F}_{\overline{x}},$$

where $X_{\overline{x'}}$ denotes the geometric fiber $X \times_{X'} \overline{x'}$.

Proof. In this situation we have

$$X \times_{X'} \operatorname{Spec} \mathcal{O}_{X',\overline{x'}} = \coprod_{\overline{x} \in X_{\overline{x'}}} \operatorname{Spec} \mathcal{O}_{X,\overline{x}},$$

and now the result follows from this and the lemma. $\square$

4.2. **The mapping cylinder.** Let $j\colon U \to X$ be an open immersion, and let $i\colon Z \to X$ be any closed immersion onto the complement of U. Suppose $\mathcal{F}$ is a sheaf on X. We can associate to X

- a sheaf $j^s \mathcal{F}$ on U,
- a sheaf $i^s \mathcal{F}$ in Z, and
- a morphism $(i^s \mathcal{F}) \to i^s j_s (j^s \mathcal{F})$.

This last is induced from $\mathcal{F} \to j_s j^s \mathcal{F}$, which comes from the identity by adjointness of j_s and j^s.

Theorem 4.13. *There is an equivalence of categories*

$$S_{\text{ét}/X} \to \textit{triples } (\mathcal{F}_U, \mathcal{F}_Z, \phi\colon \mathcal{F}_Z \to i^s j_s \mathcal{F}_U),$$

where $\mathcal{F}_U$ is used to mean a sheaf on U and $\mathcal{F}_Z$ is used to mean a sheaf on Z.

Lemma 4.14. *If $\mathcal{F}_Z$ is any sheaf on Z, then*

$$(i_s\mathcal{F}_Z)_{\overline{x}} = \begin{cases} 0 & \text{if } x \notin i(Z) \\ (\mathcal{F}_Z)_{\overline{x_0}} & \text{if } x = i(x_0), \end{cases}$$

and if $\mathcal{F}_U$ is any sheaf on U, then

$$(j_s\mathcal{F}_U)_{\overline{x}} = (\mathcal{F}_U)_{\overline{x_0}} \text{ if } x = i(x_0).$$

Proof. For i this is an application of Corollary 4.12; for j it is clear. □

Proof of theorem. Set $\mathcal{F} = i_s\mathcal{F}_Z \times_{i_s i^s j_s \mathcal{F}_U} j_s\mathcal{F}_U$. To see that this yields an inverse of the construction leading to Theorem 4.13, we need to verify that for any étale sheaf $\mathcal{F}$ on X the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & j_s j^s \mathcal{F} \arrow[d] \\ i_s i^s \mathcal{F} \arrow[r] & i_s i^s j_s j^s \mathcal{F} \end{tikzcd}$$

is commutative.

We check this on stalks. If $x \in U$, then $(j_s j^s \mathcal{F})_{\overline{x}} = (j^s\mathcal{F})_{\overline{x_0}} = \mathcal{F}_{\overline{x}}$ and $i_s(-)_{\overline{x}} = 0$, so the diagram becomes

$$\begin{tikzcd} \mathcal{F}_{\overline{x}} \arrow[r] \arrow[d] & \mathcal{F}_{\overline{x}} \arrow[d] \\ 0 \arrow[r] & 0, \end{tikzcd}$$

which is obviously commutative.

If $x \in Z$ then for any $\mathcal{G}$ on ét$/X$, $(i_s i^s \mathcal{G})_{\overline{x}} = \mathcal{G}_{\overline{x}}$ so the diagram becomes

$$\begin{tikzcd} \mathcal{F}_{\overline{x}} \arrow[r] \arrow[d] & (j_s j^s \mathcal{F})_{\overline{x}} \arrow[d] \\ \mathcal{F}_{\overline{x}} \arrow[r] & (j_s j^s \mathcal{F})_{\overline{x}}, \end{tikzcd}$$

which is also obviously commutative. □

Example 4.15. Suppose R is a discrete valuation ring. Let K be the field of fractions of R, let $G = \mathrm{Gal}(K^{\mathrm{sep}}/K)$, and let I_K be the inertia group of K. Let k be the residue field of R, and let $G_k = \mathrm{Gal}(k^{\mathrm{sep}}/k)$. Then we have an equivalence of categories

$$S_{\text{ét}/X} \leftrightarrow \text{triples } (N_1, N_2, \phi)$$

where N_1 is a G_k-module, N_2 is a G_K-module, and $\phi\colon N_1 \to N_2^{I_K}$.

Proof. Set $Z = \operatorname{Spec} k \hookrightarrow \operatorname{Spec} R$, and $U = \operatorname{Spec} K$. This satisfies the conditions of Theorem 4.13, so we know we have an equivalence of categories to some kind of triples. How can we interpret the resulting triples? We know the category of étale sheaves on U is equivalent to the category of G_K-modules, and the category of étale sheaves on Z is equivalent to the category of G_k-modules. This leaves only the morphism to describe.

If N_2 is a G_K-module, it corresponds to the sheaf $\mathcal{F}_2(\operatorname{Spec} K') = N_2^{G'_K}$. Now $i^s j_s \mathcal{F}_2$ corresponds to the G_k-module

$$\begin{aligned} \varinjlim_{k}{}' (i^s j_s \mathcal{F}_2)(\operatorname{Spec} k') &= (i^s j_s \mathcal{F}_2)_{\overline{k}} \\ &= (j_s \mathcal{F}_2)_{\overline{k}} \\ &= \varinjlim_{k'/k} \mathcal{F}_2(\operatorname{Spec} K \times R^h), \end{aligned}$$

where R^h is the Henselization of R over k'. But this last inverse limit is exactly $N_2^{I_K}$. ☐

The functors i^s, i_s, j^s, and j_s may be viewed as

$$\begin{aligned} &i^s \colon (\mathcal{F}_Z, \mathcal{F}_U, \phi) \mapsto \mathcal{F}_Z && j^s \colon (\mathcal{F}_Z, \mathcal{F}_U, \phi) \mapsto \mathcal{F}_U \\ &i_s \colon \mathcal{F}_Z \mapsto (\mathcal{F}_Z, 0, 0) && j_s \colon \mathcal{F}_U \mapsto (i^s j_s \mathcal{F}_U, \mathcal{F}_U, \mathrm{id}). \end{aligned}$$

We can produce two more functors

$$\begin{aligned} &i^! \colon (\mathcal{F}_Z, \mathcal{F}_U, \phi) \mapsto \ker \phi && \text{(sections with support in } Z) \\ &j_! \colon \mathcal{F}_U \mapsto (0, \mathcal{F}_U, 0) && \text{(extension by zero).} \end{aligned}$$

This definition of $j_!$ is the sheafification of the $j_!$ we had on presheaves in Remark 3.5.

The following are now formal:

- The pairs (i^s, i_s), $(i_s, j^!)$, $(j_!, j^s)$, (j^s, j_s) are pairs of adjoint functors.
- The functors i^s, i_s, j^s and $j_!$ are exact, while j_s and $i^!$ are left-exact.
- The functors j^s, j_s, $i^!$ and i_s map injectives to injectives.
- The compositions $i^s j_!$, $i^! j_!$, $i^! j_s$, and $j^s i_s$ are zero.
- The functors i_s and j_s are fully faithful, and $\mathcal{F}_{\overline{x}} = 0$ for all $x \notin Z$ if and only if $\mathcal{F} = i_s \mathcal{F}'$ for some $\mathcal{F}'$ on ét$/Z$.

5. Cohomology

On a site T we have a number of left-exact functors, namely

- $\iota \colon S_T \to P_T$,
- $\Gamma_U \colon S_T \to \mathbf{Ab}$, where $\Gamma_U \colon \mathcal{F} \mapsto \mathcal{F}(U)$,
- $f_s \colon S_T \to S_{T'}$, and
- for $T =$ ét$/X$, we have $i^!$.

We wish to examine the resulting derived functors.

Definition 5.1. *Define*

$$H^q(U, \mathcal{F}) := R^q \Gamma_U(\mathcal{F}).$$

Proposition 5.2. *Let $\mathcal{H}^q(\mathcal{F}) = R^q \iota(\mathcal{F})$, a presheaf on T. Then for every open U we have $\mathcal{H}^q(\mathcal{F})(U) = H^q(U, \mathcal{F})$.*

Proof. The functor $U \mapsto H^q(U, \mathcal{F})$ is a presheaf on X and $H^0(-, \mathcal{F}) = \iota \mathcal{F}$. By a universal δ-functor argument, $H^q(-, \mathcal{F}) = (R^q \iota)(F)$: a short exact sequence of sheaves gives a long exact sequence of $H^q(U, \mathcal{F})$ for each U, which is a long exact sequence of presheaves $(U \mapsto H^q(-, \mathcal{F}))$.

Alternatively, write $\Gamma_U = \Gamma_{U,P} \circ \iota$, where $\Gamma_{U,P} \colon P_T \to \mathbf{Ab}$ is the functor taking a presheaf $F \mapsto F(U)$. Now, $\Gamma_{U,P}$ is an exact functor, and since $\sharp$ is exact, ι takes injectives to injectives. So we can apply the Grothendieck spectral sequence:

$$R^p\Gamma_{U,P}(R^q\iota(\mathcal{F})) \implies H^{p+q}(U,\mathcal{F}).$$

But $R^p\Gamma_{U,P} = 0$ for $p > 0$, so $H^q(U,\mathcal{F}) = \Gamma_{U,P}(R^q\iota(\mathcal{F}))$, that is,

$$R^q\iota(\mathcal{F})(U) = H^q(U,\mathcal{F}).$$

□

Similarly if $f \colon T \to T'$ we have

Proposition 5.3. *The functor $R^q f_s(\mathcal{F})$ is just the sheaf associated to*

$$V \mapsto H^q(f^{-1}(V),\mathcal{F}).$$

Proof. By definition $f_s = \sharp \circ f_p \circ \iota_T$, and $\sharp \circ f_p$ is exact while ι_T preserves injectives, so we obtain a spectral sequence whose $E_2^{p,q}$ term we write down:

$$R^p(\sharp \circ f_p)(R^q\iota_T\mathcal{F}) \implies (R^{p+q}f_s)(\mathcal{F}).$$

But $R^p(\sharp \circ f_p)$ is zero if $p > 0$, so

$$R^q f_s(\mathcal{F}) = (\sharp \circ f_p)(R^q\iota_T\mathcal{F}),$$

which by the above is just the sheafification of $V \mapsto H^q(f^{-1}(V),\mathcal{F})$. □

Corollary 5.4. *If $q > 0$ then $\mathcal{H}^q(\mathcal{F})^\sharp = 0$.*

Proof. Let f be the identity function on T. Then f_s is the identity functor, which is exact, so we have $R^q f_s(\mathcal{F}) = 0$ for positive q. But applying the above, we see that this is exactly $\mathcal{H}^q(\mathcal{F})^\sharp$. □

6. Flabby sheaves and Čech cohomology

Theorem 6.1. *Let $\mathcal{A}$ be an abelian category, and let $\mathcal{C}$ be a class of objects of $\mathcal{A}$ such that*

- *every object of $\mathcal{A}$ is a subobject of an object of $\mathcal{C}$,*
- *if $A \oplus A' \in C$ for some $A' \in \mathcal{A}$ then $A \in \mathcal{C}$, and*
- *if we have an exact sequence $0 \to A' \to A \to A'' \to 0$ and A', A are in $\mathcal{C}$, then $A'' \in \mathcal{C}$.*

Then $\mathcal{C}$ contains injectives, and if $f \colon \mathcal{A} \to \mathcal{B}$ is left-exact and exact on sequences in $\mathcal{C}$, then objects in $\mathcal{C}$ are f-acyclic (so $R^q f$ may be computed with $\mathcal{C}$-resolutions).

Proof. Let I be an injective object. If $I \hookrightarrow A$, then $A = I \oplus A'$ so $I \in \mathcal{C}$.

If A is any object in $\mathcal{C}$, then take in injective resolution

$$0 \to A \to I^0 \to I^1 \to \cdots;$$

from this we can obtain many exact sequences:

$$\begin{array}{ccc} 0 \to A \to I^0 \to Z^0 \to 0 & \text{for some } Z^0 \in \mathcal{C} & \text{so } 0 \to fA \to fI^0 \to fZ^0 \to 0 \text{ exact;} \\ 0 \to Z^0 \to I^1 \to Z^1 \to 0 & \text{for some } Z^1 \in \mathcal{C} & \text{so } 0 \to fZ^0 \to fI^1 \to fZ^1 \to 0 \text{ exact;} \\ \vdots & \vdots & \vdots \end{array}$$

so the sequence

$$0 \to f(A) \to f(I^0) \to f(I^1) \to \cdots$$

is exact, and therefore

$$R^q f(A) = 0$$

for every $q > 0$. $\square$

Definition 6.2. *A sheaf $\mathcal{F} \in S_T$ is* flabby *if $H^q(U, \mathcal{F}) = 0$ for every open set U and $q > 0$.*

Observe that injective sheaves are flabby. We know that for any sheaves $\mathcal{F}$ and $\mathcal{G}$ we have $H^q(U, \mathcal{F} \oplus \mathcal{G}) = H^q(U, \mathcal{F}) \oplus H^q(U, \mathcal{G})$. Finally, if A' and A are flabby and we have an exact sequence $0 \to A' \to A \to A'' \to 0$, then the long exact sequence of cohomology shows us that A'' is also flabby. So flabby sheaves form a class suitable for Theorem 6.1.

If $\mathcal{F}$ is flabby, $\mathcal{H}^q(\mathcal{F})(U) = H^q(U, \mathcal{F}) = 0$ for $q > 0$, so (recalling the definition of $\mathcal{H}^q$), we have $R^q\iota(\mathcal{F}) = 0$ for $q > 0$. Also, $(R^q f_s)(\mathcal{F})$ is the sheaf associated to $V \mapsto H^q(f^{-1}V, \mathcal{F})$, which is the zero presheaf.

Thus $\mathcal{H}^q)$, $H^q(U, -)$ and $R^q f_s$ may all be computed using flabby sheaves.

Definition 6.3. *Let F be a presheaf and $\{U_i \to U\}$ be a covering. Then define*

$$H^0(\{U_i \to U\}, F) := \ker \left(\prod F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j) \right).$$

Note that this is a left-exact functor. So define

$$H^q(\{U_i \to U\}, F) = \big(R^q H^0(\{U_i \to U\}, -)\big)(F)$$

for $q > 0$.

Theorem 6.4. *Let $\widetilde{H}^q(\{U_i \to U\}, F)$ denote the cohomology of the Čech complex $C^q(\{U_i \to U\}, F) = \prod F(U_{i_0} \times \cdots U_{i_q})$. Then $\widetilde{H}^q = H^q$.*

Proof. From the definition it is clear that $H^0 = \widetilde{H}^0$. So, using a universal δ-functor argument, it suffices to see that $\widetilde{H}^q(\{U_i \to U\}, I) = 0$ when I is injective. This comes down to exactness of

$$\oplus Z_{U_i} \leftarrow \oplus Z_{U_i \times U_j} \leftarrow \cdots,$$

which we will omit. $\square$

Definition 6.5. *Let F be a presheaf. We define*

$$\check{H}^q(U, F) := \varinjlim_{\{U_i \to U\}} H^q(\{U_i \to U\}, F).$$

Theorem 6.6.

$$\check{H}^q(U, -) = R^q \check{H}^0(U, -).$$

Proof. To prove this via a universal δ-functor argument, we have to show $\check{H}^q(U, -)$ takes short exact sequences to long exact sequences of $\check{H}^q$. But $H^q(\{U_i \to U\}, -)$ does, and this limit (as we saw for sheafification in Theorem 2.8) is exact. $\square$

Now, if $\mathcal{F}$ is a sheaf, we get

$$\begin{tikzcd} S_T \arrow[r, "\iota"] \arrow[rr, bend right, "\Gamma_U"'] & P_T \arrow[r, "\check{H}^0 \text{ or } H^0(\{U_i \to U\}, -)"] & \mathbf{Ab}. \end{tikzcd}$$

Since ι takes injectives to injectives, we obtain a spectral sequence

$$E_2^{p,q} := H^p(\{U_i \to U\}, \mathcal{H}^q(\mathcal{F})) \Rightarrow H^{p+q}(U, \mathcal{F}),$$

and a similar spectral sequence for $\check{H}$.

Exercise 6.7. Let $\{U_0 \to X, U_1 \to X\}$ be a Zariski cover of X in E/X. Then show that there is an exact sequence, namely the Mayer-Vietoris sequence

$$\cdots \to H^q(X, \mathcal{F}) \to H^q(U_0, \mathcal{F}) \oplus H^q(U_1, \mathcal{F}) \to H^q(U_0 \cap U_1, \mathcal{F}) \to H^{q+1}(X, \mathcal{F}) \to \cdots$$

(use the above spectral sequence for the cover $\{U_0 \to X, U_1 \to X\}$).

Theorem 6.8. *If $H^q(U_{i_0} \times \cdots \times U_{i_r}, \mathcal{F}) = 0$ for every r and every $q > 0$, then $H^p(\{U_i \to U\}, \mathcal{F}) = H^p(U, \mathcal{F})$.*

Proof. Compute $H^p(\{U_i \to U\}, \mathcal{F})$ using the Čech complex. Then

$$\begin{aligned} C^p(\{U_i \to U\}, \mathcal{H}^q(\mathcal{F})) &= \prod \mathcal{H}^q(\mathcal{F})(U_{i_0} \times \cdots \times U_{i_r}) \\ &= \prod H^q(U_{i_0} \times \cdots \times U_{i_r}, \mathcal{F}) \\ &= 0 \text{ if } q > 0 \end{aligned}$$

so $H^p(\{U_i \to U\}, \mathcal{H}^q(\mathcal{F})) = 0$ and the spectral sequence degenerates. □

Corollary 6.9. *A sheaf $\mathcal{F}$ is flabby if and only if $H^p(\{U_i \to U\}, \mathcal{F}) = 0$ for every $q > 0$ and for every cover $\{U_i \to U\}$.*

The forward implication is clear from the theorem. For the reverse we will need some lemmas.

Lemma 6.10. *For any sheaf $\mathcal{F}$ and open set U we have $\check{H}^1(U, \mathcal{F}) \cong H^1(U, \mathcal{F})$, and $\check{H}^2(U, \mathcal{F}) \hookrightarrow H^2(U, \mathcal{F})$.*

Proof. Recall that we have a spectral sequence (which we did not write down explicitly)

$$E_2^{p,q} := \check{H}^p(U, \mathcal{H}^q(\mathcal{F})) \Rightarrow H^{p+q}(U, \mathcal{F}).$$

Taking the exact sequence of low degree, we obtain

$$\begin{array}{c} 0 \to \check{H}^1(U, \mathcal{F}) \to H^1(U, \mathcal{F}) \longrightarrow \check{H}^0(U, \mathcal{H}^1(\mathcal{F})) \longrightarrow \check{H}^2(U, \mathcal{F}) \to H^2(U, \mathcal{F}). \\ \| \\ (\mathcal{H}^1(\mathcal{F}))^\sharp(U) = 0 \end{array}$$ □

Lemma 6.11. *If*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

is a short exact sequence of sheaves and $H^1(\{U_i \to U\}, \mathcal{F}') = 0'$ for every cover $\{U_i \to U\}$ then

$$0 \to \iota\mathcal{F}' \to \iota\mathcal{F} \to \iota\mathcal{F}'' \to 0$$

is an exact sequence of presheaves.

Proof. Applying ι we get a long exact sequence

$$0 \to \iota\mathcal{F}' \to \iota\mathcal{F} \to \iota\mathcal{F}'' \to \mathcal{H}^1(\mathcal{F}') \to \cdots,$$

but by the previous lemma,

$$\mathcal{H}^1(\mathcal{F}')(U) = H^1(U, \mathcal{F}') = \check{H}^1(U, \mathcal{F}') = \lim H^1(\{U_i \to U\}, \mathcal{F}') = 0.$$ □

Proof of Corollary 6.9. If $0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$ is an injective resolution of $\mathcal{F}$ in S_T, we want to prove $0 \to \mathcal{F}(U) \to \mathcal{I}^0(U) \to \mathcal{I}^1(U) \to \cdots$ is exact for every U, i.e., is exact in P_T.

But we can construct a sequence of short exact sequences

$$\begin{gathered} 0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{Z}^0 \to 0 \\ 0 \to \mathcal{Z}^0 \to \mathcal{I}^1 \to \mathcal{Z}^1 \to 0 \\ \vdots \end{gathered}$$

We are assuming that $H^1(\{U_i \to U\}, \mathcal{F}) = 0$ for every cover $\{U_i \to U\}$. Applying the functor $H^0(\{U_i \to U\}, -)$ to each of the short exact sequences, we obtain long exact sequences showing that $H^q(\{U_i \to U\}, \mathcal{Z}^i) = 0$ for every $q > 0$ and every i. But then using Lemma 6.11 on the short exact sequences in succession, this implies $0 \to \iota\mathcal{Z}^i \to \iota\mathcal{I}^i \to \iota\mathcal{Z}^{i+1} \to 0$ is exact, and the result follows. □

Corollary 6.12. *Let $f\colon T \to T'$. Then if $\mathcal{F} \in S_T$ is flabby, the image $f_s\mathcal{F} \in S_{T'}$ is also flabby.*

Proof. Let $\{U_i \to U\}$ be a cover on T'. Then since $\mathcal{F}$ is flabby, the Čech complex

$$\mathcal{F}(f^{-1}U) \to \prod \mathcal{F}(f^{-1}U_i) \to \prod \mathcal{F}(f^{-1}(U_i \times_U U_j)) \to \cdots$$

is exact. But this is exactly the same complex of abelian groups as

$$f_s\mathcal{F}(U) \to \prod f_s\mathcal{F}(U_i) \to \prod f_s\mathcal{F}(U_i \times_U U_j) \to \cdots .$$ □

Example 6.13. If $U \to X$ is an E-morphism, then if $\mathcal{F}$ is flabby on E/X it follows that $\mathcal{F}|_U$ is flabby on E/U.

To see this, note that $\Gamma_U\colon S_{E/X} \to \mathbf{Ab}$ factors as $S_{E/X} \xrightarrow{-|_U} S_{E/U} \xrightarrow{\Gamma_U} \mathbf{Ab}$, and since restiction is exact, the Grothendieck spectral sequence gives $H^q_X(U, \mathcal{F}) = H^Q_U(U, \mathcal{F}|_U)$.

Corollary 6.14 (Leray spectral sequence)**.** *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ and $\mathcal{F} \in S_X$, then*

$$(R^p g_s)(R^q f_s(\mathcal{F})) \Rightarrow R^{p+q}(gf)_s(\mathcal{F}).$$

Proof. We need to show that f_s takes injectives to acyclics. But we have seen that f_s takes injective sheaves to flabby sheaves. □

As an important special case, take Z to be the topological space on a singleton, so $g_s = \Gamma(Y, -)$. Then we get

$$H^p(Y, R^q f_s(\mathcal{F})) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

This by itself is often referred to as the Leray spectral sequence.

Example 6.15. If T_G is the site of G-sets, S_G is the category of G-modules and $H^q(G, A)$ is group cohomology. If we take $\pi\colon G \to G/H$ to be the canonical morphism of sites, we get a functor $\pi_s : S_G \to S_{G/H}$. One can check that $\pi_s(A) = A^H$, so $R^q\pi_s = H^q(H, -)$. So the Leray spectral sequence says

$$H^p(G/H, H^q(H, A)) \Rightarrow H^{p+q}(G, A).$$

The sequence of low-degree terms gives inflation-restriction.

Note also that if $\check{H}(U, \mathcal{F}) = 0$ for all U then U is flabby.

Proof. We show $\mathcal{H}^q(\mathcal{F}) = 0$ by induction on q. For $q = 1$, we know $H^1(U, \mathcal{F}) = \check{H}^1(U, \mathcal{F}) = 0$. For $q > 1$, we have a spectral sequence $\check{H}^p(U, \mathcal{H}^{q-p}(\mathcal{F})) \Rightarrow H^q(U, \mathcal{F})$. For $p > 0$ the induction hypothesis gives $\mathcal{H}^{q-p} = 0$. For $p = 0$, we have $\check{H}^0(U, \mathcal{H}^q(\mathcal{F})) = \mathcal{H}^q(\mathcal{F})^\sharp(U) = 0$. □

Proposition 6.16. *Let U be an affine scheme and let $\mathcal{F}$ quasi-coherent. Then $H^p(U_{Zar}, \mathcal{F}) = 0$ for every $p > 0$.*

Sketch of proof. Check that $H^p(\{U_i \to U\}, \mathcal{F}) = 0$ for every affine cover; this implies that $\check{H}^p(U, \mathcal{F}) = 0$ (and clearly $\check{H}^p(V, \mathcal{F}) = 0$ for any affine V) so by Cartan's criterion this implies $H^p(U, \mathcal{F}) = 0$ for every $p > 0$. □

Corollary 6.17. *Let X be a separated scheme, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then*
$$\check{H}^p(X_{Zar}, \mathcal{F}) \cong H^p(X_{Zar}, \mathcal{F}).$$

Proof. Let $\{U_i \to U\}$ be a cover of open affines. Then by the preceding result and Theorem 6.8, $H^p(X, \mathcal{F}) = H^p(\{U_i \to X\}, \mathcal{F})$. Since the limit defining $\check{H}^p$ is cofiltered by the affine covers (that is, every cover has an affine refinement) the result follows. □

There is an étale version of this result:

Theorem 6.18. *Let X be a quasi-compact scheme, and suppose that every finite subset of X is contained in an open affine (for example, X could be a quasi-projective or affine scheme). Let $\mathcal{F}$ be a sheaf on $X_{\acute{e}t}$. Then*
$$H^p(X_{\acute{e}t}, \mathcal{F}) \cong \check{H}^p(X_{\acute{e}t}, \mathcal{F}).$$

For the proof, see [Mil80, Theorem 2.17].

Note that we do not have any sort of theorem assuring us that a finite fine enough cover is available to satisfy the conditions of Theorem 6.8; on the Zariski site, for quasi-projective schemes, a finite cover $\{U_i \to U\}$ by affines exists and is fine enough to allow the use of $H^p(\{U_i \to U\}, \mathcal{F})$ to compute $H^p(X_{\mathrm{Zar}}, \mathcal{F})$. This computation is often feasible in practice. On the other hand, if k is a field, then $\operatorname{Spec} k$ is certainly affine, but we saw in Example 2.6 that the étale cohomology of such a space is by no means trivial.

7. Excision and cohomology with supports

Let us return to the situation of the mapping cylinder: we have a scheme X, a (Zariski) open immersion $U \overset{j}{\hookrightarrow} X$, and a closed immersion $Z \overset{j}{\hookrightarrow} X$. We say that sections of $i_s i^! \mathcal{F}$ are sections with support in Z.

Definition 7.1. *Let $H^p_Z(X, \mathcal{F})$ denote the right derived functors of $\mathcal{F} \mapsto i_s i^! \mathcal{F}$, and call these the* cohomology groups with support in Z.

Proposition 7.2. *There is a long exact sequence*

$$0 \to (i^!\mathcal{F})(Z) \to \mathcal{F}(X) \to \mathcal{F}(U) \to \cdots$$
$$\cdots \to H^p(X, \mathcal{F}) \to H^p(U, \mathcal{F}) \to H^{p+1}_Z(X, \mathcal{F}) \to \cdots.$$

Proof. We have a short exact sequence of sheaves $0 \to j_! j^s \mathbb{Z} \to \mathbb{Z} \to i_s i^s \mathbb{Z} \to 0$. Take the associated long exact sequence of $\operatorname{Ext}^{\cdot}(-, \mathcal{F})$. But

$$\operatorname{Ext}^0(j_! j^s \mathbb{Z}, \mathcal{F}) = \operatorname{Hom}(j_! j^s \mathbb{Z}, \mathcal{F}) = \operatorname{Hom}(j^s \mathbb{Z}, j^s \mathcal{F}) = \mathcal{F}(U),$$

$$\operatorname{Ext}^0(\mathbb{Z}, \mathcal{F}) = \operatorname{Hom}(\mathbb{Z}, \mathcal{F}) = \mathcal{F}(X),$$

and

$$\operatorname{Ext}^0(i_s i^s \mathbb{Z}, \mathcal{F}) = \operatorname{Hom}(i_s i^s \mathbb{Z}, \mathcal{F}) = \operatorname{Hom}(i^s \mathbb{Z}, i^! \mathcal{F}) = (i^! \mathcal{F})(Z),$$

so the long exact sequence of Ext is exactly the above sequence. □

Theorem 7.3 (Excision theorem)**.** *Let $Z \subset X$ and $Z' \subset X'$ be closed subsets, and let $f \colon X' \to X$ be étale such that $f|_{Z'} \colon Z' \xrightarrow{\sim} Z$ and $f(X' - Z') \subset X - Z$. Then*

$$H^p_Z(X, \mathcal{F}) \xrightarrow{\sim} H^p_{Z'}(X', f^s \mathcal{F})$$

for every $p \geq 0$.

Proof. Recall that f^s is exact and preserves injectives. So we need only check the result for $p = 0$. Since $(i^! \mathcal{F})(Z) = (i_s i^! \mathcal{F})(X) = H^0_Z(X, \mathcal{F})$, we have a commutative diagram, exact along rows:

$$\begin{tikzcd}
0 \arrow[r] & H^0_Z(X, \mathcal{F}) \arrow[r] \arrow[d, "\phi"] & \mathcal{F}(X) \arrow[r] \arrow[d] & \mathcal{F}(X - Z) \arrow[d] \\
0 \arrow[r] & H^0_{Z'}(X', \mathcal{F}) \arrow[r] & \mathcal{F}(X') \arrow[r] & \mathcal{F}(X' - Z')
\end{tikzcd}$$

We want to show ϕ is an isomorphism. To show injectivity, suppose we have $\gamma \in H^0_Z(X, \mathcal{F})$ with $\phi(\gamma)$. Then its image γ_1 in $F(X)$ must map to zero when restricted to $\mathcal{F}(X - Z)$ (by the exactness of the top row) and also when mapped to $\mathcal{F}(X')$ (by mapping via ϕ). But $\{X - Z, X'\}$ form an étale cover of X, so $\gamma_1 = 0$ and $\gamma = 0$.

To show surjectivity, let $\gamma' \in H^0_{Z'}(X', \mathcal{F})$, and consider its image $\gamma'_1 \in \mathcal{F}(X')$. As before, $\{X - Z, X'\}$ form an étale cover of X, and γ'_1 agrees with 0 on $X' \times_X (X - Z) \subset X' - Z'$, so γ'_1 comes from some $\gamma_1 \in \mathcal{F}(X)$; since $\gamma_1|_{X-Z} = 0$, $\gamma_1 \in H^0_Z(X, \mathcal{F})$ and by construction γ_1 maps to γ'. □

Recall now that in the topological category, cohomology with compact supports played a role in providing a pairing on cohomology groups. We will construct an analogue of cohomology with compact supports in our situation.

Definition 7.4. *Let X be a separated variety (integral scheme of finite type over a field). We define a functor*

$$\Gamma_c(X, \mathcal{F}) := \bigcup_{Z \text{ complete}} \ker(\mathcal{F}(X) \to \mathcal{F}(X - Z)),$$

which we call sections with compact support*. But we do* not *define H_c to be the derived functors of Γ_c, as these are uninteresting (see* [Mil80, the discussion leading to Proposition III.1.29]*). Instead, if $j \colon X \hookrightarrow \overline{X}$ is an open immersion where $\overline{X}$ is complete, then we set*

$$H^p_c(X, \mathcal{F}) := H^p(\overline{X}, j_! \mathcal{F}).$$

Theorem 7.5. *If $\mathcal{F}$ is torsion, $H^p_c(X, \mathcal{F})$ is independent of $\overline{X}$ and satisfies Poincaré duality.*

We omit the proof.

In any case, $H_c^0(X, \mathcal{F}) = \Gamma_c(X, \mathcal{F})$, for any short exact sequence of sheaves $H_c^p(X, -)$ gives a long exact sequence of cohomology groups, and we have morphisms $H_Z^p(X, -) \to H_c^p(X, -)$ for any complete $Z \subset X$.

8. Comparison theorems

We will first describe a technical condition that often allows us to work with a simpler site.

Suppose $f : T \to T'$ is a morphism of topologies such that

- f^{-1} is fully faithful (that is, injective on both objects and Hom-sets), and
- if $U' \in T'$ and $\left\{U_i \to f^{-1}(U')\right\}$ is a covering in T, then there is a cover $\left\{U'_j \to U'\right\}$ in T' such that $\left\{f^{-1}(U'_i) \to f^{-1}(U')\right\}$ is a refinement of $\left\{U_i \to f^{-1}(U')\right\}$.

In this situation we will occasionally think of T' as a subcategory of T. Observe that the first condition implies that T is a finer topology than T'.

Proposition 8.1. *Given such an f, we find $\mathcal{F} \to f_s f^s \mathcal{F}$ is an isomorphism for any $\mathcal{F} \in S_T$ and f_s is exact.*

We omit the proof.

Remark 8.2. Under the stronger condition that any cover $\{U_i \to U\}$ in T such that all U_i and U are in T' is actually a cover in T', and any object in T has a cover (in T) by objects of T', then f^s and f_s are equivalences between S_T and S'_T; and also $f^s f_s \mathcal{F} \cong \mathcal{F}$.

Corollary 8.3 (of Proposition 8.1)**.** *Under the hypotheses of Proposition 8.1, for any $U \in T'$ we have*

$$H^p(U, f_s\mathcal{F}) \cong H^p(f^{-1}(U), \mathcal{F})$$

and

$$H^p(U, \mathcal{F}) \cong H^p(f^{-1}(U), f^s\mathcal{F}).$$

Proof. The Leray spectral sequence (Corollary 6.14) gives

$$E_2^{p,q} := H^p(U, R^q f_s \mathcal{F}) \Rightarrow H^{p+q}(U, \mathcal{F}),$$

but $R^q f_s = 0$ for $q > 0$. Then

$$H^p(U, \mathcal{F}') \cong H^p(U, f_s f^s \mathcal{F}') \cong H^p(f^s \mathcal{F}'). \qquad \square$$

Example 8.4. This applies to all of the following situations

- T is the big E-site and T' is the small E-site
- $T = E/X$ for E the class of all étale maps, and $T' = \text{ét}/X$ (recall $\text{ét}/X$ sets E to be the class of étale maps of finite type)
- $T = \text{ét}/X$, and $T' = E/X$ for E the class of separated étale morphisms (or affine étale morphisms)
- $T = E/X$ where E is the class of smooth morphisms, and $T' = \text{ét}/X$
- $T = \text{Fl}/X$ or X_{Fl}, and $T' = E/X$ or $T' = X_E$ for E the class of morphisms that are flat of finite type
- $T = \text{Fl}/X$ or X_{Fl}, and $T' = E/X$ or $T' = X_E$ for E the class of morphisms that are quasifinite and flat

Definition 8.5. *A site T is* noetherian *if every cover has a finite subcover.*

In this case, it is not difficult to check that it suffices to check the sheaf condition on finite covers, since a subcover is always a refinement. If T is any site, we could define T^f to be the site T with only the finite covers. We obtain a morphism $i\colon T \to T^f$. If T is noetherian, then $i_s\colon S_T \to S_{T^f}$ is thus an equivalence of categories, and $H^p(U, i_s\mathcal{F}) = H^p(U, \mathcal{F})$.

Corollary 8.6. *If T is noetherian, then $\mathcal{F}$ is flabby if and only if $H^q(\{U_i \to U\}, \mathcal{F}) = 0$ for every finite cover.*

Proof. The above shows that flabbiness in T is equivalent to flabbiness in T^f. □

Proposition 8.7. *In a noetherian topology, $H^p(U, \varinjlim_i \mathcal{F}_i) = \varinjlim_i H^p(U, \mathcal{F}_i)$ for pseudofiltered direct limits.*

Proof. Using the previous corollary, we see that a limit of flabby resolutions is again a flabby resolution. □

Example 8.8. The site ét$/X$ is not noetherian, but if X is quasi-compact then E/X is when E is the class of étale morphisms of finite presentation over X (since finite presentation maps are quasi-compact). Moreover, if X is quasi-separated, then Remark 8.2 applies and we have an equivalence of categories between sheaves on these two sites.

Corollary 8.9. *Let X be quasi-compact and quasi-separated. Then $H^p_{ét}(X, -)$ commutes with pseudofiltered direct limits.*

8.1. Comparison of étale and Zariski cohomologies.

Theorem 8.10. *If M is a quasi-coherent sheaf of $\mathcal{O}_X$-modules, then*

$$H^p(X_{Zar}, M) \cong H^p_{ét}(X, M_{ét}).$$

Proof. Let $\varepsilon\colon \text{ét}/X \to \text{Zar}/X$, so that we get a spectral sequence

$$E_2^{p,q} = H^p_{\mathrm{Zar}}(X, R^q\varepsilon_s M_{\text{ét}}) \Rightarrow H^{p+q}_{\text{ét}}(X, M_{\text{ét}}).$$

To see that the edge map $H^p(X_{\mathrm{Zar}}, M) \to H^p_{\text{ét}}(X, M_{\text{ét}})$ is an isomorphism, we want $(R^q\varepsilon_s)(M_{\text{ét}}) = 0$ for $q > 0$. But $(R^q\varepsilon_s)(M_{\text{ét}})$ is the sheafification of $U \mapsto H^q(U, M_{\text{ét}})$, so it suffices to show $H^q(U, M_{\text{ét}}) = 0$ for U affine and $q > 0$.

Let T be the subcategory of ét$/X$ of affines. Corollary 8.3 then shows that $H^q_T(U, M_{\text{ét}}) \cong H^q_{\text{ét}}(U, M_{\text{ét}})$.

The result follows immediately from:

Claim. $M_{\text{ét}}$ is flabby on T.

By construction, T is noetherian, so it suffices to check the claim on finite covers. Checking the claim for $\{Y_i \to Y\}$ is equivalent to checking it for $\{\coprod Y_i \to Y\}$, so we need only check $H^q(\{Z \to Y\}, M_{\text{ét}})$ for $Z \to Y$ affine and faithfully flat. But the Čech complex in this case is

$$0 \to M \to M \otimes_A B \to M \otimes_A B \otimes_A B \to \cdots$$

which we know is exact. □

Corollary 8.11 (of proof)**.** *If X is affine, then $H^p_{ét}(X, M_{ét}) \cong H^p_{Zar}(X, M) = 0$*

8.2. **Comparison of the étale and complex cohomologies.** Suppose X is a scheme over $\mathbb{C}$ and G is an abelian group. Then we can compute $H^p_{\text{ét}}(X, G)$ and $H^p(X(\mathbb{C}), G)$. Ideally, we would show that they were isomorphic for the group $\mathbb{Z}$. However, this does not occur. In fact, there is a profinite group $\pi_1(X)$, the étale fundamental group of X, classifying étale covers of X. With this definition, it can be shown that

$$H^1_{\text{ét}}(X, \mathbb{Z}) \cong \mathrm{Hom}_{\text{continuous}}(\pi_1(X), \mathbb{Z}),$$

which is zero since $\pi_1(X)$ is profinite and therefore has only finite quotients.

However, we can produce a useful comparison theorem by considering finite groups.

Theorem 8.12. *Let X be a smooth scheme over $\mathbb{C}$. For any finite abelian group M,*

$$H^p(X(\mathbb{C}), M) \cong H^p_{\acute{e}t}(X, M).$$

For an overview of the proof, see [Mil80, Theorem 3.12]; for the full proof, see [SGA73, Exposé IX]. In the case of H^1, it proceeds by constructing the category of coverings that are analytic local isomorphisms and proving it is equivalent to the category of étale covers (the Riemann existence theorem).

We will see a special case of this theorem as a result of our computations on curves.

9. Cohomology of $\mathbb{G}_m$

We can view the étale site as a finer topology than the Zariski site. If $\mathcal{M}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules, we have seen that the Zariski site is already fine enough to compute the "true" cohomology groups: $H^p(X_{\text{Zar}}, \mathcal{M}) = H^p(X_{\text{ét}}, \mathcal{M}_{\text{ét}})$. In this section, we will see that the Zariski site is also fine enough to describe line bundles up to isomorphism: we get no new non-isomorphic line bundles if we allow them to be constructed from étale covers instead of Zariski covers.

Definition 9.1. *Let $\mathcal{F}$ be a sheaf on E/X. Then we say $\mathcal{F}$ is* locally free of rank n *if there is a cover $\{U_i \to X\}$ such that $\mathcal{F}(U_i)$ is a free $\mathcal{O}_{U_i}(U_i)$-module of rank n for every i.*

A vector bundle of rank n on E/X *is a cover $\{U_i \to X\}$ along with a collection of elements $g_{ij} \in \mathrm{GL}_n(\mathcal{O}_{U_i \times_X U_j}(U_i \times_X U_j))$ for all i and j such that when restricted to $U_i \times_X U_j \times_X U_k$ we have*

$$g_{ij} g_{jk} = g_{ik}. \tag{*}$$

Remark 9.2. We have the usual correspondence between locally free sheaves of rank n and vector bundles of rank n.

Proposition 9.3. *Let $L(E/X)$ denote the group (under tensor product) of vector bundles of rank* 1 *on E/X. Then we have a natural isomorphism $L(E/X) \cong \check{H}^1(E/X, \mathbb{G}_m)$.*

Remark 9.4. With some care, we could define $\check{H}^1(E/X, \mathrm{GL}_n)$ for every n; we would then find that it classified vector bundles of rank n.

Proof. Observe that the condition (*) is precisely the condition for the g_{ij} to form a Čech cocycle. If they form a coboundary, then the sheaf condition will allow us to construct a global element of $\mathcal{O}^\times_{E/X}$ which provides an isomorphism with the trivial line bundle. ∎

Theorem 9.5. *Let X be a quasi-compact connected scheme. Then there is a natural isomorphism*
$$H^1(X_{Zar}, \mathbb{G}_m) \cong H^1(X_{\acute{e}t}, \mathbb{G}_m).$$

Proof. We will first simplify the space X. The morphism of sites $\varepsilon\colon \text{ét}/X \to \text{Zar}/X$ gives rise to the Leray spectral sequence (Corollary 6.14):
$$H^p(X_{\text{Zar}}, R^q\varepsilon\mathbb{G}_m) \Rightarrow H^{p+q}(X_{\text{ét}}, \mathbb{G}_m).$$
The exact sequence of low degree begins
$$0 \to H^1(X_{\text{Zar}}, \mathbb{G}_m) \to H^1(X_{\text{ét}}, \mathbb{G}_m) \to H^0(X_{\text{Zar}}, R^1\varepsilon\mathbb{G}_m),$$
so we need only show that $R^1\varepsilon\mathbb{G}_m = 0$. But (by Theorem 5.3) $R^1\varepsilon\mathbb{G}_m$ is the sheaf associated to $V \mapsto H^1(V_{\text{ét}}, \mathbb{G}_m)$. It suffices to check that this is zero on (Zariski) stalks, which amounts to showing that $H^1(V_{\text{ét}}, \mathbb{G}_m) = 0$ for $V = \operatorname{Spec} A$ where A is some local ring. In this situation, we know that $H^1(V_{\text{ét}}, \mathbb{G}_m) = \check{H}^1(V_{\text{ét}}, \mathbb{G}_m)$.

Let $\alpha \in \check{H}^1(V_{\text{ét}}, \mathbb{G}_m)$. Then there is an affine étale V-scheme U such that $\alpha \in H^1(\{U \to V\}, \mathbb{G}_m)$, that is, we can choose a cover by a single affine étale V-scheme. Writing $U = \operatorname{Spec} B$, we know A is a local ring and B is a flat and unramified extension. We know α is a Čech cocycle, and we need to show that it is a coboundary. We will omit this purely algebraic proof. □

Note that in this case we have $H^1(X_{\text{Zar}}, \mathbb{G}_m) \cong \check{H}^1(X_{\text{Zar}}, \mathbb{G}_m)$ and $H^1(X_{\text{ét}}, \mathbb{G}_m) \cong \check{H}^1(X_{\text{ét}}, \mathbb{G}_m)$. So using Proposition 9.3, this shows that $L(X_{\text{Zar}}) \cong L(X_{\text{ét}})$: no new line bundles have appeared.

When X is a connected nonsingular curve, we can work out higher cohomology groups.

Theorem 9.6. *Let X be a connected nonsingular curve over an algebraically closed field k. Then $H^p(X_{\acute{e}t}, \mathbb{G}_m) = 0$ for $p > 1$.*

Proof. Let U be a Zariski open set on X, let K be the function field of X, and let $\operatorname{Div}(U)$ be the group of Weil divisors on U. Recall that we have an exact sequence
$$0 \to \mathcal{O}_X(U)^\times \to K^\times \to \operatorname{Div}(U) \to 0$$
called the *Weil divisor exact sequence.*

Let η denote the generic point of X and g be the morphism $\operatorname{Spec} K \to X$, and for each closed point z on X, let i_x be the closed immersion $\operatorname{Spec} k \to X$. Then this exact sequence gives rise to an exact sequence of étale sheaves
$$0 \to \mathbb{G}_m \to g_s\mathbb{G}_{m,\eta} \to \oplus_x i_{x,s}\mathbb{Z} \to 0.$$

Apply the functors $H^{\cdot}(X_{\text{ét}}, -)$ to this exact sequence to give a long exact sequence. Since each x is a closed point, $i_{x,s}$ is exact (this can be checked on stalks), so for $p > 1$ we have $H^p(X_{\text{ét}}, \mathbb{G}_m) \cong H^p(X_{\text{ét}}, g_s\mathbb{G}_{m,\eta})$. To compute $H^p(X_{\text{ét}}, g_s\mathbb{G}_{m,\eta})$, apply the Leray spectral sequence (Corollary 6.14):
$$H^p(X_{\text{ét}}, R^q g_s\mathbb{G}_{m,\eta}) \Rightarrow H^{p+q}((\operatorname{Spec} K)_{\text{ét}}, \mathbb{G}_m).$$

Claim. $R^p g_s\mathbb{G}_{m,\eta} = 0$.

Assuming this claim, the Leray spectral sequence therefore shows that $H^p(X_{\text{ét}}, g_s\mathbb{G}_{m,\eta}) \cong H^p((\operatorname{Spec} K)_{\text{ét}}, \mathbb{G}_m)$. But if $G = \operatorname{Gal}(K^{\text{sep}}/K)$, then $H^p((\operatorname{Spec} K)_{\text{ét}}, \mathbb{G}_m) \cong H^p(G, K^\times)$ and since K is a function field of dimension 1 over an algebraically closed field, $H^p(G, K^\times) = 0$ for $p > 1$ by a theorem of Tsen. □

of Claim. From Proposition 5.3, we know that $R^p g_s \mathbb{G}_{m,\eta} = 0$ is just the sheaf $\mathcal{G}$ associated to $V \mapsto H^q(g^{-1}(V), \mathbb{G}_{m,\eta})$. We will evaluate this on stalks. If $\overline{x}$ is any geometric point, then $\mathcal{G}_{\overline{x}} = H^q(K_{\overline{x}}, \mathbb{G}_m)$, where $K_{\overline{x}}$ is the field of fractions of $\mathcal{O}_{X,\overline{x}}$.

Let K^{sep} be a separable closure of K, so that $\operatorname{Spec} K^{\text{sep}} \to \operatorname{Spec} K$ is a geometric point, giving a geometric point of X which we will call $\overline{\eta}$. Since every element in $\mathcal{O}_{X,\overline{\eta}}$ is algebraic over $\mathcal{O}_{X,\eta}$, we know $K_{\overline{\eta}} \subset K^{\text{sep}}$. But if we take any finite separable extension L of K, we can take the normalization of X in L and obtain an étale covering of some neighborhood on X with function field L, so $K_{\overline{\eta}} = K^{\text{sep}}$, and $\mathcal{G}_{\overline{\eta}} = H^q(K^{\text{sep}}, \mathbb{G}_m)$. But $H^q(K^{\text{sep}}, \mathbb{G}_m)$ is just group cohomology for the trivial group, so it is zero if $q > 0$.

Now, if x is a closed point, then $\mathcal{G}_{\overline{x}} = H^q(K_{\overline{x}}, \mathbb{G}_m)$, where $K_{\overline{x}}$ is the field of fractions of $\mathcal{O}_{X,\overline{x}}$, which is Henselian and has algebraically closed residue field. By a theorem of Lang, it follows that $H^q(K_{\overline{x}}, \mathbb{G}_m)$ for $q > 0$. □

10. Cohomology of a curve

In this section we follow [Mil98, Chapter 14] to explicitly compute the étale cohomology groups for a nonsingular curve over an algebraically closed field. With a little extra work, these results can be considerably generalized. We will see that these groups correspond quite closely to the complex cohomology groups, as predicted by Theorem 8.12.

Let X be a connected nonsingular curve of genus g over an algebraically closed field k.

Recall that for any scheme X, $\operatorname{Pic}(X)$ is defined to be the group of isomorphism classes of invertible sheaves, which is exactly $H^1(X_{\text{Zar}}, \mathbb{G}_m)$.

Theorem 10.1.

$$H^p(X_{\acute{e}t}, \mathbb{G}_m) = \begin{cases} k^\times & \textit{if } p = 0 \\ \operatorname{Pic}(X) & \textit{if } p = 1 \\ 0 & \textit{if } p > 1. \end{cases}$$

Proof. When $p = 0$, we know that $H^0(X_{\text{ét}}, \mathbb{G}_m)$ is $\mathcal{O}_X(X)^\times$, which is just $k^\times$. When $p = 1$, we apply Theorem 9.5; for $p > 1$, this is Theorem 9.6. □

We have a natural notion of degree for divisors, which is preserved by linear equivalence; we define $\operatorname{Pic}^0(X)$ to be the group of divisors of degree zero. This group forms an abelian variety, called the Jacobian, of dimension g.

Proposition 10.2. *The sequence*

$$0 \to \operatorname{Pic}^0(X) \to \operatorname{Pic}(X) \to \mathbb{Z} \to 0$$

is exact. If n is relatively prime to the characteristic of k, the map $z \mapsto nz$ from $\operatorname{Pic}^0(X)$ *to itself is surjective. Its kernel is a free $\mathbb{Z}/n\mathbb{Z}$-module of rank $2g$.*

Proof. The sequence is exact by construction, since the right-hand map is exactly the degree map. Since $\operatorname{Pic}^0(X)$ is an abelian variety of dimension g and n is prime to the characteristic of k, the result follows from the theory of abelian varieties. □

With these two results in hand, we are finally able to explicitly calculate some étale cohomology groups that are genuinely new.

Theorem 10.3. *Let n be an integer coprime to the characteristic of k. Then*

$$H^p(X_{\acute{e}t}, \mu_n) = \begin{cases} \mu_n(k) & \text{if } p = 0 \\ (\mathbb{Z}/n\mathbb{Z})^{2g} & \text{if } p = 1 \\ \mathbb{Z}/n\mathbb{Z} & \text{if } p = 2 \\ 0 & \text{if } p > 2. \end{cases}$$

Proof. We have the Kummer sequence

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 0.$$

Applying $H^{\cdot}(X_{\text{ét}}, -)$, we get a long exact sequence

$$\begin{aligned} 0 \to \mu_n(X) \to \mathbb{G}_m(X) \xrightarrow{n} \mathbb{G}_m(X) & \\ \to H^1(X_{\text{ét}}, \mu_n) \to H^1(X_{\text{ét}}, \mathbb{G}_m) &\xrightarrow{n} H^1(X_{\text{ét}}, \mathbb{G}_m) \\ \to H^2(X_{\text{ét}}, \mu_n) \to H^2(X_{\text{ét}}, \mathbb{G}_m) &\xrightarrow{n} H^2(X_{\text{ét}}, \mathbb{G}_m) \to \cdots \\ \to H^{p-1}(X_{\text{ét}}, \mathbb{G}_m) \to H^p(X_{\text{ét}}, \mu_n) &\to H^p(X_{\text{ét}}, \mathbb{G}_m) \to \cdots. \end{aligned}$$

Using Theorem 10.1 this sequence becomes

$$\begin{aligned} 0 \to \mu_n(k) \to k^\times \xrightarrow{n} k^\times \to H^1(X_{\text{ét}}, \mu_n) & \\ \to \operatorname{Pic}(X) \xrightarrow{n} \operatorname{Pic}(X) \to H^2(X_{\text{ét}}, \mu_n) \to 0 & \end{aligned}$$

and

$$0 \to H^p(X_{\text{ét}}, \mu_n) \to 0.$$

The result is now clear from Proposition 10.2 and the surjectivity of the map $n\colon k^\times \to k^\times$. $\square$

11. Galois module structure

So far, all our sheaves have been of abelian groups, and the resulting cohomology groups have simply been groups. However, in many situations (for example, in the context of the Weil conjectures) we have a natural Galois action on the variety in question, and we are interested in a Galois module structure on the cohomology groups. We will briefly discuss how such a Galois module structure might be constructed.

If X is a scheme in characteristic p, then we saw in [Chê04] how to produce a morphism which acts nontrivially on the cohomology groups of X. The situation was rather subtle as X was not necessarily defined over $\mathbb{F}_p$. We will address a simpler situation, where we are interested only in automorphisms which preserve the field of definition of X; as a result we can deal with a relativized situation.

Let $k \hookrightarrow k'$ be a field extension. Then $\operatorname{Spec} k' \to \operatorname{Spec} k$ is an étale map (not necessarily of finite type). The automorphism group of $\operatorname{Spec} k'$ as a scheme over k is exactly the Galois group of k' over k.

Let $S' \to S$ be an étale map, and let G be the automorphism group of S' as an S-scheme. Then if X is an S-scheme, then G acts on $X \times_S S'$, fixing X. By the functoriality of H^p, G acts as automorphisms of $H^p(E/(X \times_S S'), \mathcal{F})$, where $\mathcal{F}$ is a sheaf of Galois modules.

Appendix A. Set theory and Universes

When a site is defined on a category that is not small, many set-theoretic difficulties arise; direct and inverse limits can be a problem, coverings need to be carefully defined to ensure that they are actually sets, and so forth. One must use a certain amount of care to avoid these difficulties. This care is taken in [SGA72a], [SGA72b], and [SGA73], but it makes them very difficult to read. The goal of this appendix is to indicate to the reader the direction that is taken in those sources, and perhaps to reduce the mystery associated to the many occurences of "$\mathscr{U}$" in the text above. For further detail specifically on universes, see [SGA72a, the appendix to Exposé 1].

A.1. **Universes.**

Definition A.1. *A nonempty set $\mathscr{U}$ is called a* universe *if:*

(1) *If $x \in \mathscr{U}$ and $y \in x$, then $y \in \mathscr{U}$.*
(2) *If $x \in \mathscr{U}$ and $y \in \mathscr{U}$ then $\{x, y\} \in \mathscr{U}$.*
(3) *If $x \in \mathscr{U}$ then the power set $\mathscr{P}(x) \in \mathscr{U}$.*
(4) *If $\{x_i \text{ for } i \in I\}$ is a family of elements $x_i \in \mathscr{U}$ indexed by $I \in \mathscr{U}$, then*

$$\bigcup_{i \in I} x_i \in \mathscr{U}.$$

Example A.2. Let U be the collection of all finite sets of the form $\emptyset$, $\{\emptyset\}$, $\{\{\emptyset\}, \{\{\emptyset\}\}\}$, and so on. Then U is a universe, since only finite unions are required.

Unfortunately, the universe U just described is the only nonempty universe one can actually describe in any reasonable way: the moment one includes an infinite set, one is stuck with a hierarchy of increasingly huge sets.

Remark A.3. Every set x in a universe $\mathscr{U}$ has cardinality smaller than $\mathscr{U}$ itself by property (1). If $\mathscr{U}$ contains a set x of cardinality equal to $\mathscr{U}$, then it will also contain $P(x)$ which will have cardinality greater than $\mathscr{U}$, which is impossible. In particular, $\mathscr{U} \notin \mathscr{U}$, which is reassuring.

Remark A.4. The cardinality of any universe is a strongly inaccessible cardinal, if that clarifies matters.

Proposition A.5. *Let $\mathscr{U}$ be a universe, and x and y be elements of $\mathscr{U}$. Then*

(1) *the set $\{x\} \in \mathscr{U}$,*
(2) *if $z \subset x$, then $z \in \mathscr{U}$,*
(3) *the pair $(x, y) = \{\{x, y\}, x\} \in \mathscr{U}$,*
(4) *the union $x \cup y \in \mathscr{U}$,*
(5) *the product $x \times y \in \mathscr{U}$, and*
(6) *if $I \in \mathscr{U}$ and for each $i \in I$, $x_i \in \mathscr{U}$, then*

$$\prod_{i \in I} x_i \in \mathscr{U}.$$

A universe therefore provides a collection of "small" sets that is nevertheless closed under most reasonable set-theoretic operations. But is there an infinite universe?

Theorem A.6 (Axiom UA)**.** *For any set x, there is a universe containing x.*

This axiom seems reasonable. In fact, it can be shown that it is independent of the usual axioms of set theory. One can also substitute the axiom "every cardinal is smaller than some strongly inaccessible cardinal".

A second additional, technical, axiom is also taken:

Theorem A.7 (Axiom UB)**.** *Let P be a proposition. If there is an element $y \in \mathscr{U}$ satisfying P then the set of all elements $x \in \mathscr{U}$ satisfying P is also in $\mathscr{U}$.*

Together, the consistency of these axioms with the rest of set theory seems to be unproven, and perhaps unprovable. If we accept them, however, we can then move on to defining categories that avoid many set-theoretic problems.

A.2. **Constructions involving universes.**

Definition A.8. *A set is called* $\mathscr{U}$-small *if it is isomorphic to an element of $\mathscr{U}$. Similar terminology will be used for rings, categories, and so on.*

Definition A.9. *A category C is a* $\mathscr{U}$-category *if for each x, y in C,* $\mathrm{Hom}(x, y)$ *is $\mathscr{U}$-small.*

Example A.10. The category $\mathscr{U}$**-Set** of subsets of $\mathscr{U}$ is not $\mathscr{U}$-small (since there are too many objects) but it is a $\mathscr{U}$-category. Note also that if C is a $\mathscr{U}$-small category, the category of functors $\mathrm{Funct}(C, \mathscr{U}\textbf{-Set})$ is a $\mathscr{U}$-category, even though the set of objects is not a subset of $\mathscr{U}$, nor are the Hom-sets elements of $\mathscr{U}$.

Proposition A.11. *Let C be a $\mathscr{U}$-small category, and let D be a $\mathscr{U}$-small category (respectively, a $\mathscr{U}$-category). Then* $\mathrm{Funct}(C, D)$ *is again a $\mathscr{U}$-small category (respectively, a $\mathscr{U}$-category).*

This proposition is of interest because, in the general setting, an inverse limit of objects in C indexed by I is constructed using elements of $\mathrm{Funct}(I, C)$:

Definition A.12. *Let I be a $\mathscr{U}$-small category and let C be a $\mathscr{U}$-category. For each $x \in C$, construct the trivial category X on x (having a single object and a single arrow). Call the unique functor from I to X e_X, and call the inclusion functor i_X. Let $k_X : I \to C$ be defined by $k_X = i_X \circ e_X$. Then consider the category* $\mathrm{Funct}(I, C)$*. Let $G \colon I \to C$ be a functor. Define a presheaf $L \colon C \to \mathscr{U}$-Set by the following rule:*

$$X \mapsto \mathrm{Hom}_{\mathrm{Funct}(I,C)}(k_X, G).$$

We call L the inverse limit *of G, and we write*

$$L = \varprojlim G.$$

It often occurs that the functor L is representable, that is, $L = \mathrm{Hom}(-, L')$ for some object $L \in C$. If this occurs, we write

$$\varprojlim G = L'$$

and say that the object L' is the inverse limit *of G.*

It is fairly straightforward to verify that this definition is essentially a translation of the universal property defining an inverse limit into very general terms.

A.3. **Summary.** If we fix a universe $\mathscr{U}$, we can define many $\mathscr{U}$-categories by simply taking those objects and morphisms that are in $\mathscr{U}$ and also in some familiar category. One must verify that the Hom sets are always $\mathscr{U}$-small, but this is guaranteed if they are functions of points.

Constructing $\mathscr{U}$ versions of given categories allows one to circumvent some set-theoretic difficulties, but it does not solve them. Generally, an operation that poses some set-theoretic difficulty in a category, poses no such difficulty in the $\mathscr{U}$-version of this category, but it does pose the problem that one must verify that the resulting object lives inside $\mathscr{U}$, or at least each isomorphism class has an element inside $\mathscr{U}$. This is generally more tractable than the problem of verifying that something is a set, since one has the tools of set theory at one's disposal.

To circumvent problems with limits over index categories that are too big, for example, one generally attempts to find categories that are cofinal but $\mathscr{U}$-small.

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