On Hodge Theory and DeRham Cohomology of Variétiés

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Chapter 1

Some geometry of sheaves

1.1 The exponential sequence on a $\mathbb{C}$-manifold

Let X be a complex manifold. An amazing amount of geometry of X is encoded in the long exact cohomology sequence of the **exponential sequence** of sheaves on X:

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \stackrel{\exp}{\to} \mathcal{O}_X^\times \to 0,$$

where exp takes a holomorphic function f on an open subset U to the invertible holomorphic function $exp(f) := e^{(2\pi i)f}$ on U; notice that the kernel is the constant sheaf on $\mathbb{Z}$, and that the exponential map is surjective as a morphism of sheaves because every holomorphic function on a polydisk has a logarithm. Taking sheaf cohomology we get

$$0 \to \mathbb{Z} \to H(X) \stackrel{\exp}{\to} H(X)^\times \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^\times) \to H^2(X, \mathbb{Z}),$$

where we have written $H(X)$ for the ring of global holomorphic functions on X. Now let us reap the benefits:

I. Because of the exactness at $H(X)^\times$, we see that any nowhere vanishing holomorphic function on any simply connected $\mathbb{C}$-manifold has a logarithm – even in the complex plane, this is a nontrivial result.

From now on, assume that X is compact – in particular it homeomorphic to a finite CW complex, so its Betti numbers $b_i(X) = \dim_{\mathbb{Q}} H^i(X, \mathbb{Q})$ are finite. This also implies [Cartan-Serre] that $h^i(X, F) = \dim_{\mathbb{C}} H^i(X, F)$ is finite for all **coherent analytic sheaves** on X, i.e. locally on X F fits into an exact sequence $\mathcal{O}_U^m \to \mathcal{O}_U^n \to F \to 0$. Especially the **Hodge numbers** $h^{p,q} = h^p(X, \Omega_X^q)$ are finite.

II. $H^1(X, \mathcal{O}^\times)$ is the **Picard group** of holomorphic line bundles on X. The map $c : H^1(X, \mathcal{O}_X^\times) \to H^2(X, \mathbb{Z})$ is the **Chern class map**; the image of c modulo torsion is the **Néron-Severi group** $NS(X) \cong \mathbb{Z}^\rho$, which classifies line bundles up to **algebraic equivalence**; one says that ρ is the **Picard number** and that $b_2 - \rho$ is the **number of transcendental cycles**. Line bundles in the kernel of c are said to be **algebraically equivalent to zero** and this subgroup of $\mathrm{Pic}(X)$ is denoted by $\mathrm{Pic}^0(X)$. From the exact sequence we see $\mathrm{Pic}^0(X) \cong H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$; In case we have $2h^{0,1} = b_1$ (which will occur if X is Kahler) we are modding out a $\mathbb{C}$-vector space by a full sublattice – that is, the Picard group is a **complex torus**. Notice that if X is simply connected (and Kahler) then $\mathrm{Pic}^0(X) = 0$, and the group of line bundles is just the finitely generated discrete group $H^2(X, \mathbb{Z})$; this occurs e.g. for $X = \mathbb{P}^n$. Finally, if X is Kahler then we will see in the next chapter that we have a Hodge decomposition

$$H^2(X, \mathbb{C}) = H^2(X, \mathcal{O}_X) + H^0(X, \Omega_X^2) + H^1(X, \Omega_X).$$

The N'eron-Severi group canonically is a $\mathbb{Z}$-lattice in $H^2(X, \mathbb{C})$, and it is contained in the $(1,1)$-subspace [Griffiths-Harris ??]; thus the Picard inequality can be refined to $\rho \leq h^{1,1}$.[1] Moreover one has (still in the Kahler case) that $H^1(X, \Omega_X) \cap (\mathrm{Im}(H^2(X, \mathbb{Z})) = NS(X)$, i.e., every integral $(1,1)$-form comes from the Picard group, the **Lefschetz (1,1) Theorem**. More generally, to an i-cycle Z on X – i.e., a $\mathbb{Z}$-linear combination of closed analytic subsets of dimension i – we can associate, via a triangulation and Poincaré duality, a cohomology class $c(Z) \in H^{2d-2i}(X, \mathbb{Z})$, and one finds that the image of $c(Z) \in H^{2d-2i}(X, \mathbb{C})$ lands in the $(d-i, d-i)$-subspace. Suppose finally that X is projective. It is not quite true that every integral $(d-i, d-i)$-class needs to be represented by an algebraic cycle, but this is supposed to be true "with denominators": i.e., every element of $H^{(d-i),(d-i)} \cap H^{2d-2i}(X, \mathbb{Q})$ should be a $\mathbb{Q}$-linear combination of algebraic cycles; this is the **Hodge conjecture**.[2]

1.2 Fiber bundles, locally constant sheaves, monodromy

1.2.1 Fiber bundles as an example of descent

We review the notion of an (F, G)-bundle on a topological space, the classification via sheaf cohomology, and the special place that locally constant sheaves have among fiber bundles.

Let F be another topological space. A map $\pi : E \to X$ is said to be an F-fiber bundle over X if there is an open covering $\{U_i\}$ of X such that π_{U_i} is isomorphic, over X, to the product $F \times U_i$; such an isomorphism is called a **local trivialization** $\varphi_i : E_{U_i} \to F \times U_i$. The "data" for a fiber bundle are

[1] One still need not have equality; e.g. for any d-dimensional complex abelian variety, one has $h^{1,1} = b_2 - 2h^{1,0} = \frac{2d}{2} - 2d$, whereas the rank of the Néron-Severi group is characterized in terms of the endomorphism algebra (the Rosati-invariant subalgebra), so is generically just 1 but can be as large as $\frac{d(d+1)}{2}$, attained when A is the dth power of a CM elliptic curve.

[2] Apparently the Hodge conjecture was first formulated in its integral version but was proven false with embarrassing swiftness. I do not know the full story nor even the counterexample.

its **transition functions**: namely on the overlaps $U_i \cap U_j$ we may consider the composite $\rho_{ij} := \varphi_j \circ \varphi_i^{-1} : U_i \cap U_j \to \mathrm{Aut}(F)$. The compatibility among triple intersections is equivalent to ρ being a **one-cocycle** in $Z^1(X, \mathrm{Aut}(F)_c)$; this is a (nonabelian) Cech cohomology group, and if G is any topological group, by G_c we mean the sheaf of **continuous** G-valued functions on X. This has been formulated in the topological category, but is easily modified: if X is a real manifold and G a real Lie group, we can work with G_∞, the sheaf of smooth functions $X \to G$; if X is a complex manifold and G a complex Lie group, we can work with G_h, the sheaf of holomorphic functions $X \to G$. (Unless we are considering more than one of these categories at once, we may abusively write just G, trusting that the context will make clear whether we are working with continuous, smooth or holomorphic functions.)

On the other hand, we probably do not want the transition functions to be *arbitrary* automorphisms of F – for instance if $F = \mathbb{R}^n$ its automorphism group is an enormous (infinite-dimensional) space. This leads to (F, G)-bundles: we prescribing a **structure group** $G \leq \mathrm{Aut}(F)$ and requiring the transition functions to lie in G. It is not news, but this simple idea is miraculous in its range of applicability. For instance if $F = \mathbb{R}^n$ and we want to get real vector bundles, we take $G = \mathrm{GL}_n(\mathbb{R})$; similarly if $F = \mathbb{C}^n$ we get complex vector bundles; if $G = \mathrm{GL}_n^+(\mathbb{R})$ we get oriented vector bundles; if $G = SO_n(\mathbb{R})$ we get oriented vector bundles endowed with a Riemannian metric, and so on. The basic result is as follows:

Proposition 1 *The set of (F, G)-bundles on X is naturally in bijection with $H^1(X, G_c)$; under the correspondence the trivial F-bundle corresponds to the identity cocycle.*

Given an acquaintance with Cech cohomology (we are passing to the direct limit over refinements of covers, of course), the proof is almost immediate: we have associated a Cech class to a fiber bundle; conversely, given a cocycle $\rho_{ij} \in Z^1(\{U_i\}, G_c)$, we form the space $\coprod_i U_i \times F$ and mod out by $(u, f) \sim (u, g_{ij}(u)f)$ whenever $u \in U_i \cap U_j$. One striking aspect of the correspondence is that the fiber F appears on one side but not on the other! One take on this is that it is enough to consider **principal** bundles, i.e., where the fiber $F = G$ acting (left or right; one must choose) regularly on itself.

Another viewpoint is that we have an instance of what (following Serre in the case of Galois cohomology) I call the **first principle of descent**: we start with an "object" F_0 on a "space" X (here we have a topological space; for algebraic purposes probably the best example is the flat site of a scheme, e.g. $\mathrm{Spec}\, k$!), and a covering $\{U_i\}$ of X. Let $Y = \coprod U_i$; there is a natural surjective local homeomorphism[3] $\pi : Y \to X$. An object F on X such that $\pi^* F \cong \pi^* F_0$ is called a **twisted form** of F_0; denote by $\mathcal{T}_{Y/X}(F_0)$ the space of all twisted forms

[3]which is not necessarily a covering map – it need not be "flat," i.e., the fibers may have different cardinalities

which are trivialized over Y. In our case every (F,G)-bundle E over X admits a covering such that the pullback to Y is equal to the pullback to Y of the trivial bundle. Then:

Proposition 2 *(Descent principle) The pointed set $\mathcal{T}_{Y/X}(F_0)$ of Y/X-twisted forms is naturally in bijection with $H^1(X, \mathrm{Aut}(\pi^* F_0))$.*

We remark that there is no typo: the automorphism group of the trivial objects pulled back to Y is a sheaf (of not necessarily abelian groups) on X: its sections over Y are indeed $\mathrm{Aut}(\pi^* F_0)$ and its sections over X', $Y \to X' \to X$ are the X'-equivariant automorphisms of $\mathrm{Aut}(\pi^* F_0)$, and its sections over an arbitrary covering Z of X are the same as its sections over $Z \times_X Y$.

Corollary 3 *Any two objects F_0 and G_0 on X – however dissimilar! – such that $\mathrm{Aut}(\pi^* F_0) \cong \mathrm{Aut}(\pi^* G_0)$ will have bijectively corresponding sets of Y/X-twisted forms: $\mathcal{T}_{Y/X}(F_0) \cong \mathcal{T}_{Y/X}(F_0)$.*

Here is an application of this:

Let X be a (real or complex) manifold, and consider the set of finite rank projective $\mathcal{O}_X$-modules. One can interpret "projective" purely algebraically: for all open subsets U, $M(U)$ is a finitely generated module over the ring $\mathcal{O}_X(U)$. Also as a matter of pure algebra, every finite rank projective module over a commutative ring R becomes free over a Zariski-open subset of every point of R – this is more than enough to ensure the existence of an open cover $\{U_i\}$ such that the pullback of M to $\coprod U_i$ is a free $\mathcal{O}_X$-module. So the set of rank n projective $\mathcal{O}_X$-modules is classified by $H^1(X, \underline{\mathrm{Aut}}(\mathcal{O}_X^n))$, where $\underline{\mathrm{Aut}}(M) = \underline{\mathrm{End}}_{\mathcal{O}_X - Mod}(M,M)^\times$ is the sheaf of automorphisms of the $\mathcal{O}_X$-module M, i.e., over any open subset U we take the $\mathcal{O}_U$-module automorphisms of $M|U$. We have that $\underline{\mathrm{Aut}}(\mathcal{O}_X^n) = \mathrm{GL}_n(\mathcal{O}_X) = (\mathrm{GL}_n)_h$. Because this is the same automorphism group for a rank n holomorphic vector bundle, we conclude:

Proposition 4 *On any (real or complex) manifold X, there is a canonical bijection of pointed sets between rank n projective $\mathcal{O}_X$-modules and (real or holomorphic) rank n vector bundles on X.*

Of course, in such a situation, one would like to have an explicit bijection. The descent principle does not tell us how to write down such a bijection (but assures us that we will find one, which gives us the motivation to look). In this case it is easy to go from a vector bundle to a locally free sheaf: we just take the sheaf of local sections. The inverse is not as transparent – see [Hartshorne, pp. 128-129].

We mention in passing one more example: let X be a scheme and $G = \mathrm{PGL}_{n\,X}$, considered as a representable sheaf on the étale site of X. Since PGL_n is the common automorphism group of both $\mathbb{P}_X^{n-1}$ and $M_n(X)_X$ (matrix algebra bundle), we find a canonical correspondence between **projective bundles** on X and bundles of central simple algebras on X, i.e. **Azumaya algebras**. This leads to the interpretation of the **Brauer group of** X as classifying both geometric and algebraic objects on X. For more details, see either [Grothendieck

I,II,III] or (for a much-abbreviated version) [Clark].

A look at complex line bundles: suppose we want to study complex line bundles on a real manifold X. If we (temporarily) write $\mathcal{O}_X$ for the sheaf of $\mathbb{C}$-valued C^∞-functions on X, then we still have the exponential sequence $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 0$: even a smooth function nonzero at a point is, locally about that point, the exponential of another smooth function. But since $\mathcal{O}_X$ is **fine**, it is acyclic for sheaf cohomology, and the cohomology of the exponential sequence gives an isomorphism

$$c_1 : H^1(X, \mathcal{O}_X^\times) \xrightarrow{\sim} H^2(X, \mathbb{Z}).$$

That is, in the smooth category, a complex line bundle is determined by its Chern class.

When X is a complex manifold, this need not be the case, as we saw in the previous section: the kernel of c_1 is the complex torus $H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$, whose dimension is the first Betti number $b_1(X)$. Thus we see that $c_1(L)$ determines L if and only if $b_1(X) = 0$, in particular if X is simply connected.

1.2.2 Locally constant sheaves

The aim of this section is to define locally constant sheaves, understand their relation to fiber bundles, and give their monodromy classification.

Definition: A **locally constant sheaf** of abelian groups F on X is a sheaf for which there admits a cover $\{U_i\}$ of X such that $F|_{U_i}$ is isomorphic to a constant sheaf. If X is connected, the stalks of a locally constant sheaf are mutually isomorphic to a common abelian group Λ.

Relation with fiber bundles: we can use the principle of descent to associate a fiber bundle to a locally constant sheaf: A locally constant sheaf with group Λ is a twisted form of the constant sheaf with group Λ, whose sheaf of automorphisms is just $\mathrm{Aut}(\Lambda)_c$. However, since Λ is merely an "abstract" abelian group, $\mathrm{Aut}(\Lambda)$ is given the **discrete** topology, and $\mathrm{Aut}(\Lambda)_c$ means the sheaf of **locally constant** functions from X to the group $\mathrm{Aut}(\Lambda)$. We can similarly speak of locally constant sheaves with G-structure, where $G \leq \mathrm{Aut}(\Lambda)$ – in particular taking, $\Lambda = \mathbb{R}^n$ or $\mathbb{C}^n$ and $G = \mathrm{GL}_n(\mathbb{R})$ or $\mathrm{GL}_n(\mathbb{C})$, we have a notion of locally constant sheaves of vector spaces.

Write $\underline{G}$ for the constant sheaf with group G. Since $\underline{\mathrm{GL}_n} \hookrightarrow (\mathrm{GL}_n)_c$ (or $(\mathrm{GL}_n)_\infty$ or $(\mathrm{GL}_n)_h$), the mapping

$$H^1(X, \underline{\mathrm{GL}_n}) \to H^1(X, (\mathrm{GL}_{n\ c/\infty/h})$$

shows that any locally constant sheaf of vector spaces can be viewed as a continuous, smooth or holomorphic vector bundle on X, albeit one of a very special form.

Example: Consider the sheaf of differentials Ω^1 on $\mathbb{P}^1/\mathbb{C}$ as in [Hartshorne], [C-K]. In terms of transition functions, it is given by the standard covering $U_1 = \mathbb{P}^1 - \infty = \mathbb{A}^1[x]$, $U_2 = \mathbb{P}^1 - 0 = \mathbb{A}^1[y]$ and with transition function $\rho_{12} : U_1 \cap U_2 \to \mathcal{O}^{\times}_{U_1 \cap U_2}$ given by $-1/x^2$. This is not a locally constant function! Moreover, since its divisor has degree -2, the line bundle is nontrivial. We will soon see that there are no nontrivial locally constant sheaves on $\mathbb{P}^1(\mathbb{C}) = S^2$, so that the line bundle Ω^1 cannot be given by locally constant transition functions.

We should also give an example of a locally constant sheaf that is not constant! Let $X = S^1$, and let $E \to X$ be the Mobius band, which *a priori* is a real line bundle on X in the broader sense of the previous section, famously nontrivial. But the structure group can be reduced from $\mathbb{R}^\times$ to $\mathbb{Z}/2\mathbb{Z}$, i.e., it is a locally constant sheaf. This is also well known and easy to check: indeed when we make the Mobius band out of two strips U_1 and U_2 with two components of intersection $U_1 \cap U_2 = V_{12}^1 \coprod V_{12}^2$, at one end we glue U_1 to U_2 identically, and at the other hand we glue by a uniform half twist – i.e., the first transition function is 1 and the second is -1.

Of course we could untwist the Mobius band by pulling back via $z^2 : S^1 \to S^1$, and this leads us to suspect that the reason there are no nonconstant locally constant sheaves on S^2 is that it is simply connected. This is true and leads us directly to the considerations of the next section.

1.2.3 Monodromy

In this section, we will need covering space theory to be applicable to X, so we suppose that X is connected, locally path-connected and semi-locally simply connected – in particular, it has a universal cover $\tilde{X} \to X$.

Let $f : X \to Y$ be a continuous map of topological spaces and F a locally constant sheaf on Y. Then f^*F is a locally constant sheaf on X.

Let F be a locally constant sheaf on X with fibers isomorphic to Λ. Fix a basepoint $x \in X$, and let $\gamma : [0,1] \to X$ be a loop based at x. Then by the remark, γ^*F is a locally constant sheaf on $[0,1]$. But we claim that any locally constant sheaf on the unit interval is constant. By an immediate compactness argument, it comes down to showing: if we have a sheaf F on $I = I_1 \cup I_2$ a union of overlapping intervals such that $F|_{I_1}$ and $F|_{I_2}$ are both isomorphic to constant sheaves, then so is F itself. But this is itself a kind of descent argument, involving the familiar ([Hartshorne], [Alon])

Lemma 5 *(Glueing lemma) If $\{U_i\}$ is an open cover of X and we have sheaves F_i on each U_i and the data of an isomorphism $\varphi_{ij} : F_i|_{U_i \cap U_j} \to F_j|_{U_i \cap U_j}$ satisfying the conditions $\varphi_{ii} = 1$, $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$, then there is a unique sheaf F on*

X together with isomorphisms $\psi_i : F|_{U_i} \xrightarrow{\sim} F_i$ such that $\psi_j \circ \psi_i^{-1} = \varphi_{ij}$.

We leave it to the reader to check that the glueing lemma implies the following generalization of our claim: let $X = U_1 \cup U_2$ such that $U_1 \cap U_2$ is connected. Then a sheaf F on X which restricts to a constant sheaf on U_1 and on U_2 is already constant on X.

Back to the case of our loop $\gamma : [0,1] \to X$. We now know that there is an isomorphism $\Psi : \gamma^* F \cong \underline{\Lambda}_{[0,1]}$: in particular we have $\Psi(0) : (\Gamma^* F)_0 \cong \Lambda$ and $\Psi(1) : (\Gamma^* F)_1 \cong \Lambda$. On the other hand, the stalks at 0 and 1 are identified with the stalks at $\gamma(0) = x = \gamma(1)$. It follows that the trivialization Ψ gives rise to an automorphism $\Psi(1) \circ \Psi(0)^{-1}$ of the stalk of F at x.

Exercise: Suppose $\gamma_1 \sim \gamma_2$ are homotopic paths. Show that the induced automorphisms are the same. (Hint: View the homotopy as giving a morphism $[0,1] \times [0,1] \to X$, divide the square into sufficiently small nicely overlapping squares on which the pulled back sheaf is constant, and argue as in the previous exercise.)

It follows that we have defined a homomorphism $\pi_1(X,x) \to \mathrm{Aut}(F_x)$, called the **monodromy representation**.

Theorem 6 *The monodromy representation gives a categorical equivalence between G-structured Λ-locally constant sheaves on X and G-compatible $\pi_1(X)$-module structures on Λ.*

Proof: We shall construct the inverse functor. Our hypotheses are such as to ensure that there is a universal cover $\tilde{X} \to X$, and since the theorem, if true, implies that the pullback of any locally constant sheaf to $\tilde{X}$ will be constant, this suggests our strategy: given the data of $\rho : \pi_1(X,x) \to \mathrm{Aut}\,\Lambda$, we will construct a locally constant sheaf F_ρ on X by descent from a constant sheaf on $\tilde{X}$. Indeed, let $\tilde{E} = \tilde{X} \times \Lambda$ (Λ is viewed as a discrete space), and consider the quotient space $E := \tilde{E}/\sim$, where $(x,f) \sim (gx, gf)$ for all $g \in \pi_1(X,x)$. Clearly projection onto the first factor gives a map $\pi : E \to X$. Since $\pi_1(X,x)$ acts discretely on $\tilde{X}$, every point of $\tilde{X}$ has a neighborhood $\tilde{U}$ such that $\tilde{U} \times F$ is mapped homeomorphically onto its image – i.e., $\pi : E \to X$ is a fiber bundle over F. We leave it as an exercise to show that the associated sheaf of local sections to π is a locally constant sheaf, and that this construction is indeed inverse to our association of a representation to a locally constant sheaf.

Remark: It is useful to recall that there were two steps to the proof of the monodromy theorem, the first being an argument that every locally constant fiber bundle became trivial when pulled back to the universal cover, and the second being an interpretation of such bundles as being equivalent to (G-)$\pi_1(X)$-module structures on Λ. Note also that it is certainly not always the case that a fiber bundle on X must trivialize on its universal cover (again recall Ω_1 on

$\mathbb{P}^1(\mathbb{C}) = S^2$), but this happens often enough that it is worth abstracting the second part of the argument as follows:

Proposition 7 *Let X be a topological space with universal cover $\pi : \tilde{X} \to X$ and fundamental group $\mathfrak{g}$. Then the pointed set of (topological, smooth or holomorphic) (F, G)-bundles which trivialize over $\tilde{X}$ is isomorphic to the group cohomology set $H^1(\mathfrak{g}, \pi^*(G))$.*

Compare with [Mumford, pp. 22-23] for an analogoue valid for all sheaves $\mathcal{F}$ on X (but with a somewhat weaker conclusion). The point is that the automorphism group (sheaf) G of a locally constant sheaf is a trivial $\mathfrak{g}$-module, but this is not necessarily true for more general fiber bundles.

Example: In the topological category, every fiber bundle over a contractible paracompact base is trivial, a consequence of the following basic result.

Theorem 8 *(Covering Homotopy Theorem) Let $\pi : E \to Y$ be an (F, G)-bundle over a paracompact base. Let $g_1,\ g_2 : X \to Y$ be homotopic maps. Then the pullbacks $g_1^*\pi$ and $g_2^*\pi$ are isomorphic. In particular, every (F, G)-bundle over a contractible base is trivial.*

For the proof see e.g. [Milnor-Stasheff]. It follows that (F, G)-bundles over a $K(\pi, 1)$-space[4] are classified by $\mathrm{Hom}(\pi, G)$. There is, up to homotopy equivalence, a unique $K(\pi, 1)$-space for each group π, but we are rather lucky if it is finite-dimensional – for instance, there is an isomorphism $H^\bullet(\pi, \Lambda) \cong H^\bullet(K(\pi, 1), \Lambda)$ from the group cohomology of π (with coefficients in the trivial π-module Λ) to the singular cohomology of the Eilenberg-MacLane space [Brown], so for instance, for any $n > 1$, $H^k(K(\mathbb{Z}/n\mathbb{Z}, 1), \mathbb{Z}/n\mathbb{Z}) \neq 0$ for every even k. (One knows that in fact $K(\mathbb{Z}/2\mathbb{Z}, 1) = \mathbb{R}P^\infty = \varinjlim_n \mathbb{R}P^n$, and the other $K(\mathbb{Z}/n\mathbb{Z}, 1)$'s are infinite-dimensional "lens spaces.") But for all n, $T^n := S^1 \times \ldots \times S^1$ is $K(\mathbb{Z}^n, 1)$. It follows that the (F, G)-bundles on an n-dimensional real torus are classified by $\mathrm{Hom}(Z^n, G) = G^n$.

Example: A coherent analytic sheaf on a **Stein manifold** is acyclic for sheaf cohomology; this is a theorem due to Serre which is the analytic analogue (but proved first!) of the acyclity of coherent sheaves on affine varieties. In particular, a nonsingular affine analytic space is a Stein manifold – so $\mathbb{C}^n$ is a Stein manifold. If we further assume that $b_1(X) = h_1(X, \mathbb{Z}) = 0$, then the exponential sequence gives $H^1(X, \mathcal{O}_X^\times) = 0$, whence:

Proposition 9 *Let X be a complex manifold with fundamental group $\mathfrak{g}$ whose universal cover $\tilde{X}$ is Stein. Then holomorphic line bundles on X are classified by the group cohomology group $H^1(\mathfrak{g}, \mathcal{O}_{\tilde{X}}^\times)$.*

In particular this result applies to **complex tori**, and we get the fact that we can represent any line bundle on $\mathbb{C}^n/\Lambda$ as a collection of functions $\Lambda \to \mathcal{O}_{\mathbb{C}^n}^\times$

[4] A space with a contractible universal cover and fundamental group π. These are also called **Eilenberg-MacLane** spaces.

satisfying the cocycle condition, i.e., by **theta functions**.

Consider now the place of locally constant line bundles on an abelian variety among all line bundles: these are given by homomorphisms $\Lambda \to \mathbb{C}^\times$. The group of all such is just $(\mathbb{C}^\times)^{2n}$, which is not quite what we expect to see. The problem is that the map which associates to a locally constant line bundle its associated holomorphic line bundle is neither surjective nor injective. To repair matters, one considers the composite "change of structure groups"

$$S^1 \to \mathbb{C}^\times \to H(\mathbb{C}^n)^\times,$$

and it turns out that the image in the Picard group of $\mathbb{C}^n/\Lambda$ of the locally constant sheaves with structure group S^1 coincides with the image of the locally constant sheaves with structure group $\mathbb{C}^\times$, and moreover the map $T^{2n} = \mathrm{H}^1(\Lambda, S^1) \to H^1(\Lambda, \mathcal{O}_{\mathbb{C}^n}^\times)$ is *injective*; indeed the image of the composite is precisely the Picard variety of line bundles algebraically equivalent to zero. These statements are not immediate; rather, they are much of the content of the **Appell-Humbert theorem** classifying line bundles on a complex torus. We will however be able to see later that every locally constant line bundle on a complex manifold has vanishing Chern class, by showing that it admits a flat connection. This brings us to the next section.

1.3 Flat connections, especially Gauss-Manin

Let $E \to X$ be a (say complex) vector bundle on a real manifold X. In this section, we plunge to the core of differential geometry (but of course for our own nefarious, ultimately algebraic, purposes) by defining a **connection** on E: it is a $\mathbb{C}$-linear morphism of sheaves $E \to \Omega^1(E) := \Omega^1 \otimes E$ satisfying the Leibniz rule

$$D(fg) = f \cdot dg + gD(f),$$

where d denotes the usual exterior derivative. Note that D is of course **not** an $\mathcal{O}_X$-module map: the special case to keep in mind is the trivial line bundle $L_0 \cong \mathcal{O}_X$ on X; then d itself gives a connection on L_0, and differentiation is by its nature $\mathbb{C}$-linear but not $\mathcal{O}_X$-linear.

The matrix of one-forms: it is quite easy to write down connections locally. Namely, over any trivializing open subset U for E, choose a local frame $e = (e_1, \ldots, e_d)$ – i.e., sections $e_i \in \Gamma(U, E)$ such that $e_1 \wedge \ldots \wedge e_d$ is a nowhere vanishing section of the line bundle $\Gamma(U, \Lambda^d E)$. Because of the Leibniz rule, D is determined by its action on e, and can in these local coordinates be given simply by a $d \times d$ matrix with entries in Ω^1, via

$$De_i = \sum_{j=1}^{d} \theta_{ij} e_i.$$

Viewing $E|_U = \mathcal{O}^d_{X\,U}$ via e, and writing a section $s \in \Gamma(U, E)$ in vector form as $s = \sum_{i=1}^d s_i e_i = s_e \cdot e$, a short calculation gives the matrix equation

$$Ds_e = (d + \theta)s_e.$$

There is no condition to be imposed on the matrix θ, so indeed every connection can be given in local coordinates as $d + M$ where $M \in M_d(\Omega^1)$. In particular, the difference of any two connections is an $\mathcal{O}_X$-linear map.

Globally, every vector bundle over a paracompact base admits a connection: in local coordinates we can take $D = d$, and smooth via a partition of unity.

The curvature matrix: It is defined in local coordinates e on U as

$$\Theta = \Theta_{D,e} = d\theta + \theta \wedge \theta,$$

i.e., it is an $n \times n$-matrix of two-forms on U. We have the equation

$$(d + \theta_e)(d + \theta_e)s_e = \Theta s_e.$$

In other words, the curvature matrix gives an $\mathcal{O}_X$-linear map $E \to \Omega^2(E)$ which looks for all the world like $D \circ D$. Indeed it is as soon as we extend the connection to a map $D : \Omega^i(E) \to \Omega^{i+1}(E)$, via

$$D(\eta_e) = d(\eta_e) + \theta_e \wedge \eta_e,$$

or globally by continuing to enforce the Leibniz rule:

$$D(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg \omega} \omega \wedge D(\eta).$$

We say the connection D is **flat** if $\Theta = 0$.[5]

There is a clear "formal" reason to be interested in flat connections: it says precisely that the sequence

$$0 \to E \xrightarrow{D} \Omega^1(E) \xrightarrow{D} \Omega^2(E) \xrightarrow{D} \dots \tag{1.1}$$

is a **complex** of sheaves on X, a kind of generalized DeRham complex giving a resolution of the vector bundle E. We will see in Chaper 2 that the Hodge theorem can be generalized to a theorem about such a complex of sheaves.

But there are more immediate, geometric reasons to be interested in flat connections: suppose that E is a line bundle, so that Θ is just a 2-form on X. Notice that it is closed: $\theta \wedge \theta = -\theta \wedge \theta = 0$, so $d\Theta = d(d\theta + \theta \wedge \theta) = 0$. Therefore, via the DeRham theorem, $\Theta \in H^2(X, \mathbb{R})$.

[5] One also says that a connection D with $D^2 = 0$ is **integrable**, which could be preferred on the grounds that it uses a term not already ubiquitous in algebraic geometry (beware: every vector bundle is a flat module!) On the other hand, I think the differential geometers have us beat on this point: calling something which has zero curvature flat makes more sense than calling something for which tensoring with that thing is exact flat.

Proposition 10 *Suppose that $E \to X$ is a complex line bundle on a real manifold X. The curvature two-form Θ of a line bundle E is the image of the Chern class $c_1(E)$ under the natural map $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})$.*

For a proof of this, see [Griffiths-Harris] or [Wells]. In particular, a line bundle is algebraically equivalent to zero if and only if it admits a flat connection.

More generally, (almost) the entire theory of characteristic (Chern) classes of complex vector bundles on a real manifold in terms of the curvature matrix Θ. The key observation is that, if (E, D) has fiber dimension > 1, it need not be that $d\Theta = 0$ but if $P : M_n(\mathbb{C}) \to \mathbb{C}$ is any polynomial function in the matrix entries with the invariance property $P(YXY^{-1}) = P(X)$ for all matrices X and Y, then $dP(\Theta) = 0$, so that $P(\Theta) \in H^{\bullet}_{DR}(X)$. Taking $P = \sigma_k$, the kth elementary symmetric function of the eigenvalues, gives the kth Chern class $c_k(E) \in H^{2k}_{DR}(X)$, up to a scaling. The reason for the "almost" in the first sentence of this paragraph, is: since $H^{2k}_{DR}(X) \cong H^{2k}(X, \mathbb{R})$, there is a slight loss of information over the topologically defined Chern classes $c_k \in H^{2k}(X, \mathbb{Z})$: a vector bundle which admits a flat connection is such that all of its topological Chern classes are *torsion*, but not in general identically zero. For all this, see Appendix C of [Milnor-Stasheff].

1.3.1 Flat connections versus locally constant sheaves

Let us look once again at the relationship between locally constant sheaves and fiber bundles. On the geometric side – or in terms of transition functions – we saw that a locally constant sheaf of complex vector spaces *is* a vector bundle with an impressively small structure group. On the sheaf side, this is not quite true: by definition, the stalks of a $\Lambda = \mathbb{C}^n$-locally constant sheaf are all isomorphic to $\mathbb{C}^n$, whereas the corresponding locally free sheaf has stalk at P isomorphic to the much larger group $\mathcal{O}^n_{X,P}$. But this is easily remedied: to go from a $\mathbb{C}^n$-locally constant sheaf F on X to the locally free sheaf corresponding to the corresponding vector bundle, we just take

$$F \mapsto F \otimes_{\mathbb{C}} \mathcal{O}_X.$$

(Depending upon what we mean by $\mathcal{O}_X$, this makes sense and is correct in the topological, smooth, and holomorphic categories.)

But now we have another instance of descent: because it came from F, $\mathcal{F} = F \otimes \mathcal{O}_X$ can be canonically endowed with a connection: namely, in local coordinates, we take $\sigma = \sum s_i e_i \in \mathcal{F}$, and define

$$D(\sigma) := \sum ds_i \otimes e_i \in \mathcal{F} \otimes \Omega_X.$$

The point being: this expression is independent of the coordinates, because any other trivialization is obtained from the first by a transition matrix with constant coefficients; such changes of variables are d-linear.

Moreover this connection D is flat, since in local coordinates it is just d, and indeed $d^2 = 0$. We now get the result that we mentioned earlier during our discussion of line bundles on abelian varieties:

Corollary 11 *A locally constant line bundle on a complex manifold is algebraically equivalent to zero.*

Conversely, if $(E \to X,\ D)$ is a vector bundle endowed with a flat connection, we define a sheaf F_D by taking for $F_D(U)$ the **horizontal** sections over U, namely the kernel of $D|_U$. One can show (using Frobenius' integrability criterion for distributions; see [Voisin]) that F_D is a $\mathbb{C}^n$-locally constant sheaf. These two constructions are mutually inverse to each other, i.e., one has:

Proposition 12 *There is a bijective correspondence (in either the smooth or the holomorphic category) between vector bundles on a (real/complex) manifold X endowed with a flat connection and $\mathbb{C}^n$-locally constant sheaves on X.*

Note that one important consequence of this is that a *smooth* complex vector bundle which can be endowed with a flat connection necessarily admits a canonical structure of a *holomorphic* vector bundle.

1.3.2 The Gauss-Manin connection

Let $\pi : X \to B$ be a proper submersion of (smooth or complex) manifolds. The implicit function theorem guarantees that the fibers $\pi^{-1}(b)$ are themselves manifolds, allowing us to think of π as giving a **family** of manifolds over the base B. In fact, our hypotheses ensure that, in the smooth category, we have a complete understanding of the local behavior of such a family:

Theorem 13 *(Ehresmann Lemma) Let $\pi : X \to B$ be a proper smooth submersion of real manifolds over a contractible pointed base $(B, 0)$, and write $X_0 := \pi^{-1}(0)$. Then there exists a diffeomorphism over the base $T : X \xrightarrow{\sim} X_0 \times B$.*

Actually we need this result only locally, where it is a special case of the existence of tubular neighborhoods. For a proof of the global case (involving some differential topology), see [Demailly].

In other words, in the smooth category all such families are locally constant (aha!). Imagine now a holomorphic family satisfying the same hypotheses; of course it need not be holomorphically locally trivial (there are moduli spaces, after all), but the fact that it is smoothly locally trivial allows us to view the family as a **deformation** of the complex structure on a fixed fiber.

Let $\underline{A}$ be the constant sheaf on X for some group A (think of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\ldots$). Let $H^k_A := R^k\pi_*(\underline{A})$, the kth derived functor of the pushforward. One knows that $R^k\pi_*(F)$ is the sheaf associated to $U \mapsto H^k(\pi^{-1}, F|_{\pi^{-1}(U)})$. Since B is locally contractible, the Ehresmann Lemma implies that $H^k(X_0 \times U, \underline{A}) \cong$

$H^k(X_0, \underline{A})$ for a fundamental system of neighborhoods $X_0 \times U$ of B at 0. That is, $H^k_A = R^k\pi_*(A)$ is a locally constant sheaf, isomorphic in a neighborhood of 0 to $H^k(X_0, A)$.

Definition: The corresponding flat connection $\Delta : H^k \to \Omega^1(H^k)$ is called the **Gauss-Manin connection**.

When $A = \mathbb{C}$, using the remark at the end of the previous section we may view H^k as a holomorphic vector bundle on B. We will use this structure at the end of the next chapter to give a meaning to the holomorphy of the Hodge filtration.

Chapter 2

Hodge theory and DeRham cohomology: the analytic case

2.1 Introduction

Let $X/\mathbb{C}$ be a (smooth, proper, irreducible) algebraic variety of dimension d. Classically, the algebraic geometry of X was developed alongside the algebraic topology of the associated $\mathbb{C}$-manifold $X(\mathbb{C})$ – in particular the intersection theory of algebraic cycles ($\mathbb{Z}$-linear combinations of irreducible subvarieties) was understood to take place in the cohomology ring $H^\bullet(X(\mathbb{C}), \mathbb{Z})$ via a **cycle class map** $c : Z_i(X) \to H^{2d-2i}(X(\mathbb{C}), \mathbb{Z})$. It is critically important that the singular cohomology groups are nonvanishing up to dimension $2d$. Since the Zariski topology on (the associated scheme of) X is a d-dimensional Noetherian space, by Grothendieck's vanishing theorem [C-K], we have that for any sheaf F on X $H^i(X, F) = 0$ for all $i > d$, and it seems like the Zariski cohomology groups are hopelessly incapable of capturing the topological data of the Betti cohomology groups.

But we are giving up on the sheaf cohomology groups too easily: although no single sheaf F can play the role of a constant sheaf on X^{an}, we may still be able to read the data of the singular cohomology groups off of the cohomology of **a family** of sheaves on X. Indeed consider the family of sheaves $\Omega^i_{X/\mathbb{C}}$ of "regular i-forms," defined for all $i \in \mathbb{N}$. These are **coherent sheaves** of $\mathcal{O}_X$-modules on the scheme X: recall that $\Omega^0_{X/\mathbb{C}} = \mathcal{O}_X$ itself; $\Omega^1_{X/\mathbb{C}}$ is the globalization of the module of differentials. For any affine open subscheme given by a $\mathbb{C}$-algebra A, $\Omega^1_{A/\mathbb{C}}$ is the A-module generated by symbols da for $a \in A$ and subject to the relations $d(a+b) = d(a) + d(b), d(ab) = adb + da(b)$, $dc = 0$ for $c \in \mathbb{C}$.[1] This process is compatible with localization, so we can glue to get a coherent $\mathcal{O}_X$-module $\Omega^1_{X/\mathbb{C}}$. Indeed, $\Omega^1_{X/\mathbb{C}}$ is locally free of dimension d if and only if $X/\mathbb{C}$ is nonsingular, and is nothing but the **cotangent bundle**. For $i > 1$, we define $\Omega^i_{X/\mathbb{C}} := \Lambda^i \Omega^1_{X/\mathbb{C}}$, i.e., just the globalization of the exterior powers of modules. So if $X/\mathbb{C}$ is nonsingular, $\Omega^i_{X/\mathbb{C}}$ will be a locally free sheaf on X of rank $\binom{d}{i}$ – especially, $\Omega^d_{X/\mathbb{C}}$ is an invertible sheaf on X, the **canonical bundle**. Moreover, working purely at the level of exterior powers of modules, we have an exterior derivative $d : \Lambda^i M \to \Lambda^{i+1} M$ which, famously, satisfies $d^2 = 0$. Therefore we have

$$\mathcal{O}_X = \Omega^0_X \to \Omega^1_X \to \ldots \to \Omega^d_X \to 0,$$

the **DeRham complex** of $X/\mathbb{C}$.

Consider all possible cohomology groups $H^p(X, \Omega^q)$: they must vanish when $p > d$ or when $q > d$. Because we have assumed X is complete, the algebraic analogue of the Cartan-Serre finiteness theorem [Hartshorne, ???] tells us that the cohomology groups of any coherent sheaf on X are finite dimensional $\mathbb{C}$-vector spaces. We put $h^{p,q} = \dim_{\mathbb{C}} H^p(X, \Omega^q)$, and we are ready for the following celebrated theorem, implying in particular that the Betti numbers can be calculated from cohomology of coherent sheaves.

Theorem 14 *(Hodge Theorem) The Betti numbers of X^{an} are determined by the coherent cohomology of the sheaves Ω^i: for all n, we have*

$$\dim_{\mathbb{C}} H^n(X, \mathbb{C}) = \sum_{p+q=n} h^{p,q}.$$

Moreover, $h^{p,q} = h^{q,p}$.

In the next two sections we give the proof of this theorem, or rather the proof modulo some (not at all trivial) analytic and differential geometric facts. In fact, part of the point of giving the proof is to appreciate its essentially **non-algebraic** nature.

2.2 Summary of Hodge Theory on Riemannian manifolds

Let (M, g) be a compact oriented $\mathbb{R}^n$-manifold endowed with a Riemannian metric g. Every (paracompact!) real manifold can be so endowed – the easy way to do this is to take a locally finite covering of M by subsets homeomorphic to $\mathbb{R}^n$, endow each of these with the standard Euclidean metric, and add up all these individual metrics, smoothing with a partition of unity. Another way to prove this result is to realize M as a submanifold of $\mathbb{R}^N$ by Whitney embedding,

[1] A useful special case is that if $A = C[X_1, \ldots, X_n]/(f_j)$ is a finite-type $\mathbb{C}$-algebra, $\Omega_{A/\mathbb{C}}$ is the finitely generated A-module with generators dX_i for $1 \leq i \leq n$ and relations $d(f_j) = 0$.

take the Euclidean metric on $\mathbb{R}^N$ and restrict to N.[2] For now, we write $\Lambda_M^p = \Lambda_M^p(\mathbb{R})$ for the space of C^∞ p-forms on M with real coefficients. Let dV be the volume n-form on M associated to the metric g – in local coordinates $x_1, \ldots, x_n$, it is given by

$$dV = \sqrt{\det(g_{ij})}dx_1 \wedge \ldots \wedge dx_n.$$

We define, using the metric, the **Hodge star** operator

$$\star : \Lambda_M^p \to \Lambda_M^{n-p},$$

defined as follows: in a neighborhood U about every point M admits an orthonormal frame $e_1, \ldots, e_n$ of sections of the tangent bundle – i.e., $g(e_i(x), e_j(x)) = \delta_{ij}$ for all $x \in U$. Every i-form can be written as a sum of terms $\sum_I f_I(x)de_I$, where $I \subset \{1, \ldots, n\}$ is a subset of cardinality i and $de_I = \bigwedge_{i \in I} de_i$. Define $I^\star = \{1, \ldots, n\} \setminus I$, the complementary subset, and finally define

$$\star(\sum_I f_I(x)de_I) = \sum_I f_I(x)de_{I^\star}.$$

This allows us to endow Λ_M^i with an inner product, namely

$$\langle \alpha,\ \beta \rangle = \int \star(\alpha \wedge (\star\beta))dV.$$

The completion of this real inner product space is denoted $L^2(\Lambda_M^p)$, the Hilbert space of square-integrable p-forms on M. It is built into our definition that the Hodge star operator is a Hilbert space isometry $L^2(\Lambda_M^p) \to L^2(\Lambda_M^{n-p})$. We put $L^2(\Lambda_M) := \bigoplus_p L^2(\Lambda_M^p)$ (Hilbert space direct sum, i.e., $\langle\ ,\rangle = \sum_p \langle\ ,\rangle_p$.)

Because of this, it makes sense to speak of the adjoint operator to the exterior derivative d on $L^2(\Lambda_M)$, denoted $d^\star$. One can check that it exists and is given by $(-1)^{n+1} \star\ \circ\ d\ \circ\ \star$.

Finally, we define the Laplace-Beltrami operator on $\Lambda_M^\bullet$ as

$$\Delta = d \circ d^* + d^* \circ d.$$

We remark that if M is the (noncompact; in this case we should take the completion of the space of compactly supported smooth forms) manifold $\mathbb{R}^n$ endowed with the Euclidean metric $ds^2 = \sum dx_i^2$ then, up to a sign, the Laplacian of a zero form is the familiar $\sum_i \frac{\partial^2 f}{\partial x_i^2}$.)

In general, we define $\mathcal{H}^p(M) = \ker(\Delta)$, the **harmonic** p-forms.

[2]The latter approach raises the issue of whether every compact Riemannian manifold arises as a submanifold of Euclidean space. The answer is yes; this is the celebrated Nash Embedding Theorem [Nasar].

Lemma 15 *For any $s \in \Lambda^p_M$, we have*

$$\langle \Delta s,\ s\rangle = ||ds||^2 + ||d^* s||^2$$

.

Moreover, $s \in \Lambda^p_M$ is harmonic iff $ds = d^* s = 0$.

Proof: The formula is immediate: $\langle \Delta s,\ s\rangle = \langle d(d^\star s) + d^\star(ds), s\rangle = \langle d(d^\star s), s\rangle + \langle d^\star(ds), s\rangle = \langle d^\star s, d^\star s\rangle + \langle ds, ds\rangle = ||ds||^2 + ||d^\star s||^2$. It clearly follows that a harmonic form is both d-closed and $d^\star$-closed. Conversely, if $ds = d^\star s = 0$, then

$$\langle \Delta s, \Delta s\rangle = \langle dd^\star s + d^\star ds, dd^\star s + d^\star ds\rangle = 0.$$

Theorem 16 *(Hodge theorem for Riemannian manifolds)*
a) For all p, there is an orthogonal decomposition $\Lambda^p_M = \mathcal{H}^p(M) \oplus \operatorname{Im} d \oplus \operatorname{Im} d^\star$.
b) Since d and d^* are adjoint, $\operatorname{Ker} d = (\operatorname{Im} d^*)^\perp$, and we conclude that $Z^p_{DR}(M) := \operatorname{Ker}(d : \Lambda^p_M \to \Lambda^{p+1}_M)$ is naturally isomorphic to $\mathcal{H}^p(M) \oplus \operatorname{Im} d$. That is, each DeRham cohomology class contains a unique harmonic representative.

"Proof": It is easy to see that $\mathcal{H}^p(M)$, $\operatorname{Im} d$ and $\operatorname{Im} d^*$ are mutually orthogonal subspaces of Λ^p_M: indeed $\langle ds,\ d^* t\rangle = \langle d^2 s, t\rangle = 0$. Moreover (using that harmonic forms are d-closed and $d^\star$-closed), since $\operatorname{Im} d^* = (\operatorname{Ker} d)^\perp$, no harmonic form is in the image of d^*; similarly, no harmonic form is in the image of d. To show that this subspace is all of Λ^p_M is another matter entirely. For this we need to know that Δ is an **elliptic operator** on M.

For completeness, we indicate briefly the definition of an elliptic differential operator: a differential operator of order at most m between vector bundles E and F on M with Riemannian metrics is a thing which can in local coordinates be written as a matrix $\sum_{I\ :\ |I| \le m} a^I_{ij}(x) D^I$, where e.g. $D^{(1,2)} = \frac{\partial}{\partial x_1}\frac{\partial^2}{\partial x_2^2}$. The associated **symbol** is obtained by dropping all the lower order terms and formally replacing the D^I's with $(\zeta)^I = (\zeta_1, \ldots, \zeta_n)^I = \zeta_1^{i_1} \cdots \zeta_n^{i_n}$, so

$$\sigma(D)(x, \zeta) = \sum_{I:|I|=m} a^I_{ij}(x) \zeta_1^{i_1} \cdots \zeta_n^{i_n}.$$

The operator is elliptic if for all $x \in M$ and all $\zeta \in \mathbb{R}^n \setminus O$, the symbol $\sigma(D)(x, \zeta)$ is an invertible matrix. For instance, since the homogeneous form $x_1^2 + \ldots + x_n^2$ has no nontrivial real zeros, the classical Laplacian on $\mathbb{R}^n$ is elliptic. It is not so hard to see that the general Laplace-Beltrami operator is elliptic; see e.g. [Demailly]. The hard part is the following result, which is an entirely serious theorem in the realm of PDEs, using Sobolev spaces, Garding's inequality, and so on.

Theorem 17 *(Finiteness theorem for elliptic operators) Let P be an elliptic operator on the sections of a vector bundle $E \to M$, whose fibres are equipped with an inner product. Then the $\Gamma(M, E) = \operatorname{Im}(P) \oplus \operatorname{Ker} P^*$, where the first summand is a closed subspace of finite codimension.*

Theorem 4 (proved by Hodge, of course, for the Laplace-Beltrami operator; later the general theory of elliptic operators developed around his proof) finishes the proof for us, since $\Delta = \Delta^*$ is self-adjoint and

$$\operatorname{Im}\Delta = \operatorname{Im}(d \circ d^* + d^* \circ d) \subset \operatorname{Im} d + \operatorname{Im} d^*.$$

An application: Let $\rho : \tilde{M} \to M$ be a degree N unramified cover of a compact smooth manifold M. One knows that $\chi(\tilde{M}) = N\chi(M)$ for truly topological reasons (pull back a sufficiently fine triangulation of M), but it is not as clear that we have inequalities $b_i(\tilde{M}) \geq b_i(M)$. But we claim that indeed $H^i(\rho)$ is an injection for all i, and harmonic cohomology gives an easy proof of this: indeed it is certainly true that pullback map is injective on the level of differential forms (as follows immediately from the chain-rule and that ρ is a submersion). Now choose any Riemannian metric on M and pull it back to $\tilde{M}$; since ρ is unramified, $\tilde{M}$ is locally isometric to M; since the Laplace-Beltrami operator is local by construction, it follows that the Laplacian **commutes** with pullback of differential forms. We conclude that the harmonic forms on M map monomorphically into the harmonic forms on $\tilde{M}$, whence the claim.[3]

2.3 A quick proof of the DeRham Theorem

For comparison, we recall the **DeRham Theorem**, which gives a canonical isomorphism between the DeRham cohomology $H^\bullet_D R(M)$ of a real manifold M and the singular cohomology with $\mathbb{R}$-coefficients. It is instructive to note that in constrast to the hard analysis of the Hodge theorem, the DeRham theorem can be proved using **only** the machinery of sheaf cohomology.

On the one hand we have the **DeRham resolution** of the constant sheaf $\underline{\mathbb{R}}$ on X:

$$0 \to \underline{\mathbb{R}} \xrightarrow{\iota} \Lambda^0_M \xrightarrow{d} \Lambda^1_M \ldots \to \Lambda^n_M \to 0.$$

Certainly $d^2 = 0$ – even at the level of presheaves. Moreover, upon restriction to any star-shaped domain, closed forms are exact (Poincar' Lemma), so as a sequence of sheaves it is exact, i.e., it gives a resolution of $\underline{\mathbb{R}}$. But the sheaf of sections of any vector bundle on a manifold is soft (indeed it is fine: we have partitions of unity), hence acyclic for sheaf cohomology (as discussed in [C-K]). This shows that the DeRham cohomology naturally isomorphic to $H^\bullet(X, \underline{\mathbb{R}})$.

What about the singular cohomology? Let X be a locally contractible topological space and G an abelian group. We define a sheaf $\mathcal{S}^p(G)$ as follows: for any open subset U, we put $S^p(G)(U) := \operatorname{Hom}_{\mathbb{Z}}(S_p(U, \mathbb{Z}), G)$, where $S_p(U, \mathbb{Z})$ is the usual group of U-valued singular p-chains. We have coboundary maps $\delta : S^p(G)(U) \to S^{p+1}(U)$. Here's the punchline: of course this is not an exact

[3] In fact there is a proof using only DeRham cohomology: we must show that if $\rho^*(\omega)$ is exact, then so was ω. Writing $\rho^*(\omega) = d\theta$, it need not be the case that θ "descends" to M, but its "norm" (in the Galois-theoretic sense!) does; we leave the details to the reader.

sequence at the level of presheaves – indeed, taking U-sections, the cohomology is precisely the singular cohomology $H^\bullet_{sing}(U,G)$. But by the assumed local contractibility, on stalks we get an exact sequence. Therefore, letting $\mathcal{S}^p(G)$ be the sheafification of $U \mapsto S^p(G)(U)$, we have a long exact sequence of sheaves. Moreover, the kernel – which does not need to be sheafified – of $S^0(G)(U) \to S^1(G)(U)$ is canonically identified with the constant sheaf $\underline{G}$, so we find that $G \to \mathcal{S}^\bullet(G)$ is a resolution of $\underline{G}$. Moreover we claim it is a soft resolution: $S^0(G)(U) = \mathrm{Hom}_{\mathbb{Z}}(S_0(U,\mathbb{Z}),G) = \mathrm{Hom}_{\mathbb{Z}}(\bigoplus_{u\in U}\mathbb{Z}[u],G) = \bigoplus_{u\in U}\mathrm{Hom}(\mathbb{Z}[u],G) = \bigoplus_{u\in U} G = C^0(G)$, the canonical flasque sheaf (of discontinuous sections) associated to the constant sheaf $\underline{G}$. So $S^0(G)$ – which is already a sheaf – is flasque, and flasque sheaves are soft. Moreover, taking now $G = \mathbb{R}$, the $\mathcal{S}^i(\mathbb{R})$'s are modules over $\mathcal{S}^0(\mathbb{R})$, via the cup-product. But in general, a sheaf of modules F over a soft sheaf of rings R is soft. Indeed, take a section s of F over a closed subset K of M. By definition of the sections of a sheaf over closed subsets, s extends to some open neighborhood U of K. Since $K \cap (X \setminus U) = \emptyset$, we can define a section ρ of R over $K \cup (X \setminus U)$ by making it identically equal to the unity 1 on K and identically 0 on $X \setminus U$. Since R is assumed to be soft, ρ extends to all of X, and the product ρs gives an extension of s to all of X. Thus the singular resolution $\underline{\mathbb{R}} \to \mathcal{S}^\bullet(\mathbb{R})$ is also an acyclic resolution and can be used to compute the cohomology of X.

In fact, we can make the isomorphism between DeRham cohomology and singular cohomology **explicit**, as follows: first, we may as well work with differentiable p-chains (the above argument goes through verbatim). Then we have a commutative diagram

$$\underline{\mathbb{R}} \to \Lambda^\bullet_X$$

$$\underline{\mathbb{R}} \to \mathcal{S}^\bullet(X,\mathbb{R})$$

given by integration of p-forms against p-chains. Since both complexes are acyclic and the left-hand map is an isomorphism, the general theory of acyclic resolutions shows that the induced map on cohomology must be an isomorphism. This is the usual form of DeRham's theorem.

2.4 Hodge theory for complex & Kahler manifolds

Suppose X is now a $\mathbb{C}^n$-manifold, endowed with a Hermitian metric h. Note well that a Hermitian metric is still a C^∞-object – it has nothing to do with the $\mathbb{C}$-structure on X and indeed (unsing partitions of unity, as above) any $\mathbb{C}^n$-bundle on a real manifold can be endowed with a Hermitian metric. Viewing X as an $\mathbb{R}^{2n}$-manifold via local coordinates $z_1, \overline{z_1}, \dots, z_n, \overline{z_n}$, we consider $\Lambda^\bullet_X = \Lambda^\bullet_X(\mathbb{C})$ the sheaves of $\mathbb{C}$-valued C^∞-differential forms on X, which are local expressions of the form $f_I(z)dz_I \wedge d\overline{z}_J$ – note well that $f_I(z)$ is a $\mathbb{C}$-valued merely C^∞ function. By definition of a complex manifold, transitions between coordinate systems preserve the decomposition into z-coordinates and $\overline{z}$-coordinates:

this allows us to decompose the exterior derivative as $d = \partial + \overline{\partial}$, where e.g. on zero forms $\partial f = \sum_{i=1}^n \frac{\partial f}{\partial z_i} dz_i$ and $\overline{\partial} f = \sum_{i=1}^n \frac{\partial f}{\partial \overline{z_i}} d\overline{z_i}$. We thus visibly get a decomposition of Λ_X^r into $\bigoplus_{p+q=r} \Lambda_X^{p,q}$, the sheaf of $\mathbb{C}$-valued $\mathbb{C}^\infty$ "(p,q)"-forms.

We also have a Hermitian Hodge-star operator defined by

$$u \wedge (\overline{\star v}) = \langle u,\ v \rangle dV,$$

where the volume form is associated to the "underlying" Riemannian metric – the real part of a Hermitian metric gives a Riemannian metric. The Hodge star operator gives a $\mathbb{C}$-linear isometry $\Lambda^{p,q} \to \Lambda^{n-q,n-p}$, and in this way we have not one but three Laplacians. The first is Δ, which is just obtained tensoring from $\mathbb{R}$ to $\mathbb{C}$ the Laplacian on the underlying real manifold. We also have Δ_1 and Δ_2 (slightly nonstandard notation, but the standard notation, $\Box$ and $\overline{\Box}$, seems rather silly), obtained by using ∂ (respectively $\overline{\partial}$) in place of d:

$$\Delta_1 = \partial \circ \partial^* + \partial^* \circ \partial,$$

$$\Delta_2 = \overline{\partial} \circ \overline{\partial}^* + \overline{\partial}^* \circ \overline{\partial}.$$

Among several identities relating these operators, we single out

$$\partial^* = - \star \overline{\partial} \star,\ \overline{\partial}^* = - \star \partial \star .$$

We work with Δ and Δ_2, defining

$$\mathcal{H}^p(X) = \mathcal{H}^p(X, \mathbb{C}) = \ker(\Delta : \Lambda^n \to \Lambda^n)$$

and

$$\mathcal{H}^{p,q}(X) = \mathcal{H}_2^{p,q}(X) = \ker(\Delta_2 : \Lambda^{p,q} \to \Lambda^{p,q}).$$

We speak of the **harmonic n-forms** and **harmonic (p,q)-forms** respectively.

Now we have three different versions of (p,q)-cohomology: the harmonic cohomology $\mathcal{H}_2^{p,q}(X)$; the coherent analytic sheaf cohomology $H^q(X, \Omega^p)$, and finally the **Dolbeault cohomology**, i.e., the "$\overline{\partial}$-DeRham cohomology":

$$H_{\overline{\partial}}^{p,q}(X) := H^q((\Lambda_X^{p,\bullet}, \overline{\partial}).$$

There is also a $\overline{\partial}$-analogue of the DeRham theorem: namely we have the **Dolbeault resolution**

$$0 \to \Omega_X^p \to \Lambda_X^{p,0} \xrightarrow{\overline{\partial}} \Lambda_X^{p,1} \xrightarrow{\overline{\partial}} \dots \xrightarrow{\overline{\partial}} \Lambda_X^{p,n} \to 0,$$

and since the sheaves $\Lambda_X^{p,q}$ are fine, we conclude

$$H^q(X, \Omega_X^p) = \frac{(\mathrm{Ker}(\Lambda_X^{p,q} \xrightarrow{\overline{\partial}} \Lambda_X^{p,q+1})}{\mathrm{Im}(\Lambda_X^{p,q-1} \xrightarrow{\overline{\partial}} \Lambda_X^{p,q})} = H_{\overline{\partial}}^{p,q}(X).$$

The analogue of Theorem 16 for Δ_2 is:

Theorem 18 *For all (p,q), there is an orthogonal decomposition*

$$\Lambda_X^{p,q} = \mathcal{H}_X^{p,q} \oplus \operatorname{Im} \overline{\partial} \oplus \operatorname{Im}(\overline{\partial}^*).$$

Corollary 19 *On a complex manifold, Dolbeault, harmonic and coherent cohomology coincide:*

$$H^p(X, \Omega^q) = \mathcal{H}_X^{p,q} = H_{\overline{\partial}}^{p,q}(X).$$

But we still do not know how any of these cohomology groups compute $H^\bullet(X, \mathbb{C})$. Indeed, they need not, until we add an extra hypothesis.

Our Hermitian metric, $\sum_{ij} h_{ij} z_i \overline{z_j}$ can be written as $h = S + iA$, where S is symmetric and A is skew-symmetric; put $\Omega := 1/2A$, a real-valued $(1,1)$-form. One says that h is a **Kahler metric** if $d\Omega = 0$. (Notice that any Hermitian metric on a one-dimensional $\mathbb{C}$-manifold is automatically Kahler.) A $\mathbb{C}$-manifold is said to be **Kahler** if it admits a Kahler metric.

The property of a metric being Kahler is preserved upon passage to submanifolds, so any submanifold of a Kahler manifold is Kahler. Moreover, $\mathbb{CP}^n$ has a canonical Kahler metric, the **Fubini-Study metric**; we conclude that any compact complex manifold which is algebraic is a Kahler manifold.

Theorem 20 *(Kahler identities) Let (X,h) be a Kahler metric on a complex manifold. Then*

$$\Delta_1 = \Delta_2 = 1/2\Delta.$$

Again this theorem has too much content for us to review here (the standard proof requires some representation theory of $\mathfrak{sl}_2(\mathbb{C})$) but unlike the purely analytic Theorem 17, it is discussed in *every* reputable text on Hodge theory, e.g. [Wells], [Griffiths-Harris], [Voisin I], [Demailly]. But it is certainly what we need: it tells us that on a Kahler manifold we have a unique notion of a harmonic form, so that

$$\mathcal{H}_X^n = \bigoplus_{p+q=n} \mathcal{H}_X^{p,q}.$$

Actually more is true: since (even without the Kahler condition), $\Delta_1 = \overline{\Delta_2}$, on a Kahler manifold we get that $\overline{\Delta_2} = \Delta_1 = \Delta_2$, so if a (p,q)-form is harmonic, so is its complex conjugate (q,p)-form. Thus we have canonical isomorphisms $H^{p,q}(X, \mathbb{C}) \cong \overline{H^{q,p}(X, \mathbb{C})}$, and in particular $h^{p,q} = h^{q,p}$.

Finally, we should discuss the *invariance* of the Hodge decomposition: *a priori* the direct sum decomposition $H^n(X, \mathbb{C}) = \mathcal{H}^n = \bigoplus_{p+q=n} \mathcal{H}^{p,q}$ seems to depend upon the choice of Kahler metric, but one can show that this is not the case. Probably the best way to see this is to observe that $\mathcal{H}^{p,q}$ can be intrinsically defined in terms of the **Hodge filtration** on the DeRham complex as $F^p \cap \overline{F^q}$; we will explore this viewpoint in Chapter 3. For an elementary proof involving yet a fourth kind of (p,q)-cohomology, namely the **Bott-Chern** cohomology groups

$$H_{BC}^{p,q}(X,\mathbb{C}) = \frac{\mathrm{Ker}(d : \Lambda^{p,q}(X) \to \Lambda^{p+q+1}(X))}{\partial\overline{\partial}(\Lambda^{p-1,q-1}(X))},$$

(which are at least *a priori* independent of the Kahler metric), see [Demailly, pp. 40-42]. In summary, we have "proved":

Theorem 21 *(Hodge theorem for Kahler manifolds) Let $X/\mathbb{C}$ be a compact Kahler manifold. Then there is a canonical isomorphism*

$$H^n(X,\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X,\mathbb{C})$$

satisfying $H^{q,p} = \overline{H^{q,p}}$.

Remark: Let (E,D) be a vector bundle on X endowed with a flat connection. We have a notion of DeRham cohomology with coefficients in E, namely in the exact sequence (1.1) of Section 1.3, take global sections and then cohomology; we denote this $H_{DR}^{\bullet}(X,E)$. Simply by replacing d everywhere by D, one can redo all the constructions of this section, getting especially $\Delta_1(E) = \Delta_2(E) = 1/2\Delta(E)$ and at last an isomorphism

$$H_{DR}^n(X,E) = \bigoplus_{p+q=n} H^{p,q}(X,E)$$

satisfying $H^{p,q}(X,E) = \overline{H^{q,p}(X,E)}$. This generalization is not so important for us here, but what we have done is the complex-analytic analogue of taking crystalline cohomology of **crystals** rather than cohomology of the structure sheaf.

Finally, if $X/\mathbb{C}$ is projective nonsingular variety, then as mentioned above the associated complex manifold $X(\mathbb{C})$ is compact Kahler. We must appeal to Serre's GAGA theorem: there is a natural analytification functor from coherent sheaves on $X/\mathbb{C}$ in the algebraic sense to coherent sheaves on $X(\mathbb{C})$ in the analytic sense, such that coherent cohomology computed algebraically is canonically isomorphic to coherent cohomology computed analytically. At last we get our algebraic Theorem 14!

2.5 Implications for the topology of compact Kahler manifolds

The Hodge Theorem is intriguing even at the level of algebraic topology: it places constraints on the Betti numbers of compact Kahler manifolds that need not be satisfied for more general compact complex manifolds (in particular, the Kahler hypothesis is essential in the Hodge theorem and not just an artifice of the proof).

For instance, $h^{1,0} = h^{0,1} = 1/2b_1$ for any compact Kahler manifold. Thus the group of line bundles algebraically equivalent to zero $\mathrm{Pic}^0(X) = H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$ really is a complex torus, as promised in Section 1.1. When X is projective $\mathrm{Pic}^0(X)$ admits a Riemann form, i.e., is an abelian variety, the Picard variety. So $b_1(X) = 0$ implies the triviality of the Picard variety of X. Interpreting $b_1(X)$ in the sense of étale cohomology, this statement makes sense purely algebraically, i.e., in all characteristics. In Chapter 4, we will gain a profound appreciation for the "nonobviousness" of this algebraic statement (i.e., it can be false in positive characteristic!) In fact the equality $h^{1,0} = 1/2b_1$ is already "nonobvious" for complex manifolds:

Example (Hopf surfaces): Consider $X = (\mathbb{C}^2 \setminus \{0\})/\Gamma$, where for some fixed $\lambda \in (0,1)$, $\Gamma = \lambda^{\mathbb{Z}}$, viewed as a group of homotheties of $\mathbb{C}^2$. Each element of Γ is a $\mathbb{C}$-manifold automorphism of $(\mathbb{C}^2 \setminus \{0\}$, so the quotient X is a $\mathbb{C}$-manifold. Since $\mathbb{C}^2 \setminus \{0\}$ is diffeomorphic to $\mathbb{R}^{>0} \times S^1$, we see that X is diffeomorphic to $S^1 \times S^3$. Using the Kunneth formula, we compute the Betti numbers of X: $b_0 = 1, b_1 = 1, b_2 = 0, b_3 = 1, b_4 = 1$. X is definitely not a Kahler manifold! Actually, the $b_2 = 0$ is also enough to ensure that a complex manifold is non-Kahler: one can show that the top wedge power of the fundamental form Ω is a positive scalar multiple of the volume form – in particular $[\Omega^n] \neq 0 \in H^{2n}_{DR}(X, \mathbb{C})$, which implies that **every** wedge power of Ω must be cohomologically nontrivial, and so all the even Betti numbers of a Kahler manifold are **positive**.

Fundamental groups of compact Kahler manifolds: Of course, that $b_1(X)$ must be even is saying something about $\pi_1(X)$, namely that the free rank of its abelianization is even. What if we want to know about $\pi_1(X)$ itself? It is well-known that every finitely presented group arises as the fundamental group of a compact $\mathbb{R}^4$-manifold. Less well-known but still true is that every finitely presented group is the fundamental group of a compact $\mathbb{C}^3$-manifold, so the above restriction on $\pi_1(X)$ for Kahler manifolds is actually rather surprising. Say that a group is a **Kahler group** if it arises as the fundamental group of a compact Kahler manifold.

Question 22 *Which finitely presented groups are Kahler groups?*

This is analogous to the question of which finite groups are Galois groups over $\mathbb{Q}$ and to the question of which finitely presented groups are π_1 of a compact $\mathbb{R}^3$-manifold but, purely on its own terms, seems more interesting than both, since the conjectured answer to the first question is "all of them" and to the second is "very few." In contrast, the frontier between Kahler and non-Kahler groups is remarkably rugged. For instance, we will see in Chapter 4 that every finite group is the fundamental group of an algebraic variety (even in characteristic p – this is a theorem of Serre). Since the class of Kahler groups is clearly closed under products and certainly $\mathbb{Z}^{2g}$ is a Kahler group (the fundamental group of a genus g curve or equally well of its Jacobian), we can completely characterize the abelianizations Kahler groups. To see that this is not enough: the class of

Kahler groups is closed under passage to subgroups (by covering space theory; a cover of a Kahler manifold is Kahler), so the free group on two generators is not Kahler (even though its abelianization is), since it contains free subgroups on every *odd* number of generators.

One might ask why we study Kahler groups instead of fundamental groups of projective manifolds. The answer is that so far no one has ever found a Kahler group which is non-projective; moreover, it is conjectured that the complex structure on a Kahler manifold can be *deformed* (in the sense of Section 1.3.2) to a projective complex structure, which would imply that the two classes are the same. In practice, most constructions of Kahler groups can be done with algebraic manifolds, while non-existence arguments tend to work for the larger class of Kahler manifolds. For much more on this fascinating question, see [Amoros et. al.].

Chapter 3

Algebraic DeRham Cohomology

3.1 Souping up the Hodge Theorem: spectral sequences and hypercohomology

In the previous chapter we proved the Hodge theorem for smooth, projective complex varieties – but only by translating the statement into a statement about Kahler manifolds. We now want to recast a portion of the Hodge theorem in terms of a statement about degeneration of spectral sequences. The translated statement, namely, "The Hodge to de Rham spectral sequence for a smooth projective complex variety degenerates at the E_1 term" is itself purely algebraic, so it is at least meaningful to ask whether it is true in characteristic p.

3.1.1 Spectral sequence of a double complex

Let $(K^{p,q}, d' + d'')$ be a double complex with horizontal and vertical differentials d' and d''. We assume it is concentrated in the first quadrant, i.e. $K^{p,q} = 0$ unless $p,\ q \geq 0$. From the double complex we pass to the associated **total complex**, $K^n := \bigoplus_{p+q=n} K^{p,q}$, endowed with the the differential $d = d' + (-1)^q d''$. On the total complex one has a decreasing filtration

$$F^p K^n := \bigoplus_{p \leq j \leq n} K^{j,n-j}.$$

This induces a filtration on the cohomology groups $H^\bullet(K^\bullet)$ of the total complex, namely

$$F^p H^l(K^\bullet) := \mathrm{Im}(H^l(F^p K^\bullet) \to H^l(K^\bullet)).$$

There is a spectral sequence

$$E_1^{p,q} = H^q((K^{p,\bullet}, d'')) \implies H^{p+q}(K^\bullet).$$

Recall this means that for all $r \geq 1$ we have differentials $d_r : E_r^{p,q} \to E^{p+r,q-r+1}$, such that, inductively, $E_{r+1} = H^\bullet(E_r)$. Since the complex is concentrated in the first quadrant, for any given (p, q) eventually the head or the tail of every "arrow" lies outside of the first quadrant, so that the process stabilizes pointwise: $\lim_{r\to\infty} E_r^{p,q} = E_\infty^{p,q}$ exists. The convergence means that $E_\infty^{p,q} = G^p H^{p+q}(K^\bullet)$, the pth graded piece of the filtration.

Finally, we say the spectral sequence **degenerates** at the E_r-term if all the differentials d_{r+i}, for all $i \geq 0$, are zero. Then indeed $E_r^{p,q} = E_\infty^{p,q}$. One simply says the spectral sequence **degenerates** if it degenerates at the E_1-term (or at the first term under consideration; in a slightly different context, many spectral sequences start with the E_2-term).[1]

3.2 The Hodge to DeRham spectral sequence

Let $X/\mathbb{C}$ be a complex manifold – not yet assumed to be compact or Kahler. Dolbeault's theory provides us with a double complex, namely $K^{p,q} = \Lambda_X^{p,q}$. Our two differentials are just ∂ and $\overline{\partial}$ – or, to adhere strictly with the sign conventions of the previous section, take $d'' =(-1)^q\overline{\partial}$; we will not be so careful about this – with total differential d. Notice that the asasociated total complex is in degree n $\bigoplus_{p+q=n} \Lambda^{p,q}$ with differential d – i.e, the $\mathbb{C}$-valued DeRham complex $\Lambda_X^\bullet$. The associated spectral sequence is called the **Hodge to DeRham** spectral sequence: let's look at it. The E_1 terms are $E_1^{(p,q)} = H^q((\Lambda^{p,\bullet}, \overline{\partial})) = H^{p,q}(X,\mathbb{C}) = \mathcal{H}^{p,q}(X) = H^q(X,\Omega^p)$, the Hodge groups. So we can write the spectral sequence as

$$H^{p,q}(X,\mathbb{C}) \implies H_{DR}^{p+q}(X,\mathbb{C}).$$

So for any complex manifold, the Hodge groups are related to the DeRham cohomology – somehow. The question is: does this spectral sequence degenerate (immediately)?

Suppose now that X is compact; then by the finiteness theorem of Serre we know that $H^{p,q}(X,\mathbb{C})$ are all finite dimensional $\mathbb{C}$-vector spaces; we may write $h^{p,q}$ for their dimensions and $b_n := \dim H_{DR}^n(X,\mathbb{C})$ for the Betti numbers. Now if the spectral sequence degenerates, we can sum along the line $x + y = n$ to get the nth Betti number: i.e., degeneration implies $\sum_{p+q=n} h^{p,q} = b_n$, the greater part of the Hodge theorem. But in fact the converse is true: notice that since a spectral sequence involves repeated passage to subquotients, the dimensions of the $\mathbb{C}$-vector spaces $E_r^{p,q}$ are nonincreasing functions of r, and that a single differential is nonzero is precisely the condition for some subquotient to be proper.

[1] To be sure: in contrast to most instances in mathematics (and in life), degeneration of a spectral sequence is a joyous occasion: it means that two quantities which abstract nonsense says are related, albeit in a very complicated way, are actually related in the simplest possible way, aka the way in which you wanted them to be related.

In other words if the spectral sequence does not degenerate we must have for some n that $\sum_{p+q=n} h^{p,q} > b_n$. In summary:

Proposition 23 *Let X be any compact complex manifold. The Hodge to DeRham spectral sequence degenerates at the E_1-term iff for all n we have $\sum_{p+q=n} h^{p,q} = b_n$.*

So the following is an immediate consequence of the Hodge theorem:

Theorem 24 *The Hodge to DeRham spectral sequence of a compact Kahler manifold degenerates at the E_1 term.*

Remark: The part of the Hodge theorem that says that a compact Kahler manifold has $H^{p,q}(X, \mathbb{C}) = \overline{H^{q,p}(X, \mathbb{C})}$ is therefore not guaranteed by the degeneration of this spectral sequence. Indeed, it turns out that if $X/\mathbb{C}$ is any compact complex surface, the spectral sequence degenerates. Moreover X will be Kahler iff b_1 is even; otherwise it turns out that $h^{1,0} = h^{0,1} + 1$ [BPV]. Notice that, together with Serre duality, this computes the Hodge diamond of the Hopf surfaces studied in Chapter 2.

3.3 Hypercohomology

Let us at long last return to the algebraic category: to help us do this, suppose X/k is a smooth projective variety over an algebraically closed field of positive characteristic p. We still have Hodge numbers, defined via coherent cohomology: $h^{p,q} := \dim_k H^q(X, \Omega^p)$. However we do not have anything like DeRham resolution of the constant sheaf $\underline{\mathbb{C}}$, because indeed constant sheaves on Noetherian spaces are flasque and do not need to be resolved. Nor do we have the Dolbeault double complex $\Lambda_X^{p,q}$. Nevertheless, we can still construct a **Hodge to DeRham** spectral sequence whose E_1 term is $H^q(X, \Omega^p)$ by using a construction of pure homological algebra: hypercohomology.

Namely, let $S^\bullet$ be a bounded below (cohomological) complex of sheaves on a topological space.[2] Choose $I^\bullet$ an **injective resolution** of the complex $S^\bullet$: by definition, this means a morphism of complexes $\varphi : S^\bullet \to I^\bullet$ to a complex of injective objects such that $H^\bullet(\varphi) : H^\bullet(S^\bullet) \to H^\bullet(I^\bullet)$ is an isomorphism (a so-called **quasi-isomorphism** of complexes). Note well that this generalizes the notion of an injective resolution of a single sheaf as soon as we identify the sheaf S with the complex $S \to 0 \to 0 \to \ldots$. We need two facts about resolutions of complexes of sheaves whose analogues in the case of a single sheaf are familiar from [CK]: first, that injective resolutions exist, and second that they are unique up to homotopy; for the proofs of these facts (which require no new ideas), see e.g. [Iversen].

[2] It will be clear that we could work in more generality: in an arbitrary abelian category with enough injectives and with some left-exact functor R.

So, given $S^\bullet$ our complex of sheaves, we define its **hypercohomology** groups $\mathbb{H}^n(S^\bullet) := H^n(\Gamma(X, I^\bullet))$; observe that this too generalizes the definition of cohomology groups of a single sheaf, and are similarly independent of the choice of injective resolution.

In our algebraic setting we have the DeRham complex

$$\Omega_X^\bullet : \ \mathcal{O}_X = \Omega_X^0 \to \Omega_X^1 \to \ldots \to \Omega_X^d \to 0,$$

and we define the **algebraic DeRham cohomology of** X to be the hypercohomology of the DeRham complex:

$$H_{DR}^n(X/k) = \mathbb{H}^n(\Omega_{X/k}^\bullet).$$

In the remainder of this section we explain the following two important facts:

• Why the algebraic DeRham cohomology coincides with the analytic DeRham cohomology in the complex case.

• How to construct a purely algebraic Hodge to DeRham spectral sequence

$$H^q(X, \Omega_X^p) \implies H_{DR}^{p+q}(X). \tag{3.1}$$

When $k = \mathbb{C}$, the Poincaré Lemma holds for holomorphic differentials:

$$0 \to \underline{\mathbb{C}} \to \Omega_X^0 \to \Omega_X^1 \to \ldots \to \Omega_X^n \to 0$$

so that $\Omega_X^\bullet$ is a resolution – not acyclic! – of the constant sheaf $\underline{\mathbb{C}}$. But consider: to take the cohomology of $\underline{\mathbb{C}}$, we take any injective resolution of $\underline{\mathbb{C}}$. Since $\Omega_X^\bullet$ is itself a resolution of $\mathbb{C}$, taking an injective resolution $I^\bullet$ of the complex $\Omega_X^\bullet$, the fact that $\Omega^\bullet \to I^\bullet$ is a quasi-isomorphism precisely means that $\ker(I^0 \to I^1) \cong \ker(\Omega^0 \to \Omega^1) \cong \underline{\mathbb{C}}$ and that thereafter the complex $I^\bullet$ is exact, so that $I^\bullet$ is itself an injective resolution of $\underline{\mathbb{C}}$ and $H^n(X, \underline{\mathbb{C}}) = \mathbb{H}^n(\Omega_X^\bullet)$. So algebraic DeRham cohomology computes DeRham cohomology in the complex case.

Finally, any time we have a complex of sheaves $S^\bullet$ we will get a **hypercohomology spectral sequence**

$$H^q(X, S^p) \implies \mathbb{H}^{p+q}(X, S^\bullet) \tag{3.2}$$

Indeed we take for each S^p an injective resolution $S^p \to I^{p,\bullet}$: these successive injective resolutions form the columns of a double complex. Moreover, since we have a natural bijection in the homotopy category between $\mathrm{Hom}(S^p, S^{p+1})$ and $\mathrm{Hom}_{complexes}(I^{p,\bullet}, I^{p+1,\bullet})$ we can choose essentially unique horizontal maps from one injective resolution to the next. The associated total complex is a complex of injective sheaves quasi-isomorphic to $S^\bullet$ – draw a picture! – i.e., upon taking global sections and then cohomology we have computed the hypercohomology of $S^\bullet$. It follows that if we take global sections of the entire complex, we get a double complex with $E_1^{p,q}$ =Ker($\Gamma(X, I^{p,q}) \to \Gamma(X, I^{p,q+1}))/\operatorname{Im}(\Gamma(X, I^{p,q-1}) \to \Gamma(X, I^(p,q)) = H^q(X, S^p)$. This shows that in general, there is a hypercohomology spectral sequence as in (2) above. Applying it to $\Omega_X^\bullet$, we get a Hodge to DeRham spectral sequence, which, although purely algebraic in nature, coincides in the complex case with the Hodge to DeRham spectral sequence constructed using Dolbeault cohomology groups.

3.4 Relative Hodge theory of Kahler manifolds

In this section we will say a bit about the Hodge theory of a smooth family $\pi : X \to S$. This material belongs at the end of Chapter 2, but because we will use the language of spectral sequences, we have chosen to put it here instead. The source for most of the material in this section was Section 10 of [Demailly]; our remarks about smooth versus holomorphic families from Section 1.3 will be helpful here.

A clue to the fact that one should be able to consider a much more general Hodge theory can be found already in the fact that one has not merely sheaves of differentials for varieties but sheaves of *relative* differentials $\Omega_{X/S}$ associated to an arbitrary morphism of schemes $X \to S$. Moreover $X \to S$ is smooth of dimension d if and only if $\Omega_{X/S}$ is a vector bundle of rank d on X. We assume for the remainder of the section that we have a proper smooth family of complex manifolds over a connected base $\pi : X \to S$. The first basic result is the following

Theorem 25 *(Kodaira-Spencer Semicontinuity Theorem)[Demailly] Let $X \to S$ be a proper smooth $\mathbb{C}$-analytic map and $E \to X$ a locally free sheaf on X; put $h^q(t) := h^q(X_t, E_t)$. Then the $h^q(t)$ are upper-semicontinuous functions on S, and more precisely, so is*

$$h^q(t) - h^{q-1}(t) + \ldots + (-1)^q h^0(t),\ 0 \le q \le n = \dim X_t.$$

Corollary 26 *Let $X \to S$ be a smooth, proper morphism of $\mathbb{C}$-analytic spaces whose fibres X_t are Kahler manifolds. Then the Hodge numbers $h^{p,q}(X_t)$ of the fibres are constant. Moreover, in the Hodge decomposition*

$$H^k(X_t, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X_t, \mathbb{C}),$$

the mappings $t \mapsto H^{p,q}(X_t, \mathbb{C})$ give C^∞ (but in general not holomorphic) subbundles of the bundle $tH^k(X_t, \mathbb{C})$.

Proof: By the Ehresmann Lemma, the Betti numbers $b_k = H^k(X_t, \mathbb{C})$ are constant. Since $h^{p,q}(X_t, \mathbb{C}) = h^q(X_t, \Omega^p_{X_t})$ is by the theorem an upper semicontinuous function of t and

$$h^{p,q}(t) = b_k - \sum_{r+s=k, (r,s) \neq (p,q) h^{r,s}(X_t),}$$

they are clearly lower-semicontinuous as well. So they are continuous, and hence constant.

Theorem 27 *(Kodaira)[Voisin I] For our smooth, proper holomorphic family $\pi : X \to S$, the Kahler locus – i.e., the subset of $s \in S$ such that $\pi^{-1}(s)$ is Kahler – is open. Indeed, if ω_0 is a Kahler metric on the fiber $\pi^{-1}(s_0)$, then on a neighborhood of s_0 one can endow the fibers with Kahler metrics $\omega(s)$ such that $s \mapsto \omega(s)$ is C^∞.*

More precise and more general results are available, using Grauert's direct image theorems. Recall that if $f : X \to Y$ and E is a sheaf on X, the higher direct image sheaves $R^k f_* E$ on Y are given as the sheafification of $U \mapsto H^k(f^{-1}(U), E)$. We have a hyperanalogue: if $A^\bullet$ is a complex of sheaves, we have complexes $\mathbb{R}^q f_*(A^\bullet)$,

$$U \mapsto \mathbb{H}^k(f^{-1}(U), A^\bullet).$$

We have the following fundamental result:

Theorem 28 *(Direct image theorem) Let $\sigma : X \to S$ be a proper morphism of $\mathbb{C}$-analytic spaces and $A^\bullet$ a bounded complex of coherent sheaves of $\mathcal{O}_X$-modules. Then:*
a) $\mathbb{R}^k \sigma_* A^\bullet$ is a complex of ocherent sheaves on S.
b) Every point of S admits a neighborhood $U \subset S$ on which there exists a bounded complex $W^\bullet$ of $\mathcal{O}_S$-modules whose sheafified cohomology $\mathcal{H}^k(W^\bullet)$ are isomorphic to the complexes $\mathbb{R}^k \sigma_* A^\bullet$).
c) If σ has equidimensional fibers, the hypercohomology of the fiber X_t with values in $A^\bullet_t := A^{bullet} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_t}$ ($\mathcal{O}_{X_t} = \mathcal{O}_X / \sigma^* \mathfrak{m}_{S,t}$) is given by

$$\mathbb{H}^k(X_t, A^\bullet_t) = \mathbb{H}^k(W^\bullet_t),$$

where $W^\bullet_t$ is the finite-dimensional complex of sheaves $W^k_t := W^k \otimes_{\mathcal{O}_S,t} \mathcal{O}_{s,t} / \mathfrak{m}_{S,t}$.
d) Under the hypothesis of c), if the hypercohomology fibrations $\mathbb{H}^k(X_t, A^\bullet_t)$ have constant dimension, the sheaves $\mathbb{R}^k \sigma_* A^\bullet$ are locally free on S.

From part b) and (the proof of) the Kodaira-Spencer theorem, one deduces:

Theorem 29 *(Semicontinuity theorem) If $X \to S$ is a proper morphism of $\mathbb{C}$-analytic spaces with equidimensional fibers and E/X is a coherent sheaf, then, putting $h^q(t) := H^q(X_t, E_t)$, we find that*

$$h^q(t) - h^{q-1}(t) + \ldots + (-1)^q h^0(t)$$

*are upper semicontinuous functions of t, (even) for the analytic **Zariski** topology on S (i.e., where the closed sets are the zero sets of finitely many analytic functions).*

Now is the time to recall (Section 1.3) that the fiber cohomologies $H^k(X_t, \mathbb{C})$ are locally constant functions of t and are thus canonically endowed with a flat connection, the Gauss-Manin connection. As we noted at the time, this implies that $t\mathrm{H}^k(X_t,\mathbb{C})$ has the canonical structure of a holomorphic vector bundle. The total cohomology $\bigoplus_k H^k(X_t,\mathbb{C})$ is called the **Hodge bundle** of $X \to S$.

Consider now the relative DeRham complex $\Omega^\bullet_{X/S}, d_{X/S})$ of $X \to S$. This complex furnishes us with a resolution of $\sigma^{-1}(\mathcal{O}_S)$,

$$\mathbb{R}^k\sigma_*\Omega^\bullet_{X/S} = R^k\sigma_*(\sigma^{-1}(\mathcal{O}_S)) = R^k\sigma_*(\sigma^{-1}\mathcal{O}_S) = (R^k\sigma_*\mathbb{C}_X) \otimes_{\mathbb{C}} \mathcal{O}_S. \tag{3.3}$$

The last – important! – equality comes from the $\mathcal{O}_S(U)$-linearity for the cohomology caluclated on $\sigma^{-1}\mathcal{O}_S$. In other words, $\mathbb{R}^k\sigma_*\Omega^\bullet_{X/S}$ *is* the locally free $\mathcal{O}_S$-module associated to the locally constant sheaf $t \mapsto H^q(X_t,\mathbb{C})$.

We get a spectral sequence of hypercohomology

$$E_1^{p,q} = R^q\sigma_*\Omega^p_{X/S} \implies G^p\mathbb{R}^{p+q}\sigma_*\Omega^\bullet_{X/S} = G^pR^{p+q}\sigma_*\mathbb{C}_X.$$

(This spectral sequence is obtained from the prior (general) hypercohomology spectral sequence by sheafifying.) Since the cohomology of $\Omega^p_{X/S}$ along the fibres X_t is nothing but the constant rank guy $H^q(X_t,\Omega^p_{X_t})$, part d) of the direct image theorem shows that the $R^q\sigma_*\Omega^p_{X/S}$ are locally free. Finally, the filtration $F^pH^k(X_t,\mathbb{C}) \subset H^k(X_t,\mathbb{C})$ is obtained at the level of locally free $\mathcal{O}_S$-modules by taking the image of the $\mathcal{O}_S$-linear map

$$\mathbb{R}^k\sigma_*F^p\Omega^\bullet_{X/S} \to \mathbb{R}^k\sigma_*\Omega^\bullet_{X/S},$$

a coherent subsheaf (in fact locally free, because of the constancy of rank of the fibres). From this and equation (6) one gets:

Theorem 30 *(Holomorphy of the Hodge filtration) The Hodge filtration $F^pH^k(X_t,\mathbb{C}) \subset H^k(X_t,\mathbb{C})$ is a holomorphic subbundle, with respect to the holomorphic structure defined by the Gauss-Manin connection.*

In general, $H^{p,q}(X_t,\mathbb{C}) = F^pH^k(X_t,\mathbb{C}) \cap \overline{F^qH^k(X_t,\mathbb{C})}$ has no reason to be a holomorphic **subbundle** of $H^k(X_t,\mathbb{C})$, even though $H^{p,q}(X_t,\mathbb{C})$ has a natural structure of a holomorphic vector bundle, obtained either from the cohrent sheaf $R^q\sigma_*\Omega^p_{X/S}$ or as a quotient of $F^pH^k(X_t,\mathbb{C})$. Otherwise put, it is the Hodge decomposition which need not be holomorphic.

Chapter 4

Some characteristic p algebraic geometry – not finished yet!

4.1 Comparing topological invariants

Let us now try to compare our various numerical invariants in positive characteristic: specifically, let X/k be a smooth proper, connected variety over an algebraically closed field of characteristic p, of dimension d. For $0 \leq n \leq 2d$, we have three different kinds of Betti numbers: the Hodge Betti numbers $b'_n := \sum_{p+q=n} h^{p,q} = \sum_{p+q=n} \dim H^q(X, \Omega^p)$; the DeRham Betti numbers $b''_n := \dim H_D R^n(X) = \dim \mathbb{H}^n(\Omega^\bullet_X)$, and finally the ℓ-adic Betti numbers $b := \dim H^n(X_{\text{ét}}, \mathbb{Q}_\ell)$.

(Actually, in the case when X comes from good reduction in characteristic zero, we should introduce yet a fourth kind of Betti number, the "Betti" Betti numbers, or those which come from basechanging the generic fibre to the complex numbers and taking the literal Betti cohomology. But these agree with the ℓ-adic Betti numbers when the former are defined.)

Well, what can we say?

To summarize the discussion of the last subsection, we do know $b'_n \geq b''_n$, with equality iff the Hodge to DeRham spectral sequence degenerates.

We do know $h^{p,q} = h^{n-p,n-q}$ – this is Serre duality. But beware – we certainly do not know $h^{p,q} = h^{q,p}$, even if the spectral sequence degenerates.

Suppose that X lifts to characteristic zero, i.e., suppose there is a proper smooth variety $\mathcal{X}/W(k)$, the ring of Witt vectors of k whose special fibre is our X. On the generic fibre all three of our Betti numbers are defined and are equal (Hodge theorem + DeRham theorem). Because of the smoothness, the DeRham sheaves $\Omega^i_{X/W(k)}$ are still locally free sheaves over $W(k)$ – in particular they are **flat**, so the **semicontinuity theorem** applies, telling us that the dimensions of the cohomology groups can only jump up in passage from the generic fibre to the special fibre. Moreover the ℓ-adic Betti numbers are **continuous** under these hypotheses (ℓ-adic cohomology is enviably well-behaved), so this says that the existence of a smooth lifting implies $b'_n \geq b_n$ for all n, i.e., the Hodge Betti numbers are largest of all.

Example: The Hodge to DeRham spectral sequence always degenerates for algebraic curves: by Serre duality, this statement is equivalent to $b''_1 = 2g$. We can see this directly: we certainly know all the Hodge numbers; there are only four possibly nonzero ones, and they are $h^{0,0} = h^{1,1} = 1$, $h^{1,0} = h^{0,1} = g$ (by definition!). Starting with d_2, the "geometry of vectors in the plane"[1] implies that all the differentials are zero, so we just have to worry about the horizontal arrows d_1. One of them is $d_1 : H^0(\mathcal{O}_X) \to H^0(\Omega^1_X)$ and indeed the differential of a (necessarily constant) global function is zero. The second arrow $d_1 : H^1(\mathcal{O}_X) \to H^1(\Omega^1_X)$ is indeed the map functorially induced on H^1 by the last arrow. As King Lear wisely said, "Nothing can come of nothing," so this differential is zero as well. So all three Betti numbers agree.

Example: One also has $b_\bullet = b'_\bullet = b''_\bullet$ for abelian varieties (of any dimension). We will see one reason why this must be the case at the end of the section.

Igusa's example: Igusa constructed a (liftable) surface X/k with $h^{0,1} > \dim \operatorname{Pic}(X)$. This was distressing, since over $\mathbb{C}$ one can construct the Picard variety from the cohomogy of

$$0 \to \underline{\mathbb{Z}} \hookrightarrow \mathcal{O}_X \overset{2\pi i exp}{\to} \mathcal{O}_X^\times \to 0,$$

namely as the quotient of the $\mathbb{C}$-vector space $H^1(X, \mathcal{O}_X)$ by the complete lattice $H^1(X, \mathbb{Z})$, and hence we have $h^{0,1} = \dim \operatorname{Pic}^0(X)$ in characteristic zero. It turns out that in characteristic p it is still true that $\dim \operatorname{Pic}^0(X) = h^{1,0} = g$, the number of global oneforms, so we must have $h^{0,1} > h^{1,0}$ and hence $b''_1 > 2 \dim \operatorname{Pic}^0(X)$. But (as it turns out), [2] since $\mathcal{X}/W(k)$ is smooth, $\operatorname{Pic}^0(X)/W(k)$ is smooth, so the dimension of its special fibre is the same as the dimension of the general fibre, and we have $b''_1 > 2 \dim \operatorname{Pic}^0(\mathcal{X}_\eta) = b_1$.

Serre's example: It is similar but simpler than Igusa's. He takes a surface in

[1] This is much funnier if you happen to be teaching Math 133.

[2] What we are sweeping under the rug is that, over an algebraically closed field of characteristic zero, the Picard variety is the connected component at the identity of the Picard scheme, which represents the functor of line bundles on X. However in positive characteristic the connected component of the Picard scheme need not be **reduced**; by definition, the Picard variety is obtained by extracting the reduced subvariety, and in fact the passage from Picard scheme to Picard variety is the cause of the inequality!

$\mathbb{P}^3/W(k)$ with a fixed-point free group of automorphisms isomorphic to $\mathbb{Z}/p\mathbb{Z}$, where p is the characteristic of k. Classically – i.e., in characteristic 0 using analytic techniques – one knows that a hypersurface of dimension at least 2 is simply connected (one of the Lefschetz theorems); in other words, we have a map $\tilde{S} \to S$ from the universal cover with covering group $\mathbb{Z}/p\mathbb{Z}$. In particular, the fundamental group of S_η is $\mathbb{Z}/p\mathbb{Z}$, so $b_1 = 0$. Arguing as in the above example, we must have the Picard variety vanishing, so $h^{1,0} = 0$. Again we have $h^{0,1} > h^{1,0}$. For both surfaces we have $b_i'' = b_i$;, so the Hodge and DeRham Betti numbers are both "too large" compared to the ℓ-adic Betti numbers.

4.2 "New" geometry in characteristic p

Example (W. Lang's thesis): Let k be an algebraically closed field of characteristic $p > 0$. A **quasi-elliptic surface** is a nonsingular surface X/k which admits a dominant morphism $X \to C$ (C a smooth curve) whose generic fibre is a (geometrically integral) **nonsmooth** curve of genus one. (Compare with the definition of an elliptic surface, which is the same except that the generic fibre is nonsingular of genus one.)

The first thing to notice is that quasi-elliptic surfaces can only exist in positive characteristic. Indeed, in characteristic zero a morphism $f : X \to Y$ of nonsingular varieties is **generically smooth**, i.e., there exists an open subset U of X such that $f|U$ is smooth (this is equivalent to the generic fibre being smooth) – see [Hartshorne, 3.10.5 3.10.7], but indeed the result is immediate once one remembers that if $k(V)/k$ is a finitely generated field extension of dimension d, then $\dim_k \Omega_{k(V)/k} = d$ iff $k(V)/k$ has a separable transcendence basis of cardinality d – no worries about this in characteristic 0! In general, what can be said is that the generic fibre is a **nonsingular** curve over the imperfect field $k(C)$, and – alas! – over an imperfect field nonsingularity (i.e., regularity of the local rings) and smoothness (the Jacobian condition) are distinct notions. We can see this explicitly [Hartshorne, Exercise III.10.1]: consider

$$y^2 = x^p - t,$$

which we may equally well view as giving (the affine model of) a hypersurface in $\mathbb{P}^3$ or as a curve X over $k(\mathbb{P}^1)$. This curve is geometrically singular: as soon as we pass to the field extension $k' = k(t^{1/p})$, the equation becomes

$$y^2 = (x - t^(1/p))^p,$$

and certainly $(t^{1/p}, 0)$ is a singular point. To see that over k it is not smooth, consider the differential condition: $\Omega^1_(X/k)$ is generated by dx and dy and subject only to the relation $0 = d(y^2 - x^p - t) = 2ydy$ – at the unique (scheme-theoretic, not k-valued!) point $P \in X$ with $y(P) = 0$, this means that the module of differentials is free of rank $2 > 1$. On the other hand, this strange point P is not a singular point (over k). Indeed, I claim that $m_P = (y)$, so that the local ring $\mathcal{O}_P$ is a DVR and is hence nonsingular. And indeed, $\mathcal{O}_X/(y) = k[x,y]/(y^2 - x^p + t, y) = k[x]/(x^p - t)$ which is a purely inseparable **field** extension of k.

Taking $p = 3$ in the preceding discussion, we do indeed get a quasi-elliptic surface $X/\mathbb{P}^1$ via $y^2 = x^3 - t$. (In fact, quasi-elliptic surfaces exist only in characteristics 2 and 3. The generic fibre, being nonsingular of arithmetic genus 1, must be analytically irreducible – in other words, the preceding considerations require us to have a cusp and not a regular double point. And it is intuitively clear that cuspidal curves can behave in weird ways only when $p = 2$ or 3.) One of the results in Lang's thesis computes the Hodge and DeRham Betti numbers of a quasi-elliptic surface $\pi : X \to C$ fibred over an elliptic curve C/k for which there exists a section $s : C \to X$ such that $s(C)$ is contained in the smooth locus of X. Then, if the line bundle $R^1\pi_*\mathcal{O}_X$ on C has degree 1, one has

$$h^{0,0} = h^{2,2} = 1,\ h^{0,1} = h^{1,0} = 2,\ h^{1,1} = 4,\ h^{0,2} = h^{2,0} = 1,\ h^{1,2} = h^{2,1} = 2.$$

$$b_0'' = 1,\ b_1'' = 3,\ b_2'' = 4,\ b_3'' = 3,\ b_4'' = 1.$$

So we have strict inequality wherever possible: $b_1' = 4 > 3 = b_1''$ and $b_2' = 6 > 4 = b_2'' = 4$, $b_3' = 4 > 3 = b_3''$ (of course $b_0' = b_0'' = b_4' = b_4'' = 1$ for any surface).

Example (Enriques surfaces in characteristic 2): A recent book by Dolgachev concerns the geometry of Enriques surfaces (recall that, over $\mathbb{C}$, an Enriques surface is a surface of Kodaira dimension 0 which admits an unramified double cover which is a K3 surface). It turns out that the majority of the book deals with characteristic 2, where Enriques surfaces have a "new" complexity (one presumes) undreamt of by Enriques. We mention only the example of a **supersingular** Enriques surface: namely $H^1(\mathcal{O}_X) \cong k \cong H^2(\mathcal{O}_X)$, and Frobenius induces the zero endomorphism on $H^1(\mathcal{O}_X)$ (which is why they are called supersingular). Since over $\mathbb{C}$ K3 surfaces are simply connected, Enriques surfaces have $\pi_1 = \mathbb{Z}/2\mathbb{Z}$ and hence $b_1 = 1/2h^{0,1} = 1/2h^{1,0} = 0$. Moreover, one has over $\mathbb{C}$ that the canonical bundle $K_X = \Omega^2$ is nontrivial, but its square K_X^2 is (so X has Kodaira dimension 0). So supersingular Enriques surfaces are really weird, and indeed exist only in characteristic 2. One finds for them that $b_1'' = 1$ but $b_1' = 2$, so again we have a failure of the Hodge to DeRham spectral sequence to degenerate.

4.3 The Deligne-Illusie Theorem

We have now examined a handful of examples of degeneration / nondegeneration of the Hodge to DeRham spectral sequence. The reader may well have noticed that each case of nondegeneration occurred in the presence of some weird geometric phenomenon which was only possible in positive characteristic.

To make this precise, say a smooth projective variety X/k over an algebraically closed field of characteristic p can be lifted to characteristic zero if there exists a smooth scheme $\mathcal{X}/W(k)$ over the ring of Witt vectors of k whose special fibre is X. There are various criteria for varieties to be liftable, e.g. a theorem of Grothendieck [SGA1] says that varieties with both $h^{0,2} = 0$ and $h^2(X, T_X) = 0$ ($T_X = \Omega_X^\vee$ is the tangent bundle) are liftable, so in particular curves are liftable. On the other hand, (say principally polarized) abelian varieties are liftable as well, since the moduli space of principally polarized abelian varieties of dimension g is smooth over $\mathbb{Z}$. It can be shown that (at least when $\dim X < p$) liftability **implies** degeneration of the spectral sequence. Since we know the spectral sequence degenerates in characteristic zero, this sounds vaguely plausible. But Deligne and Illusie proved something much stronger:

Theorem 31 *(Deligne-Illusie) Let X/k be a smooth projective variety over an algebraically closed field of characteristic $p > 0$ and of dimension $d < p$. If X lifts even to $W_2(k)$, then the Hodge to DeRham spectral sequence degenerates.*

Of course liftability to $W_2(k)$ certainly does not imply liftability all the way to $W(k)$, so their proof cannot possibly use the Hodge theorem for Kahler manifolds. So consider a variety X over a number field K: for all but finitely many places v of K, X does extend to a smooth scheme over $\mathcal{O}_K$, and by the previous theorem one knows that in characteristic p the spectral sequence degenerates. Deligne and Illusie exploit this degeneration to show that the spectral sequence of the generic fibre degenerates. In summary, they give a **purely algebraic** proof of the (degeneration of the spectral sequence part of the) Hodge theorem for algebraic $\mathbb{C}$-manifolds by reducing to characteristic p, where the degeneration of the spectral sequence is in general false! Those who prefer to keep their distance from the "pathologies" of algebraic geometry in positive characteristic would do well to remember this remarkable success story.

4.4 Serre's Mexico paper

Witt vector cohomology Let A be a ring of characteristic p; we have defined $W_n(A)$, the Witt vectors of length n with coefficients in A, and $W(A) := \varprojlim_n W_n(A)$. We have F and V: first a **Frobenius** $(a_0, \ldots, a_{n-1}) \mapsto (a_0^p, \ldots, a_{n-1}^p)$; and second a **Verschiebung** (or **décalage**) $V : W_n(A) \to W_{n+1}(A)$, $(a_0, \ldots, a_{n-1}) \mapsto (0, a_0, \ldots, a_{n-1})$. We also have **restriction** $R : W_{n+1}(A) \to W_n(A)$, $(a_0, \ldots, a_n) \to (a_0, \ldots, a_{n-1})$. Some identities satisfied are:

$$(Vx) \cdot y = V(x \cdot FRy), \; RVF = FRV = RFV = p \cdot .$$

Signalons que $W_1(A) = A$. The constructions of $W_n(A)$, $W(A)$ and the F, V, R are functorial in A and very nice: in fancy language, W_n is a functor represented by a **ring scheme** over $\operatorname{Spec} \mathbb{F}_p$ whose underlying scheme is isomorphic to $\mathbb{A}^n/\mathbb{F}_p$, but with a "twisted" addition and multiplication given by the usual universal polynomials. Similarly W is "represented by an ind-scheme" over

Spec $\mathbb{F}_p$, but for our purposes the extra erudition in these statements comes devoid of extra content.

The point is that if $(X, \mathcal{O}_X)$ is a locally ringed space of characteristic p, then we have a sheaf of rings $\mathcal{W}_n$ on X via $U \mapsto W_n(\Gamma(U, \mathcal{O}_X))$ and $\mathcal{W}$ on X as the inverse limit of the $\mathcal{W}_n$. Notice that $W_1 = \mathcal{O}_X$. For all $n \geq m$ we have a short exact sequence of sheaves on X

$$0 \to \mathcal{W}_m \stackrel{V^{n-m}}{\to} \mathcal{W}_n \stackrel{R^m}{\to} \mathcal{W}_{n-m} \to 0. \tag{4.1}$$

Taking $m = 1$, we get

$$0 \to \mathcal{O} \stackrel{V^{n-1}}{\to} \mathcal{W}_n \stackrel{R}{\to} \mathcal{W}_{n-1} \to 0. \tag{4.2}$$

It follows that $\mathcal{W}_n$ is an n-fold (nontrivial!) extension of the structure sheaf $\mathcal{O}_X$.

Suppose now that X is a variety over an algebraically closed field k of characteristic p. Then $\mathcal{O}$ is a sheaf of k-modules so $\mathcal{W}_n(\mathcal{O})$ is a sheaf of $W_n(k)$-modules. Put $\Lambda := W(k)$; we get that the cohomology groups $H^q(X, \mathcal{W}_n)$ are canonically $\Lambda_n = \Lambda/(p^n)$-modules. In fact more is true: they are $\mathcal{D}_n := \Lambda_n\langle F,\ V\rangle$-modules, where we mean the non-commutative polynomial ring over Λ_n generated by indeterminates F and V and subject to the relations $F(\lambda v) = F\lambda F(w),\ V(\lambda w) = F^{-1}\lambda V(w),\ R(\lambda w) = \lambda R(w), RFV = FRV = RFV = p.$ [3]

The $\mathcal{W}_n$ are not $\mathcal{O}_X$-modules ($\mathcal{W}_n$ is a sheaf of rings of characteristic p^n, while $\mathcal{O}_X = \mathcal{W}_1$ is of characteristic p), they behave in much the same way (in sickness and in health). We have:

Proposition 32 *Let X be a scheme of characteristic p.*
a) $H^q(X, \mathcal{W}_n) = 0$ for $q > \dim X$.
b) If X is affine, $\mathcal{W}_n$ is acyclic for sheaf cohomology.
c) If X is projective, $H^q(X, \mathcal{W}_n)$ is a finite-length Λ_n-module.

In all cases, the results follow immediately by induction and the long exact cohomology sequence of (2).

The restriction maps R^{n-m} induce morphisms $H^q(X, \mathcal{W}_n) \to H^q(X, \mathcal{W}_{n-m})$, which form an inverse system of abelian groups. We denote $H^q(X) := H^q(X, \mathcal{W}) := \varprojlim_n H^q(X, \mathcal{W}_n)$, the **Witt vector cohomology** of X; they are canonically $\Lambda = W(k)$-modules. Note well that the notation $H^q(X, \mathcal{W})$ is truly abusive: we have an inverse limit of cohomology groups, not a cohomology group of the sheaf $\varprojlim \mathcal{W}_n$ (which would not be the same, just as in ℓ-adic cohomology).

[3] D is for Dieudonné, of course.

4.4.1 Torsion

Recall that we said in the introduction that if $\ell \neq p$ the Weil cohomology theory $H^q(X, \mathbb{Z}_l)$ loses information about the p-torsion phenomena in characteristic p. In contrast, the Witt vector cohomology – which *a priori* is not Weil, by Proposition 18a) – is very sensitive to p-torsion phenomena. So Witt vector cohomology is interesting, even if it is not always well-behaved.

Consider the coboundary operators in the long exact cohomology sequence of (1) above:

$$\ldots \to H^q(X, \mathcal{W}_m) \overset{V^{n-m}}{\to} H^q(X, \mathcal{W}_n) \overset{R^m}{\to} H^q(X, \mathcal{W}_{n-m}) \overset{\delta^q_{n,m}}{\to} H^{q+1}(X, \mathcal{W}_m) \to \ldots$$

$$\delta^q_{n,m} : H^q(X, \mathcal{W}_{n-m}) \to H^{q+1}(X, \mathcal{W}_m),\ n \geq m.$$

These operators are F^{n-m}-semilinear.

When $n \geq 2m$, the ideal $V^{n-m}(\mathcal{W}_m) \leq \mathcal{W}_n$ has square zero, which allows us to calculate the effect of $\delta^q_{n,m}$ on the cup product:

$$\delta^q_{n,m}(x \cdot y) = \delta^r_{n,m}(x) \cdot F^{n-m}R^{n-2m}y + (-1)^r F^{n-m}R^{n-2m}x \cdot \delta^s_{n,m}(y),$$

where $x \in H^r(X, \mathcal{W}_{n-m}),\ y \in H^s(X, \mathcal{W}_{n-m})$ and $r + s = q$.

We say that X has **no homological torsion in dimension** q if the maps $\delta^q_{n,m} = 0$ for all $n \geq m$. Clearly this is equivalent to $R^m : H^q(X, \mathcal{W}_n) \to H^q(X, \mathcal{W}_{n-m})$ is a surjection for all $n \geq m$.

Examples: A variety of dimensin d has no cohomological torsion in dimension d. Every variety has no cohomological torsion in dimension 0 (since $\mathcal{W}_n \to \mathcal{W}_{n-1}$ is surjective as a morphism of **pre**sheaves – we have multiplicative representatives). It follows that algebraic curves have no homological torsion, and this fact should be viewed as serving to "explain" why algebraic curves in characteristic p have no "pathologies."

The morphisms β_n: A variation on the above theme will give us endomorphisms on the classical cohomology algebra $H^*(X, \mathcal{O})$. Indeed, we define

$$\beta^q_1 : H^q(X, \mathcal{O}) \to H^{q+1}(\mathcal{O})$$

as the $\delta^q_{2,1}$ map associated to

$$0 \to \mathcal{O} \to \mathcal{W}_2 \to \mathcal{O} \to 0.$$

We have $\beta^2_1 = 0$, so we may define new cohomology groups $H^q(X, \mathcal{O})_2 := \ker(\beta^q_1)/\operatorname{Im}(\beta^{q-1}_1)$. In general, we define

$$Z^q_n := \operatorname{Im}(H^q(X, \mathcal{W}_n) \overset{R^{n-1}}{\to} H^q(X, \mathcal{O})) = \ker(H^q(X, \mathcal{O}) \overset{\delta^q_{n,n-1}}{\to} H^{q+1}(X, \mathcal{W}_{n-1})).$$

$$B_n^q := \ker(H^q(X,\mathcal{O}) \overset{V^{n-1}}{\rightarrow} H^q(X,\mathcal{W}_n)) = \mathrm{Im}(H^{q-1}(X,\mathcal{W}_{n-1}) \overset{\delta^{q-1}_{n,1}}{\rightarrow} H^q(X,\mathcal{O})).$$

The Z_n^q give a decreasing filtration on $H^q(X,\mathcal{O})$, the B_n^q give an increasing filtration on $H^q(X,\mathcal{O})$ and $B_n^q < Z_n^q$ for all n. For $n = 1$ we have $B_1^q = 0$, $Z_1^q = H^q(X,\mathcal{O})$. In general, put $H^q(X,\mathcal{O})_n := Z_n^q/B_n^q$. If $x \in Z_n^q$, choose a $y \in H^q(X,\mathcal{W}_n)$ such that $R^{n-1}y = x$, and put $z := \delta^q_{n+1,1}(y) \in H^{q+1}(X,\mathcal{O})$. By passage to the quotient we get a well-defined homomorphism

$$\beta_n^q : H^q(X,\mathcal{O})_n \to H^{q+1}(X,\mathcal{O})_n$$

such that $\ker(\beta_n^q) = Z^q_{n+1}/B_n^q$, $\mathrm{Im}(\beta_n^{q-1}) = B^q_{n+1}/B_n^q$. In order for X to have no torsion, it is necessary and sufficient for the β_n^q to all be zero.

Remark: Serre says in his paper that “there is every reason to think” that $H^*(X,\mathcal{O})$ admits reduced Steenrod powers and that the operation β_1 coincides with one of these powers. In the endnotes to his *Oeuvres* he adds that these powers indeed have been defined (by Epstein). Malheureusement, I have no idea what reduced Steenrod powers are.

Case of projective varieties: assuming that X/k is projective (or even complete) much simplifies the preceding section: since then $H^q(X,\mathcal{O})$ is a finite-dimensional k-vector space, it follows that the filtrations Z_n^q and B_n^q must stabilize: we denote by Z^q_∞, B^q_∞ their limiting values. In particular, the $\beta_n^q = 0$ for all $n \gg 0$.

We return to the inverse limits of the cohomology groups, which we defined earlier but said nothing about.

Lemma 33 *The inverse limit of exact sequences of finite length modules is exact.*

Applying this lemma to the long exact sequence

$$\ldots H^q(X,\mathcal{W}_N) \overset{V^N}{\rightarrow} H^q(X,\mathcal{W}_{N+n}) \to H^q(X,\mathcal{W}_n)$$

we get

$$\ldots \to H^q(X) \overset{V^n}{\rightarrow} H^q(X) \to H^q(X,\mathcal{W}_n) \overset{\delta_n^q}{\rightarrow} H^{q+1}(X) \to \ldots \tag{4.3}$$

Take $n = 1$ in the above. The image of $H^q(X)$ in $H^q(X,\mathcal{O})$ is just Z^q_∞ (this uses the lemma). It follows that X has no torsion in dimension q if and only if $\delta_1^q = 0$ – the other δ_n^q's are then automatically zero. For arbitrary n, equation (9) shows that the image of $H^q(X)$ in $H^q(X,\mathcal{W}_n)$ is $H^q(X)/V^nH^q(X)$. It follows that $H^q(X) = \varprojlim_n H^q(X)/V^nH^q(X)$; we gather that $H^q(X)$ is complete and separated for the filtration by powers of V.

Put now $T_n^q = \ker(V^n : H^q(X) \to H^q(X))$; by (9), T_n^q is the image of δ^q_{n-1} – so it is fair to think of it as torsion – and is a finite length Λ-submodule of $H^q(X)$ (indeed a Λ_n-submodule, hence a torsion Λ-module). Evidently $T^q_n \subset T^q_{n+1}$, and a little diagram-chasing shows that $T^q_{n+q}/T^q_n \cong Z^{q-1}_{n+1}/Z^{q-1}_\infty$. In particular, the T^q_n stabilize, and we denote the limit (union) by T^q. That T^q be zero is equivalent to X having no torsion in dimension $q-1$. Moreover, the length of T^q is preicsely

$$l(T^q) = \sum_{n=1}^{\infty} l(Z^{q-1}_n/Z^{q-1}_\infty) = \sum_{n=1}^{\infty} n \cdot l(\mathrm{Im}(\beta^{q-1}_n)). \tag{4.4}$$

Proposition 34 ***Suppose*** *that $H^q(X)$ is a finite-type Λ-module. Then its torsion submodule is T^q, and $L^q = H^q/T^q$ is a free Λ-module of rank $l(L^q/VL^q) + l(L^q/FL^q)$.*

Proof (Does it feel like Iwasawa theory yet?): Let n be such that $T^q = T^q_n$. Then $V^n = 0$ on T^q; since $FV = p$, we conclude that T^q is p^n-torsion, so that at least T^q is contained in the torsion submodule T' of H^q. (We already observed this above.) Consider now the endomorphism $V' : T'/T^q \to T'/T^q$ that V induces on the quotient; by definition of T^q, V' is injective. But, since H^q is a finite-type module over the PID Λ, its torsion submodule T' is finite-length, and an injective monomorphism is bijective. So $T' = VT' + T^q$. Applying V^n,

$$V^n T' = V^{n+1} T' = \dots.$$

Since $\cap V^n H^q = 0$, we get $V^n T' = 0$, so $T' < T^q$ and $T^q = T'$, proving the first part of the proposition. For the second, $L^q = H^q/T^q$ is obviously free, of rank equal to the k-dimension of $L^q/pL^q = L^q/FVL^q$. Since V is a semi-linear isomorphism from L^q to VL^q, we have

$$\dim_k(L^q/FVL^q) = l(L^q/VL^q) + l(VL^q/FVL^q) = l(L^q/VL^q) + l(L^q/FL^q).$$

Proposition 35 *If H^q/FH^q has finite length, H^q has finite type.*

Corollary 36 *The following are equivalent:*
a) Every $H^q(X)$ is a finite-type Λ-module.
b) For all q, $S^q = \varprojlim_n H^q(X, \mathcal{W}_n/F\mathcal{W}_n)$ has finite length.

Proof: Passing to the limit over the exact sequence

$$\dots H^q(X, \mathcal{W}_n) \xrightarrow{F} H^q(X, \mathcal{W}_n) \to H^q(X, \mathcal{W}_n/F\mathcal{W}_n) \to H^{q+1}(X, \mathcal{W}_{n\to} \dots$$

we get the exact sequence

$$\dots H^q \xrightarrow{F} H^q \to S^q \to H^{q+1} \to \dots.$$

If the H^q are finite-type, then Proposition 20 implies that the cokernel of F has finite length, hence so does its kernel, so S^q has finite length. Conversely, if S^q has finite length, so does the cokernel of F and we apply the previous Proposition.

Corollary 37 *If X has neither $q-1$ nor q-torsion and*

$$F : H^q(X, \mathcal{O}) \to H^q(X, \mathcal{O})$$

is surjective, then H^q is a free Λ-module of rank equal to $h^q(X, \mathcal{O})$.

Proof: Since X has no q-torsion, $Z^q_\infty = H^q(X, \mathcal{O})$, and the hypothesis on F implies that $F : H^q/VH^q \to H^q/VH^q$ is surjective. By an induction on n, the same is true for $F : H^q/V^nH^q \to H^q/V^nH^q$, and (using our lemma about inverse limits of finite length modules), we get $FH^q = H^q$. Since X has no torsion in dimension $q-1$ we have $T^q = 0$ and $H^q = L^q$. By applying the last two propositions the result follows.

At this point, the reader should suspect that it is **not** easy to show that the H^q are, in general, finite-type Λ-modules. Indeed, it's impossible. Serre first gave as a counterexample a cuspidal (singular!) curve. Later he gave the more distressing counterexample of a supersingular abelian surface, which we shall be discussing presently.

4.4.2 H^1 of a normal projective variety

Let A be a perfectly arbitrary commutative ring (not necessarily of characteristic p) and consider an element $\alpha = (a_0, \ldots, a_{n-1})$ of $W_n(A)$. To α we associate the one-form

$$D_n(\alpha) = da_{n-1} + a_{n-2}^{p-1} da_{n-2} + \ldots + a_0^{p^{n-1}-1} da_0.$$

If A has characteristic 0, the components $a^{(0)},\ a(1), \ldots a^{(n-1)}$ are defined, and we have

$$D_n(\alpha) = \frac{1}{p^{n-1}} da^{(n-1)}.$$

From this we deduce

$$D_n(\alpha + \beta) = D_n(\alpha) + D_n(\beta)$$

and

$$D_n(\alpha\beta) \equiv D_n(\alpha) b_0^{p^{n-1}} + a_0^{p^{n-1}} D_n(\beta) \mod p.$$

By "prolongation of algebraic identities," we deduce that these equations remain true in characteristic p, the latter simplifying to

$$D_n(\alpha\beta) = D_n(\alpha) \cdot F^{n-1}R^{n-1}\beta + F^{n_1}R^{n-1}\alpha \cdot D_n(\beta).$$

In particular, let X be a normal variety and $A = \mathcal{O}_x$ be the local ring at some point x; we get a homomorphism $D_n : W_n(\mathcal{O}_x) \to \Omega^1_x$. If $\alpha \in FW_n(\mathcal{O}_x)$ – i.e., if the components of α are pth powers, then evidently $D_n(\alpha) = 0$. Conversely, one knows that the relation $da = 0$ implies that a is a pth power in $k(X)$; more generally, $D_n(\alpha) = 0$ implies that each component a_i of α is of the form b_i^p, $b_i \in k(X)$; but $b_i^p = a$ implies b_i is integral over $\mathcal{O}_x$, hence itself belongs to $\mathcal{O}_x$ (this is precisely what the assumption of normality is made to ensure). It follows that the kernel of D_n is $FW_n(\mathcal{O}_x)$. Sheafifying, we get:

Lemma 38 *The map D_n defines, by passage to the quotient, a monomorphism of sheaves $\mathcal{W}_n/F\mathcal{W}_n \hookrightarrow \Omega^1_x$.*

Suppose now that X is projective normal. By the Lemma, $H^0(X, \mathcal{W}_n/F\mathcal{W}_n)$ is a subvector space of the finite-dimensional k-space $H^0(X, \Omega^1)$; we deduce that $\dim_k H^0(X, \mathcal{W}_n/F\mathcal{W}_n)$ is *bounded* as $n \to \infty$. Let ν be the maximum value, and put $g = \dim Z^1_\infty = \dim(\operatorname{Im} H^1 \to H^1(X, \mathcal{O}))$.

Proposition 39 *The Λ-module $H^1 = H^1(X, \mathcal{W})$ is a free module of rank at most $g + \nu$, with equality holding if X has no torsion in dimension 1.*

Proof: The $H^0(X, \mathcal{W}_n/F\mathcal{W}_n)$ form an increasing sequence of subspaces of $H^0(X, \Omega^1)$, so there is an integer n_0 such that $h^0(X, \mathcal{W}_n/F\mathcal{W}_n) = \nu$ for all $n \geq n_0$. From the exact sequence of sheaves

$$0 \to \mathcal{W}_n \xrightarrow{F} \mathcal{W}_n \to \mathcal{W}_n/F\mathcal{W}_n \to 0,$$

we deduce the exact sequence

$$0 \to H^0(X, \mathcal{W}_n/F\mathcal{W}_n) \to H^1(X, \mathcal{W}_n) \xrightarrow{F} H^1(X, \mathcal{W}_n).$$

Since $H^1(X, \mathcal{W}_n)$ is a finite-length Λ-module, we gather that

$$\ell(H^1(X, \mathcal{W})n)/FH^1(X, \mathcal{W}_n)) = \nu, \ n \geq n_0.$$

Since $H^1/FH^1 = \varprojlim H^1(X, \mathcal{W}_n)/FH^1(X, \mathcal{W}_n)$, one also has $\ell(H^1/FH^1) \leq \nu$, so that by Proposition 21, H^1 is a finite-type Λ-module. Moreover, since there is no 0-torsion, we have $T^1 = 0$, whence $L^1 = H^1$ and $H^1/VH^1 = Z^1_\infty$; applying Proposition 20, we get that H^1 is Λ-free of rank $\dim Z^1_\infty + \ell(H^1/FH^1) \leq g + \nu$, giving the first part of the Proposition. Suppose now that X has no one-dimensional torsion, so the homomorphisms $R : H^1(X, \mathcal{W}_{n+1}) \to H^1(X, \mathcal{W}_n)$ are surjetive, hence also the quotient maps

$$R : H^1(X, \mathcal{W}_{n+1})/FH^1(X, \mathcal{W}_{n+1}) \to H^1(X, \mathcal{W}_n)/FH^1(X, \mathcal{W}_n).$$

But if $n \geq n_0$ both of sides have length ν, meaning that the map is a bijection. Passing to the limit, one can say the same for the homomorphism $H^1/FH^1 \to H^1(X, \mathcal{W}_n)/FH^1(X, \mathcal{W}_n), so\ell(H^1/FH^1) = \nu$. Applying again Proposition 20 one concludes that the rank of H^1 equals $g + \nu$.

Remark: Even when X has one-dimensional torsion, one can calculate the rank of H^1: it is $g + \nu - \ell(T^2/FT^2)$.

4.4.3 Case of algebraic curves

Here X/k will denote a complete, irreducible nonsingular curve over an algebraically closed field k of characteristic $p > 0$. Let $K = k(X)$. We view K as a constant sheaf on X with $\mathcal{O}$ as a subsheaf, getting the Cartier exact sequence

$$0 \to \mathcal{O} \to K \to K/\mathcal{O} \to 0.$$

Since K is constant and X is irreducible, it is flasque, and $H^1(X, K) = 0$; thus a piece of the long exact cohomology sequence gives

$$K \to H^0(X/K/\mathcal{O}) \to H^1(X, \mathcal{O}) \to 0.$$

Let R be the algebra of **repartitions** of X, so that an element $r \in R$ is a family $\{r_x\}_{x \in X(k)}$ with $r_x \in K$ such that for almost every x, $r_x \in \mathcal{O}_x$.[4] The repartitions with $r_x \in \mathcal{O}_x$ for all x form a subring $R(0)$, and $R/R(0) = H^0(X, K/\mathcal{O})$. We then regard the above exact sequence as giving an isomorphism

$$R/(R(0) + K) \cong H^1(X, \mathcal{O}). \tag{4.5}$$

We get now the elegant formulation of Serre duality via **residues**: the pairing $\langle\ ,\rangle : R \times H^0(X, \Omega^1)$ given by

$$\langle r, \omega\rangle - \sum_{x \in X(k)} \mathrm{res}_x(r_x \omega)$$

induces a duality modulo $R(0) + K$, i.e., exhibits $H^1(X, \mathcal{O})$ and $H^0(X, \Omega^1$ as dual k-vector spaces.

4.4.4 The Hasse-Witt matrix

The goal is to find the matrix of the semilinear endomorphism $F : H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O})$ with respect to some suitable basis of the (g-dimensional!) vector space $H^1(X, \mathcal{O})$. The first remark is that under the isomorphism of (11), F corresponds to taking the pth-power in R. On the other hand, using the duality between $R/(R(0) + K)$ and $H^0(X, \Omega^1)$, there are g points $P_1, \ldots, P_g \in X(k)$ with corresponding uniformizing parameters $t_1, \ldots, t_g$, such that the repartitions $r_i = \{r_{i,x}\}$, $r_{i,x} = 0, x \neq P_i,\ r_{i,P_i} = 1/t_i$ give a basis for $R/(R(0) + K)$. (One needs only to choose them such that the divisor $P_1 + \ldots + P_g$ is *nonspecial*, i.e. has vanishing H^1 in the Riemann-Roch theorem.)

Let $A = (a_{ij})$ be the matrix of F with respect to this basis $\{r_i\}_{i=1}^g$. By definition, we have

$$r_i^p \cong \sum_{j=1}^{g} a_{ij} r_j \mod (R(0) + K),\ 1 \leq i \leq g.$$

So there are global functions $g_i \in K$ such that

$$g_i \equiv r_i^p - \sum_{j=1}^{g} a_{ij} r_j \mod R(0).$$

In other words, each g_i is regular away from the P_i and at P_j has principal part $\delta_{ij}/t_j^p - a_{ij}/t_j$. This recovers the classical definition of the **Hasse-Witt** matrix

[4] This construction is due to Chevalley, i.e., the same person who introduced adeles.

of X.

We will now have use for the notion of a "semilinear Jordan decomposition"; this goes as follows. Let F be a p-linear endomorphism of a finite dimensional k-vector space V. There is a canonical direct sum decomposition

$$V = V_s \oplus V_n$$

into F-stable subspaces, such that F is *nilpotent* on V_n and *bijective* on V_s. This is to be thought of as analogous to the linear algebra fact that one can decompose a linear transformation on a finite-dimensional vector space into a nilpotent part (corresponding to the zero eigenspace) together with an invertible part (corresponding to everything else). We denote the dimensions of V_s and V_n by $\sigma(V)$, $v(V)$ respectively. One shows first that V_s possesses a basis $e_1, \ldots, e_\sigma$ such that $F(e_i) = e_i$ for all i; the $v \in V$ such that $F(v) = v$ are then the $\mathbb{Z}/p\mathbb{Z}$-span of $e_1, \ldots, e_\sigma$; clearly then we get a (finite)elementary p-group V^F of order p^σ. From the existence of such a basis, we see that the map $1 - F$ on V is surjective.

Let V' be the dual vector space to V. Then we have F', a p^{-1}-linear endomorphism of V' defined by adjunction:

$$\langle Fv, v' \rangle = \langle v, F'v' \rangle^p, \ v \in V, \ v' \in V'.$$

We get a similar decomposition of the dual space

$$V' = V'_s \oplus V'_n,$$

such that if $\{e'_i\}$ is the dual basis to $\{e_i\}$, we have $F'(e'_i) = e'_i$ and similarly $V'^F = \langle e'_i \rangle_{\mathbb{Z}/p\mathbb{Z}}$; the groups V^F and V'^F are in Pontrjagin duality.

These considerations apply to $V = H^1(X, \mathcal{O})$, $V' = H^0(X, \Omega^1)$. We write simply σ, v for $\sigma(V)$, $v(V)$; clearly $g = \sigma + v$. In fact the integer σ is just the *rank* of the matrix $AA^p \cdots A^{p^{g-1}}$ (so in the elliptic curve case it is just the rank of A).

Proposition 40 *(Cartier) For all m, the image of*

$$D_m : H^0(X, \mathcal{W}_m / F\mathcal{W}_m) \to H^0(X, \Omega^1)$$

is equal to the kernel of F'^m.

It follows that, for $m \gg 0$, the image of D_m is equal to the nilpotent component $H^0(X, \Omega^1)_n$, hence has dimension v. It follows that this v *coincides* with the $\nu = \max h^0(X, \mathcal{W}_m / F\mathcal{W}_m)$ defined above. Using Proposition 25 (which applies because curves have no torsion), we conclude:

Theorem 41 *The Λ-module $H^1(X, \mathcal{W})$ is free of rank $g + v = 2g - \sigma$. In particular, it depends only on the Hasse-Witt matrix of X.*

Example: Let X/k be a genus one curve – i.e., an elliptic curve, since k is algebraically closed. Then the Hasse-Witt matrix is a scalar, so (for a fixed elliptic curve; apologies to the mod p modular forms posse) the only basis-invariant piece of information we can extract is whether or not it is zero. Of course, the Hasse invariant is zero iff the curve is supersingular, although to see this involves[5] an identification of $H^1(A, \mathcal{W})$ with the **formal part** (i.e., the nonétale part) of the Dieudonné module. In particular, E/k is supersingular iff $h^1(X, \mathcal{W}) = 2$; ordinarily (pun intended), $h^1(X, \mathcal{W}) = 1$.

4.4.5 The case of abelian varieties

This subsection briefly summarizes some results from Serre's later paper *Quelque propriétés des variétés abéliennes en caractéristique* p. As many of these results are much better known (indeed there is substantial intersection between this paper and the final chapter of Serre's book *Groupes algébriques et corps de classes*, and the complement is now part of the Cartier-Dieudonné theory of p-divisible groups of abelian varieties).

Let A/k be an abelian variety of dimension d over an algebraically closed field of characteristic p. The key point is the complete determination of the cohomology algebra $H^*(A) = \sum_n H^n(A, \mathcal{O}_A)$, namely it is what you want it to be:

Theorem 42 *$H^*(A)$ is the exterior algebra on the k-vector space $H^1(A, \mathcal{O}_A)$, which is of dimension d.*

Concerning the proof we say only this: the hard part is the $h^1(A, \mathcal{O}_A) = \dim A$ part (which is part of the basic work on commutative algebraic groups done by Rosenlicht in the 1950's).[6] Knowing this, the rest of the theorem is purely algebraic, coming down to the fact that $H^*(A)$ has the natural structure of **Hopf algebra**.

Another important theorem which is an essentially formal consequence of this is:

Theorem 43 *Abelian varieties have no homological torsion – i.e. the groups $T^q = 0$, or equivalently the maps R on Witt vector cohomology are all surjective.*

Accordingly the formula $H^1(A, \mathcal{W}) = d + \nu$ of the previous section holds. The quantity ν can be interpreted as promised in the last section: form the "augmented Tate module" $T_p(A) = H^1(A, \mathcal{W}) \oplus T_{p,\text{ét}}(A)$, where the second term is the Tate module in the naive sense: i.e., really the inverse limit of the p-power torsion in $A(k)$. Notice that $T_p(A)$, unlikely-looking agglomeration of dissimilar objects though it is, is at least a finite free $\Lambda = W(k)$-module.

Theorem 44 *The rank of $T_p(A)$ is $2d$.*

[5] At least, it does for me; there must be more elementary ways to proceed.

[6] The title of his paper is indeed *Some basic theorems on algebraic groups*, Am. J. Math. 78 (1956), pp. 401-443.

Write a for the rank of the étale part of the Tate module – so that e.g. $A(k)[p] \cong (\mathbb{Z}/p\mathbb{Z})^a$. One knows that – in grave contrast to characteristic zero! – $a \le d$; when equality holds A is said to be **ordinary**. We conclude that $\nu = 2d - a - d = d - a$. In particular, this justifies the claim made in the last section that $\nu > 0$ means that our elliptic curve is supersingular. (But beware: starting in dimension 2 there is "room" between ordinary varieties and supersingular varieties.)

Serre's example: Let $X = E \times E$ be the square of a supersingular elliptic curve. It follows from the above discussion that F acts as zero on $H^1(X, \mathcal{O}) = H^1(E, \mathcal{O}_E) \times H^1(E, \mathcal{O}_E)$ (Kunneth formula!).

Lemma 45 *For all n, F acts as zero on $H^2(X, \mathcal{W}_n)$.*

Proof: We shall use the abbreviation $H_n^q := H^q(X, \mathcal{W}_n)$; especially, $H_1^q = H^q(X, \mathcal{O}_X)$. Due to the absence of homological torsion, we have a short exact sequence

$$0 \to H^q(X, \mathcal{W}_{n-1}) \xrightarrow{V} H^q(X, \mathcal{W}_n) \xrightarrow{R} H^q(X, \mathcal{O}) \to 0.$$

We go by induction on n, $n = 0$ being trivial. Since F and V commute, by induction F is zero on VH_{n-1}^2. We know that H_1^2 has dimension 1, with a basis being given by cup-product of two elements $x, x' \in H_1^1$. Choose elements $y, y' \in H_n^1$ such that $R^{n-1}y = x$, $R^{n-1}y' = x'$. Since $\mathcal{W}_n$ is a sheaf of rings, can compose with the natural map $\mathcal{W}_n \oplus \mathcal{W}_n \to \mathcal{W}_n$ (given by the product) to view the cup product as a law of composition on H_n^*. In particular, we have $y.y' \in H_n^2$ whose image under R^{n-1} is $x.x'$. The Λ-module H_n^2 is therefore generated by VH_{n-1}^2 and by $y.y'$, so it is enough to show that $F(y.y') = 0$.

Because of the supersingularity of X, we have $Fy = Fy' = 0 \in H_1^1$. Using the exact sequence, there are $z, z' \in H_{n-1}^1$ such that $Fy = Vz$, $Fy' = Vz'$. Since $F(y.y') = Fy.Fy'$, we have $F(y.y') = Vz.Vz'$. Using the identity $Va.b = V(a.FRb)$, we get that

$$Vz.Vz' = V(z \cdot FRVz');$$

since $FRV = p$, we get

$$F(y.y') = Vz.Vz' = V(pz.z').$$

Since $z.z' \in H^{2\,n-1}$, it is, by induction, killed by F, and *a fortiori* by $p = RVF$, so indeed $F(y.y') = 0$, completing the proof.

Corollary 46 *The Λ-module $H^2(X, \mathcal{W})$ is a p-torsion Λ-module which is **not** of finite-type.*

Proof: Passing to the limit on the above claim, F is zero on $H^2(X, \mathcal{W})$. Since $p = FV$ it follows that $H^2(X, \mathcal{W}) = H^2(X, \mathcal{W})[p]$. So everything in sight is thus a k-vector space. But since $\dim_k H^2(X, \mathcal{W}_1) = 1$, by induction using the above exact sequence we have $\dim_k H^2(X, \mathcal{W}_n) = n$. This means that $\dim_k H^2(X, \mathcal{W})$ is infinite!

4.5 The Cartier operator on differential forms

The proof of Proposition 26 involves a fundamental operator on differential forms due to Cartier, valid for varieties of arbitrary dimension. In the case of curves – which we limit ourselves to at present – it was already anticipated by Tate.

Let $x \in X(k)$ and $t \in \mathcal{O}_x$ such that dt is nonvanishing at x. The functions $1, t, \ldots, t^{p-1}$ therefore form a basis for $\mathcal{O}_x$ considered as a module over $\mathcal{O}_x^p$ (a p-basis – this certainly uses that $\mathcal{O}_x$ has dimension one). In other words, every function $f \in \mathcal{O}_x$ can be written uniquely as

$$f = f_0^p + f_1^p t + \ldots + f_{p-1}^p t^{p-1}, \; f_i \in \mathcal{O}_x. \tag{4.6}$$

The coefficients f_i^p can be written down; they are linear combinations of the successive derivatives $d^k f/dt^k, \; 0 \leq k \leq p-1$; in particular $f_{p-1}^p = -d^{p-1}f/dt^{p-1}$. Let $\omega = f dt$ be an element of Ω_x^1, and put

$$C(\omega) = f_{p-1} dt; \tag{4.7}$$

the operation $C : \Omega_x^1 \to \Omega_x^1$ just defined is the **Cartier-Tate operator**. One shows that it does **not** depend on the choice of t. Moreover, taking $f \in K$ and not necessarily in $\mathcal{O}_x$, one extends C to an operator defined on *all* (meromorphic) differentials on X.

Proposition 47 *(Cartier) a) $C(\omega_1 + \omega_2) = C(\omega_1) + C(\omega_2)$.*
b) $C(f^p \omega) = f C(\omega)$.
c) $C(df) = 0$.
d) $C(f^{p-1} df) = df$.
e) The sequence $0 \to \mathcal{W}_m / F\mathcal{W}_m \overset{D_m}{\to} \Omega_1 \overset{C_m}{\to} \Omega^1 \to 0$ is exact for all $m \geq 1$.

Proof: The first three parts are immediate from the (well-)definition of C. For d) see [Tate, Lemma 1]. Now for e): because of b) and d) it is clear that C is surjective, so we need to show only that $\ker(C^m) = \mathrm{Im}(D_m)$. For $m = 1$ this means that $C(\omega) = 0$ implies that $\omega = df$, which is clear from (12) and (13), as we can integrate both perfect pth powers – $d(g^p t) = d(g^p)t + g^p dt = g^p dt$ – and powers of t up to t^{p-2} (but not t^{p-1}!) Then we can reason by induction on m, using the formula

$$C D_m \alpha = D_{m-1} R \alpha, \; \alpha \in \mathcal{W}_m. \tag{4.8}$$

Because of (14), $\mathrm{Im}(D_m) \subset \mathrm{Ker}(C^m)$. Conversely, if $\omega \in \Omega_x^1$ is such that $C^m(\omega) = 0$, then by the induction hypothesis there exists $\beta \in W_{m-1}(\mathcal{O}_x)$ such that $D_{m-1}\beta = C(\omega)$. Choose $\alpha \in W_m(\mathcal{O}_x)$ such that $R\alpha = \beta$, (14) gives $C(\omega - D_m \alpha) = 0$, so $\omega - D_m \alpha = df$; putting $\alpha' = \alpha + V^{m-1} f$, we have $\omega = D_m \alpha + df = D_m(\alpha' - V^{m-1} f) + df = D_m \alpha' - df + df = D_m \alpha'$, completing the proof.

Proposition 48 *The homomorphism $C : H^0(X, \Omega^1) \to H^0(X, \Omega^1)$ coincides with the transpose F' of F.*

Proof: We must show that if ω is a differential form and r is a repartition, that we have

$$\langle r^p, \omega \rangle = \langle r, C\omega \rangle^p.$$

Using the residue characterization of the duality pairing, this becomes:

$$\sum_{x \in X(k)} \mathrm{res}_x(r_x^p \omega) = \sum_{x \in X(k)} \mathrm{res}_x(r_x C\omega)^p,$$

which in turn is a consequence of the following (easy) formula:

$$\mathrm{res}_x(\pi) = \mathrm{res}_x(C\pi)^p$$

valid for any differential π.

Notice that Proposition XX is an immediate consequence of Proposition 33e) and Proposition 34.

Remark: The proper generalization of the Cartier operator to varieties of arbitrary dimension involves **closed** differential forms, a condition which we do not see in dimension 1.

4.5.1 Divisor classes of order p

Let $\mathrm{Pic}\, X$ denote the group of divisor classes on X (i.e., divisors modulo linear equivalence); let $J(X)$ denote the image of the degree zero divisors, and $J(X)[p] = (\mathrm{Pic}\, X)[p] < J(X)$ the p-torsion subgroup.

Proposition 49 *The group $J(X)[p]$ is canonically isomorphic to the additive group of global differentials $\omega \in H^0(X, \Omega^1)$ satisfying $C(\omega) = \omega$. In particular, $J(X)[p]$ is a finite group of order p^σ, σ the "invertible dimension of F' on $H^0(X, \Omega^1)$.*

Proof: First we define a mapping $\theta : J(X)[p] \to H^0(X, \Omega^1)$. For $d \in J(X)[p]$, let D be a representative divisor; since $pd = 0$, there is a function $f \neq 0$ such that $pD = (f)$. Put $\omega := df/f$, the "logarithmic differential" of f. A miracle occurs: if we change D to an equivalent divisor $D + (g)$, this multiplies f by g^p, hence does not change df/f (don't try this in characteristic zero!); hence df/f is actually an **invariant** of d, say $\theta(d)$. It remains to be checked that df/f is globally regular (as they used to say, a differential of the first kind). Take $x \in X(k)$; the equation $pD = (f)$ shows that we can write $f = t^p u$, where $u \in \mathcal{O}_x^\times$ (think of a divisor as a Cartier(!) divisor), so $df/f = du/u$ which is regular at x. Therefore we have indeed defined a map $\theta : J(X)[p] \to H^0(X, \Omega^1)$, which is immediately checked to be a monomorphism of groups (if $d \neq 0$, f is not itself a pth power, so $df/f \neq 0$). Using the formulas of Proposition 33, we have

$$C(df/f) = C(f^{p-1} df/f^p) = C(f^{p-1}/df)/f = df/f.$$

Convesely, a differential form such that $C(\omega) = \omega$ is of the form df/f by a theorem of Jacobson; if ω is regular, the order of f at any point of X is divisible by p, so that $(f) = pD$, so that $\omega = \theta(d)$, completing the proof.

Remarks: Since $\log_p \# J(X)[p] = a$, the a-number of the Jacobian variety X discussed above, this recovers the numerology $\nu + a = d$ (following Theorem 30) in the case of Jacobian varieties.

This last proposition extends to varieties of arbitrary dimension.

4.5.2 Cyclic-p^n-coverings of algebraic varieties

4.5.3 Basic facts about the quotient of a variety by a finite group of automorphisms

Let Y/k be an algebraic variety (we do not need to assume that k is algebraically closed) and $G \leq \mathrm{Aut}(Y/k)$ a group of automorphisms. In general, trying to construct the quotient Y/G is quite a delicate matter belonging to the realm of geometric invariant theory. When G is **finite**, this is supposed to be easy. Let's briefly review the construction (cf. [GACC]).

We make the technical assumption (A) that every orbit of G is contained in an affine open of Y. This is automatically true if Y is quasiprojective – it is a basic fact that every finite set is contained in an affine subset. Write $\pi : Y \to Y/G = X$ the projection map. We define $\mathcal{O}_X = \pi_*(\mathcal{O}_Y)$. The basic fact is that $(X, \mathcal{O}_X)$ is an algebraic variety and $Y \to X$ is a morphism. One sees this by glueing, reducing to the case of an affine variety, where one sees things contravariantly: if $Y = \mathrm{Spec}\, B$, then $X = \mathrm{Spec}\, A$, where $A = B^G$. One needs only check that the invariants of a finite-type reduced k-algebra under a finite group are finite-type reduced.

One has (as one wants) that Y/G is complete (resp. affine) iff Y is complete (resp. affine). (No such luck for arbitrary quotients; think of modding out GL_n by a Borel subgroup.)

One says that $\pi : Y \to X$ is a **covering** if G acts without fixed points. (Equivalently, π is a finite étale map!) We do not recopy the general facts about coverings, which can be found discussed at more length in [GACC].

4.5.4 $G = \mathbb{Z}/p\mathbb{Z}$

Let $\pi : Y \to X$ be an unramified Galois covering with group $G = \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$. There is a natural embedding $\varphi : G \hookrightarrow$ (the additive group of k). We may view π as a principal G-bundle on X – *a priori* in the flat topology, but (by descent!), $H^1_{fl}(X,) = H^1_{\text{ét}}(X,) = H^1_{Zar}(X,) = H^1(X, \mathcal{O}_X)$. It follows that, via φ each G-covering of X gives an element of $H^1(X, \mathcal{O}_X)$. Writing $\pi_1(X, G)$ for the set of all G-coverings, we get a map $f_1 : \pi_1(X, G) \to H^1(X, \mathcal{O}_X)$.

Proposition 50 *The map f_1 is an isomorphism onto the Frobenius-invariant subspace $H^1(X, \mathcal{O}_X)^F$.*

Proof (not quite the one given by Serre): Take the cohomology of the **Artin-Schreier** sequence of étale sheaves on X:

$$0 \to \underline{G} \to \overset{F-1}{\to} \to 0.$$

We use that $H^1(X, \underline{G}) = \pi_1(X, G)$.

Corollary 51 *The dimension σ of the semisimple component $H^1(X, \mathcal{O})_s$ of $H^1(X, \mathcal{O})$ is such that $\pi_1(X, \mathbb{Z}/p\mathbb{Z})$ is a finite group of order p^a.*

Corollary 52 *A variety of dimension at least 2 which is a complete intersection does not admit a (connected!) cyclic p-covering.*

Proof: In characteristic zero, we "already know this," since by the Lefschetz hyperplane theorem any d-dimensional complete intersection $V \subset \mathbb{P}^N$ has homotopy groups $\pi_1(V(\mathbb{C})) = \pi_1(\mathbb{P}^n(\mathbb{C})), \ldots, \pi_{d-1}(V(\mathbb{C})) = \pi_{d-1}(\mathbb{C})$, so in particular is simply-connected in dimension at least 2. Notice in this case that Hodge theory entails $0 = b_1(V) = 1/2h^{1,0} = 1/2h^{0,1}$, so that we have $H^1(X, \mathcal{O}) = 0$. Serre showed [FAC, SS78] in arbitrary characteristic (i.e., purely algebraically) that complete intersections in dimension at least 2 have $H^1(X, \mathcal{O}) = 0$ (and the higher-dimensional analogues implied in the complex case by the vanishing of the higher Betti numbers in the appropriate range, as above). Variant: Serre gives a different proof of Proposition 36 that seems closer to our purposes. It is based on the following

Lemma 53 *Let X/k be a variety over an algebraically closed field k, G a finite group, $Y \to X$ a G-covering, and $x \in X(k)$. Let $\mathcal{O}'_x \subset K(Y)$ be the ring of germs of functions regular at all points $y \in \pi^{-1}(x)$ of the fibre over x. This is a semilocal ring and*

$$H^0(G, \mathcal{O}'_x) = 0, \; H^i(G, \mathcal{O}'_x) = 0, \; (i > 0).$$

Proof: It is clear from the quotient construction that the G-invariant functions on $\mathcal{O}'_y$ are just $\mathcal{O}_x$. $\mathcal{O}'_x$ is visibly semilocal, with localizations $\mathcal{O}_y$, the local rings at $y \in \pi^{-1}(x)$. It follows that the completion $\hat{\mathcal{O}}'_x \cong \prod_y \hat{\mathcal{O}}_y$. Since G acts without fixed points on the fibre over x, it follows that as G-module $\hat{\mathcal{O}}'_x$ is **induced** from the e-module $\hat{\mathcal{O}}_x$. So $\hat{\mathcal{O}}'_x$ is acyclic for G-cohomology by Shapiro's Lemma. On the other hand, since $\mathcal{O}'_x$ is a finite-type $\mathcal{O}_x$-module, we have

$$\hat{\mathcal{O}}'_x = \mathcal{O}'_x \otimes \hat{\mathcal{O}}_x.$$

Moreover, since completion is flat, we deduce that

$$H^q(G, \hat{\mathcal{O}}'_x) = H^q(G, \mathcal{O}'_x) \otimes \hat{\mathcal{O}}_x,$$

and we conclude that therefore $\mathcal{O}_x$ is itself acyclic for G-cohomology.

We return now to the case of $G = \mathbb{Z}/p\mathbb{Z}$, and we will give a "Hilbert 90" treatment of Artin-Schreier theory. Namely, the function $1 \in \mathcal{O}'_x$ has trace zero, so $H^1(G, \mathcal{O}'_x) = 0$ implies (by the computation of H^1 of a finite cyclic group as the kernel of the "norm" map modulo the image of the augmentation ideal) that there is $\theta \in \mathcal{O}'_x$ such that

$$\theta^\sigma = \theta + 1,$$

where $G = \langle \sigma \rangle$. If $Y \to X$ is irreducible, θ gives a generator of $k(Y)/k(X)$. Now the construction of the associated cohomology class presents no more difficulties: take an open cover $\{U_i\}$ of X and functions θ_i regular on each $V_i = \pi^{-1}(U_i)$, and put $f_{ij} := \theta_i - \theta_j$ on $V_i \cap V_j$. Then the f_{ij} are G-invariant, so give a cocycle with values in $\mathcal{O}_X$. To see that this coycle is F-invariant, we need only remark that the $g_i = \theta_i^p - \theta_i$ are G-invariant, hence form a 0-cochain on U_i with values in $\mathcal{O}$, whose coboundary is $f_{ij}^p - f_{ij}$.

4.5.5 $G = \mathbb{Z}/p^n\mathbb{Z}$

Take now $G = \mathbb{Z}/p^n\mathbb{Z}$; we may identify G with the group $W_n(\mathbb{F}_p)$. Since $\mathbb{F}_p \hookrightarrow k$, by functoriality we have $G \hookrightarrow W_n(k)$. The latter is an algebraic group; indeed it is a commutative unipotent group – repeated extension of – which is *not* isomorphic to ^{n}. This can only happen in characteristic p and conversely any commutative unipotent group over k is isomorphic to a product of $W_n(k)$'s. In this case, we need to know that the descent argument performed earlier for continues to hold for any commutative solvable linear group (Namely, that flat morphisms with fibres isomorphic to G are in fact Zariski-locally trivial. This is rather delicate: it holds for GL_n as well – so that vector bundles can be defined Zariski locally – but most definitely *not* for PGL_n, so that the Brauer group must be defined in terms of flat (= étale, here) cohomology.) Having said this, we get a homomorphism

$$f_1 : \pi^1(X, \mathbb{Z}/p^n\mathbb{Z}) \to H^1(X, \mathcal{W}_n),$$

for which we have the following analogue of Proposition 26:

Proposition 54 *If X is projective, f_1 gives an isomorphism onto $H^1(X, \mathcal{W}_n)^F$.*

The proof equally well uses the short exact sequence of étale sheaves

$$0 \to \underline{G} \to \mathcal{W}_n \stackrel{F-1}{\to} \mathcal{W}_n \to 0.$$

Now wriete $G_k = \mathbb{Z}/p^k\mathbb{Z}$. For $n \geq m$, we have the canonical homomorphism $G_n \to G_m$, whence a homomorphism $\pi^1(X, G_n) \to \pi^1(X, G_m)$, and we want to know the image of this homomorphism.

Proposition 55 *Let α be an element of $\pi^1(X, G_m)$ with corresponding cohomology class $\xi = f_1(\alpha)$. Then α lies in the image of $\pi^1(X, G_n)$ iff the connecting map $\delta^1_{n,n-m} = 0$.*

We need the following lemma:

Lemma 56 *Let H be a finite-length Λ-module, and F a p-linear endomorphism of H. The map $\rho = F - 1 : H \to H$ is surjective.*

Proof: There is some n such that $p^n H = 0$. We go by induction on this n. If $n = 1$, H is a k-vector space of finite dimension, and we have already seen that in this situation the map ρ is surjective. The general case follows by induction (dévissage) applied to pH and H/pH.

Now we can give the proof of the proposition: if α is the image of an element $\beta \in \pi^1(X, G_n)$ corresponding to a cohomology class $\eta \in H^1(X, \mathcal{W}_n)^F$, and one sees immediately that $\xi = R^{n-m}\eta$, so that $\delta^1_{n,n-m}(\xi) = 0$. Conversely, if $\xi \in H^1(X, \mathcal{W}_n)^F$ satisfies $\delta^1_{n,n-m}(\xi) = 0$, then by definition of the Bockstein operator we may write $\xi = R^{n-m}(\eta')$ with $\eta' \in H^1(X, \mathcal{W}_n)$. Moreover, the relation $F\xi = \xi$ implies that $R^{n-m}(F\eta' - \eta') = 0$, i.e., $F\eta' - \eta' = V^m\theta$ with $\theta \in H^1(X, \mathcal{W}_{n-m})$. Applying the preceding lemma to $H^1(X, \mathcal{W}_{n-m})$, we may write $\theta = F\theta' - \theta'$, and putting $\eta := \eta' - V^m\theta'$, we get a Frobenius-invariant element restricting to ξ.

Corollary 57 *If X has no torsion in dimension one, the group $\pi^1(X, G_n)$ is the direct sum of σ isomorphic copies of $G_n = \mathbb{Z}/p^n\mathbb{Z}$.*

Proof: Write H_n for $\pi^1(X, G_n)$ viewed as a subgroup of $H^1(X, \mathcal{W}_n)$; the previous proposition together with the lack of one-dimensional torsion ensures that $R^{n-1} : H_n \to H_1$ is surjective. Its kernel is clearly VH_{n-1}. By induction on n, we deduce from this that H_n is a finite group of order $p^{n\sigma}$; since it is embedded in $H^1(X, \mathcal{W}_n)$ we have $p^n H_n = 0$. Moreover the composite

$$H_n \xrightarrow{R} H_{n-1} \xrightarrow{V} H_n$$

is multiplication by p; since R is surjective, $H_{n-1} = pH_n$ and hence $H_n/pH_n = H_1$ has p^σ elements. This shows that H_n is the direct sum of σ cyclic groups of order p^n.

Remark: Even when X does have one-dimensional torsion, one can still give an explicit formula for $\pi^1(X, G_n)$, albeit a more complicated one. The result is: let Z^1_m be the vector subspaces of $H^1(X, \mathcal{O})$ defined above. Let σ_m be the dimension of the semisimple component of Z^1_m/Z^1_{m+1}, and let τ be the dimension of the semisimple component of Z^1. Define a finitely generated abelian group H as follows:

$$H = \sum m = 1^\infty (\mathbb{Z}/p^m\mathbb{Z})^{\sigma_m} + \mathbb{Z}^r.$$

Then $\pi^1(X, \mathbb{Z}/p^n\mathbb{Z}) \cong \mathrm{Hom}(H, \mathbb{Z}/p^n\mathbb{Z})$.

4.5.6 Curves and Jacobians

Let $\phi : X \to J(X)$ be "the" "canonical" map from a curve into its Jacobian[7]. By a result of Rosenlicht, the homomorphism

$$\phi^* : H^1(J, \mathcal{O}_J) \to H^1(X, \mathcal{O}_X)$$

is bijective. It follows therefore that

$$\phi^1 : \pi^1(J, \mathbb{Z}/p\mathbb{Z}) \to \pi^1(X, \mathbb{Z}/p\mathbb{Z})$$

is also bijective – every unramified $\mathbb{Z}/p\mathbb{Z}$-cover comes from the Jacobian.

4.5.7 Serre's Example

We have the following "classical" construction:

Theorem 58 *Let G be a finite group, r a positive integer and k an(y) algebraically closed field. There exists an algebraic variety Y/k of dimension r, which is a nonsingular complete intersection, and such that G acts upon Y without fixed points. If $r = 2$ and $G = \mathbb{Z}/p\mathbb{Z}$ with $p \geq 5$, one can take Y to be a (hyper)surface in $\mathbb{P}^3$.*

We omit the proof for now, but note the following interesting consequence:

Corollary 59 *Let k be any algebraically closed field. Then every finite group occurs as the algebraic fundamental group of a nonsingular projective variety of every dimension at least 2. In particular, taking $k = \mathbb{C}$, every finite group is the ('etale = classical) fundamental group of a compact Kahler surface.*

Proof: As mentioned above, a complete intersection of dimension at least 2 is simply connected, so $\pi_1(Y/G) = G$.

Finally, Serre gives an example of a surface with $h^{1,0} + h^{0,1} > b_1$:

Theorem 60 *Let $G = \mathbb{Z}/p\mathbb{Z}$ with $p \geq 5$, and let k be an algebraically closed field of characteristic p. Let Y/k be the surface in $\mathbb{P}^3$ whose existence is guaranteed by the previous theorem; take $X := Y/G$. Then X/k is a nonsingular surface with $h^{1,0} = 0$, $h^{0,1} > 0$, $b_1 = 0$.*

Proof: The covering $Y \to X$ gives a nontrivial element of $\pi^1(X, G) = H^1(X, \mathcal{O})^F$, so indeed $h^{0,1}(X) > 0$. On the other hand by Serre duality, $h^{1,0}(Y) = h^{2,1}(Y) = \dim H^1(Y, \omega_Y)$, and since $\omega_Y = \mathcal{O}(n)$ is some multiple of the hyperplane section (precisely $n = d - 4$, where d is the degree), it follows from [Hartshorne, Ex.

[7] There are quotation maps because: one needs to choose a point of X to send to 0 in order to define the map; because of this, if k is not algebraically closed, there may well be no nontrivial morphism $X \to J(X)$ (e.g. a genus one curve without a rational point). The map which is canonical (hence rationally defined) is $\mathrm{Div}^0(X) \to J(X)$; from this perspective, the map $X \to J(X)$ involves a choice of trivialization of the principal homogeneous space $\mathrm{Div}^1(X)$ of $\mathrm{Div}^0(X)$, which is possible if and only if X has a degree one k-rational divisor.

3.5.5] that this dimension is zero, i.e., $H^0(Y, \Omega_Y) = 0$. It is a standard fact – e.g., use [Hartshorne, Prop. 2.8.11] – that since $Y \to X$ is a surjective map, the pullback on differentials is *injective*, so *a fortiori* $H^0(X, \Omega_X) = h^{1,0}(X) = 0$. (In this case, $Y \to X$ is a finite étale map, so indeed $\Omega_{Y/X} = 0$ and the pullback is a prori an isomorphism.) Finally, since as discussed above Y is simply connected, $\pi_1(X) = \mathbb{Z}/p\mathbb{Z}$ so $b_1(X) = 0$, where $b_1(X) = h^1(X_{\text{ét}}, \mathbb{Q}_l)$ for any $l \neq p$. (One knows that

$$b_1(X) = \dim_{\mathbb{Q}}((\pi_1(X)/[\pi_1(X), \pi_1(X)]) \otimes_{\mathbb{Z}} \mathbb{Q}),$$

just as in the topological case.) So we have the inequality $h^{1,0} + h^{0,1} > b_1$. One can check that $h^{0,1} = 1$. In summary, despite the vanishing of the Albanese and Picard varieties (which "comes from characteristic zero"), we have nontrivial $H^1(X, \mathcal{O}_X)$, whereas in characteristic 0 the canonical map $H^1(A(X), \mathcal{O}_{A(X)}) \to H^1(X, \mathcal{O}_X)$ is always an isomorphism – indeed over $\mathbb{C}$ the Picard variety is just the complex torus $H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$ arising from the cohomology of the exponential sequence

$$0 \to \underline{\mathbb{Z}} \stackrel{\exp}{\to} \mathcal{O}_X \to \mathcal{O}_X^\times \to 0!$$