VARIATION OF ZETA FUNCTIONS

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Contents

1. The number of zeroes of a polynomial over a finite field

In 1935 E. Artin conjectured the following theorem, which was proved almost immediately by Chevalley [4]:

Theorem 1.0.1. *Let $f(x) \in \mathbb{F}[x_1, \dots, x_n]$ be a polynomial of (maximal) degree d, where $\mathbb{F}$ is a field with $q = p^a$ elements. If $f(0, \dots, 0) = 0$ and $d < n$ then f has an additional zero. In particular, if f is a homogenous polynomial and X the corresponding projective hypersurface then $X(\mathbb{F}) \neq \emptyset$.*

Let us denote by $N(f)$ the number of zeroes of f. Then the theorem says that

$$N(f) > 0 \Rightarrow N(f) > 1.$$

The theorem was generalized by Warning [23] (in the same volume of the Journal) who proved

Theorem 1.0.2. *In the notation above*

$$N(f) \equiv 0 \pmod{p}.$$

The theorem was then generalized by Ax [2] in 1964 who proved:

Theorem 1.0.3. *Let μ be the minimal non-negative integer greater or equal to $(n-d)/d$. Then*

$$N(f) \equiv 0 \pmod{q^\mu}.$$

We remark that by inclusion-exclusion[1] one can get the following generalization:

Theorem 1.0.4. *Let $f_i \in \mathbb{F}[x_1, \dots, x_n]$ be polynomials of degree d_i, $i = 1, \dots, j$. Let $N(\{f_1, \dots, f_j\})$ be the number of common zeroes of $f_1, \dots, f_j$ in the field $\mathbb{F}$. Let μ be the minimal non-negative integer greater or equal to $(n - \sum_{i=1}^j d_i)/\sum_{i=1}^j d_i$ then*

$$N(\{f_1, \dots, f_j\}) \equiv 0 \pmod{q^\mu}.$$

The theorem was yet generalized by Katz [11] in 1971 who proved the following

Theorem 1.0.5. *Let d be the maximum of $d_1, \dots, d_j$. Let μ be the minimal non-negative integer greater or equal to $(n - \sum_{i=1}^j d_i)/d$ then*

$$N(\{f_1, \dots, f_j\}) \equiv 0 \pmod{q^\mu}.$$

Ax gave a simple proof of Theorem 1.0.2: We have

$$N(f) = \sum_{a \in \mathbb{F}^n} (1 - f(a)^{q-1}) = -\sum_{a \in \mathbb{F}^n} f(a)^{q-1},$$

where the last equality is in $\mathbb{F}$. The polynomial $f(x_1, \dots, x_n)$ is a sum of monomials $x^u := x_1^{u_1} \cdots x_n^{u_n}$, each of degree at most $d(q-1)$ and so one of the u_i, say u_{i_0} is less than $q-1$. It is enough to prove that for every such monomial $\sum_{a \in \mathbb{F}^n} x^u(a) = 0$ in $\mathbb{F}$. But $\sum_{a \in \mathbb{F}^n} x^u(a) = \prod_{i=1}^n (\sum_{a \in \mathbb{F}} a^{u_i})$. Let b be a generator of $\mathbb{F}^\times$. Note that for i_0 we get the factor 0: if $u_{i_0} = 0$ it follows from $0^0 = 1$ and if $u_{i_0} \neq 0$,

$$1 + [b^{u_{i_0}}] + [b^{u_{i_0}}]^2 + \cdots + [b^{u_{i_0}}]^{q-2} = \frac{[b^{u_{i_0}}]^{q-1} - 1}{[b^{u_{i_0}}] - 1} = 0.$$

[1]For example: $N(\{f_1, f_2\}) = N(f_1) + N(f_2) - N(f_1 f_2)$.

2. Interpretation in terms of zeta functions

Let X be a scheme of finite type over $\mathbb{F}$. Recall the definition of the zeta function $Z(X/\mathbb{F},t)$:

$$Z(X/\mathbb{F},t)=\exp\left(\sum_{r=1}^{\infty}N_r\frac{t^r}{r}\right).$$

Here N_r is the cardinality of $X(\mathbb{F}_{q^r})$. It was proven by Dwork [7] to be a rational function lying in $1+t\mathbb{Z}[[t]]$. In fact:

$$Z(X/\mathbb{F},t)=\prod_{i=0}^{2\dim(X)}\det(1-t\cdot\mathrm{Fr}|H^i_c(\overline{X},\mathbb{Q}_\ell))^{(-1)^{i+1}},$$

where H^i_c stands for étale cohomology with compact support. Further, Deligne proved in Weil II [6] that the eigenvalues of Frobenius on $H^i_c(\overline{X},\mathbb{Q}_\ell)$ are pure of weight $\leq i$ and are pure of weight i if X is smooth and projective.[2] One has

Lemma 2.0.6. *Let μ be a positive integer. The following are equivalent:*

1. *The reciprocal of every root of $Z(X/\mathbb{F},t)$ is of the form $q^\mu\times$(an algebraic integer);*
2. *For each r we have $N_r\equiv 0 \pmod{q^{r\mu}}$;*
3. *$Z(X/\mathbb{F},t)\in\mathbb{Z}[[q^\mu t]]$.*

For example, writing $Z(X/\mathbb{F},t)=\frac{\prod_i(1-\alpha_i t)}{\prod_j(1-\beta_j t)}$ and taking the logarithmic derivative we obtain

$$\begin{aligned}\sum_{r=1}^{\infty}N_r t^{r-1}&=\sum_j\frac{\beta_j}{1-\beta_j t}+\sum_i\frac{-\alpha_i}{1-\alpha_i t}\\&=\sum_{r=1}^{\infty}(\sum\beta_i^r-\sum\alpha_i^r)t^{r-1}.\end{aligned}\tag{2.1}$$

Assume for example that $N_r\equiv 0\pmod{q^{r\mu}}$, that is, under suitable normalization, $|N_r|\leq q^{-r\mu}$. Then the left hand side converges in $\mathbb{C}_p$ for $|t|<q^\mu$. This gives $|\alpha_i|,|\beta_j|\leq q^{-\mu}$. The converse follows from $N_r=\sum\beta_i^r-\sum\alpha_i^r$, using the expansion into Taylor series provided in (2.1).

Remark 2.0.7. Note that, in fact, any p-adic estimate for N_r provides one on the roots and vice-versa.

[2] An algebraic integer is *pure of weight r* if under every complex embedding it has absolute value $q^{r/2}$.

2.1. **Eigenvalues in the smooth case.** Let us assume now that X is smooth and proper of dimension n. A good case to keep in mind is X a projective non-singular variety in $\mathbb{P}^N$ defined by a homogenous ideal with generators being polynomials with $\mathbb{F}_q$ coefficients. In this case, for every Weil cohomology theory $H^*(X)$ (see [15]) we have

$$Z(X/\mathbb{F}_q, t) = \prod_{i=0}^{2n} \det(1 - t \cdot \mathrm{Fr}|H^i(X))^{(-1)^{i+1}} = \frac{P_1(t) \cdot P_3(t) \cdots P_{2n-1}(t)}{P_0(t) \cdot P_2(t) \cdots P_{2n}(t)}.$$

Remark 2.1.1. Note that because of this expression, for a projective variety X, Lemma 2.0.6 as given is of little interest because we always have the factor $\frac{1}{1-t}$ from the 0-th cohomology. However, if we denote by X_{aff} the affine cone over X (defined by the same equations but in the affine space $\mathbb{A}^{N+1}$ used to get the projective space $\mathbb{P}^N$) then it is elementary to check (see Example 2.2.2) that the following two statements are equivalent:

- For all r, $\sharp X_{\mathrm{aff}}(\mathbb{F}_{q^r}) \equiv 0 \pmod{q^{r\mu}}$;
- For all r, $\sharp X(\mathbb{F}_{q^r}) \equiv \frac{1}{1-q^r} \pmod{q^{r\mu}}$.

Important special cases of Weil cohomology are:
<u>*ℓ-adic cohomology*</u>: In this case $H^*(X) = H^*_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ and we have

$$P_i(t) = \prod_{j=1}^{b_i} (1 - \alpha_{ij} t) \in 1 + t\mathbb{Z}[t]$$

(not just in $\mathbb{Z}_\ell[t]$). This polynomial is independent of ℓ; $b_i = \dim H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ and agrees with the classical Betti number b_i if X has a lifting to characteristic zero.

<u>*Crystalline cohomology*</u>: In this case $H^*(X) = H^*_{\mathrm{crys}}(\overline{X}/W(\overline{\mathbb{F}_q}))$ and we have

$$P_i(t) = \prod_{j=1}^{b_i} (1 - \alpha_{ij} t) \in 1 + t\mathbb{Z}[t]$$

(not just in $W(\overline{\mathbb{F}_q})[t]$). This polynomial is independent of ℓ; $b_i = \mathrm{rank} H^i_{\mathrm{crys}}(\overline{X}/W(\overline{\mathbb{F}_q}))$ and agrees with the classical Betti number b_i if X has a lifting to characteristic zero.

In both cases the additional information on the P_i follows from the Riemann hypothesis for a good generalized Weil cohomology theory (see [13]) that asserts

$$|\alpha_{ij}|_{\mathrm{complex}} = q^{i/2}.$$

Again, for a Weil cohomology we have a functional equation

$$Z(X/\mathbb{F}_q, \frac{1}{q^n t}) = \pm t^{\chi} \cdot q^{n\chi/2} \cdot Z(X/\mathbb{F}_q, t),$$

where $\chi = \sum_{i=0}^{2n}(-1)^i b_i$ is the Euler characteristic (equal to the topological Euler characteristic if we have a lifting to characteristic zero). We deduce that for all i, j both α_{ij} and q^n/α_{ij} are algebraic integers, being reciprocals of roots of the polynomials P_i, P_{2n-i}. Therefore we obtain:

Corollary 2.1.2. *For every $\ell \neq p$ the algebraic numbers α_{ij} are ℓ-adic units. They have valuation $\leq \mathrm{val}_p(q^n)$ at p.*

This result and theorems like Chevalley-Warning-Ax-Katz's motivate the study of the p-adic valuations of the α_{ij} - the reciprocals of the roots of the zeta function of a, say, smooth proper variety $X/\mathbb{F}_q$.

2.2. Some easy examples.

Example 2.2.1. The projective space $\mathbb{P}^n_{\mathbb{F}_q}$.

The stratification of the projective space $\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{A}^{n-1} \cup \cdots \cup \mathbb{A}^0$ gives that for every r

$$N_r = \sharp\mathbb{P}^n(\mathbb{F}_{q^r}) = 1 + q^r + q^{2r} + \cdots + q^{nr} = \frac{1 - q^{n(r+1)}}{1 - q^r}.$$

Notice, incidentally, that the number of $\mathbb{F}_{q^r}$-points on the affine cone over $\mathbb{P}^n$, equal to $\mathbb{A}^{n+1}$, is $q^{r(n+1)}$. This easily gives the same formula.

The zeta function of $\mathbb{P}^n_{\mathbb{F}_q}$ is

$$\begin{aligned} \exp\left(\sum_{r=1}^{\infty}(1 + q^r + q^{2r} + \cdots + q^{nr})t^r/r\right) &= \prod_{i=0}^{n} \exp\left(\sum_{r=1}^{\infty}(q^i t)^r/r\right) \\ &= \prod_{i=0}^{n} \frac{1}{1 - q^i t}. \end{aligned} \tag{2.2}$$

This actually tells us that the cohomology groups $H^i_{\text{ét}}(\mathbb{P}^n, \mathbb{Q}_\ell)$ are zero for i odd and one dimensional for i even, with Frobenius Fr_q acting by q^i.

Conversely, suppose that one knows by some means the dimension of the cohomology groups of $\mathbb{P}^n$ (e.g., by base change - see a forthcoming lecture by Kolhatkar). There is a general *cycle map*: If X is a smooth projective variety of dimension n, say, over $\mathbb{F}_q$ and $Y \subset X$ is a projective subvariety of X of dimension r defined over a finite extension $L/\mathbb{F}_q$, we can associate to Y a cohomology class $[Y] \in H^{2n-2r}(\overline{X}, \mathbb{Q}_\ell)$ with the property that $\mathrm{Fr}_q[Y] = q^{n-r}[\mathrm{Fr}_q(Y)]$. For example, taking $X = \mathbb{P}^n$ and $Y_r = \mathbb{P}^r \subset \mathbb{P}^n$ for every r, we get a cohomology class $[Y_r] \in H^{2n-2r}(\overline{X}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell$ on which Fr_q acts by q^{n-r}. It is known that the classes Y_r are non-trivial[3] and hence span the cohomology. One re-derives the expression of the zeta function of $\mathbb{P}^n$.

[3]This can be deduced from the compatibility of the cycle map with intersections and the non-triviality for $r = 0$.

Example 2.2.2. If we go back to the case of a projective variety $X/\mathbb{F}_q$ and the cone over it X_{aff} we have the following easy relationship

$$\sharp X_{\text{aff}}(\mathbb{F}_{q^r}) = (q^r - 1)\sharp X(\mathbb{F}_{q^r}) + 1.$$

It follows that if $Z(X,t) = \exp(\sum N_r t^r/r) = \frac{P_1(t)\cdot P_3(t)\cdots P_{2n-1}(t)}{P_0(t)\cdot P_2(t)\cdots P_{2n}(t)}$ then

$$\begin{aligned}
Z(X_{\text{aff}},t) &= \exp(\sum((q^r-1)N_r+1)t^r/r) \\
&= \frac{\exp(\sum q^r N_r t^r/r)\exp(\sum t^r/r)}{\exp(\sum N_r t^r/r)} \\
&= \frac{Z(X,qt)}{Z(X,t)(1-t)} \qquad (2.3)\\
&= \frac{1}{P_1(t)}\cdot\frac{P_1(qt)\cdot P_3(qt)\cdots P_{2n-1}(qt)}{P_3(t)\cdot P_5(t)\cdots P_{2n-1}(t)}\cdot\frac{P_2(t)\cdot P_4(t)\cdots P_{2n}(t)}{P_0(qt)\cdot P_2(qt)\cdots P_{2n-2}(qt)}\cdot\frac{1}{P_{2n}(qt)}.
\end{aligned}$$

From this perspective it is not clear what is the cohomology (with compact support) of X_{aff}. If there is cancellation, it can only be between $P_{2j}(qt)$ and $P_{2j+2}(t)$ (because of the valuations of the eigenvalues).

To illustrate, for $X = \mathbb{P}^n$ we get perfect cancellation

$$Z(X_{\text{aff}},t) = \frac{1}{1-q^{n+1}t}$$

(for $\mathbb{A}^{n+1}$ we have $\sharp\mathbb{A}^{n+1}(\mathbb{F}_{q^r}) = q^{(n+1)r}$ and so $Z(\mathbb{A}^{n+1}/\mathbb{F}_q,t) = \exp(\sum_r q^{(n+1)r}t^r/r) = \frac{1}{1-q^{n+1}t}$).[4] If $X/\mathbb{F}_q$ is an elliptic curve with zeta function $\frac{(1-\alpha t)(1-\beta t)}{(1-t)(1-qt)}$, with $\alpha = q/\beta$ and complex absolute value $q^{1/2}$, then the zeta function of the affine cone is $\frac{(1-\alpha qt)(1-\beta qt)}{(1-qt)(1-q^2t)}\frac{(1-t)(1-qt)}{(1-\alpha t)(1-\beta t)}\frac{1}{1-t}$, equal to

$$Z(X_{\text{aff}},t) = \frac{(1-\alpha qt)(1-\beta qt)}{(1-\alpha t)(1-\beta t)(1-q^2t)}.$$

Note that no cancellation is possible. [5]

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[4]Note that the usual étale cohomology of $\mathbb{A}^n$ is 1 dimensional in dimension zero and zero otherwise. Indeed, using Künneth it is enough to prove that for $\mathbb{A}^1$. For $\mathbb{A}^1$ we use excision: If Z is a closed non-singular subset of pure codimension c of a non-singular variety X then $H^r(X,\mathbb{Z}_\ell) \cong H^r(X\setminus Z,\mathbb{Z}_\ell)$ for $0 \le r \le 2c-1$ (cf. [18, Cor. 16.2]). This gives that $H^0(\mathbb{A}^1,\mathbb{Z}_\ell)\cong\mathbb{Z}_\ell$, $H^1(\mathbb{A}^1,\mathbb{Z}_\ell)=0$, where we used the information on the cohomology groups of $\mathbb{P}^n$, deduced easily from the zeta function. To show $H^2(\mathbb{A}^1,\mathbb{Z}_\ell)=0$ one can use a general theorem saying that the étale cohomology of an affine variety is zero in dimensions greater than the dimension of the variety (loc. cit. Thm. 15.1).
On the other hand, the cohomology with compact support of $\mathbb{A}^1$ can be determined as follows: for U a non-singular variety of dimension d we have a duality (loc. cit. Aside 16.5 and Thm. 24.1)

$$H^r_c(U,\mathbb{Z}_\ell)\times H^{2d-r}(U,\mathbb{Z}_\ell(2d)) \longrightarrow H^{2d}_c(U,\mathbb{Z}_\ell(2d))\cong\mathbb{Z}_\ell.$$

In particular, we get that $H^i_c(\mathbb{A}^n,\mathbb{Z}_\ell)$ is zero in dimensions $i<2n$ and one dimensional for $i=2n$ with Frobenius acting by q^n.

[5]Because X_{aff} is affine of dimension 2, we have $H^i_c(X_{\text{aff}},\mathbb{Z}_\ell)=0$ for $i=0,1$ by [22]. Also, because $q\alpha, q\beta$ are pure of weight 3, it is probable that we have $H^i_c(X_{\text{aff}},\mathbb{Z}_\ell)=\mathbb{Z}_\ell^2$ for $i=2,3$. Finally, $H^4_c(X_{\text{aff}},\mathbb{Z}_\ell)=\mathbb{Z}_\ell$. The only issue remaining is to show there is cancellation between H^2 and H^3. Namely, that H^3 (and hence H^2) is pure of

3. The Newton polygon

Let $P(t) = \prod_{j=1}^{b}(1-\alpha_j t) = 1 + a_1 t + \cdots + a_b t^b \in \mathbb{Z}_p[t]$. Construct the lower convex polygon on the vertices

$$(0,0), (1, v_q(a_1)), \ldots, (b, v_q(a_b)),$$

where $v_q(q) = 1$. If λ is a slope of multiplicity m then there are precisely m reciprocal roots α_j such that $v_q(\alpha_j) = \lambda$. See Figure 3.1.

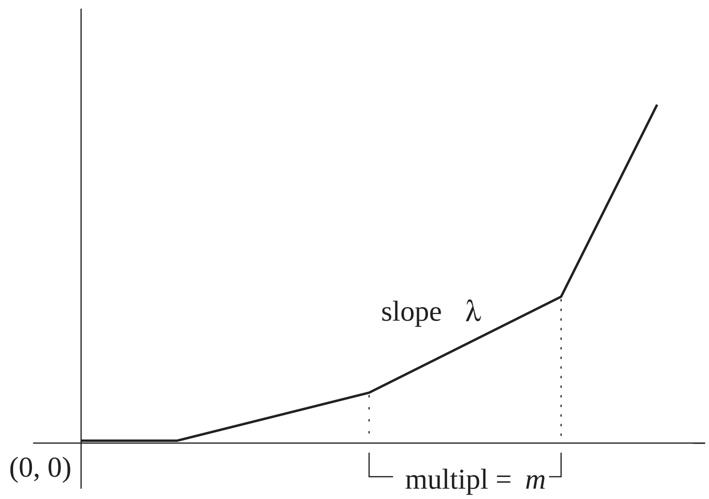

Figure 3.1. A Newton polygon. The slope λ has multiplicity m.

Example 3.0.3. Consider P_1 for the first étale cohomology of an elliptic curve over $\mathbb{F}_q$. The functional equation gives that if α is a root so is q/α. Furthermore, the relation

$$H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell) = \wedge^i H^1_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$$

holding for any abelian variety, and the fact that $\mathrm{Fr} = \mathrm{Fr}_q$ acts by q^n on $H^{2n}_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ for any smooth projective variety $X/\mathbb{F}_q$, tell us that the two roots, say α, β of P_1 satisfy $\alpha\beta = q$. We see that in any case $\beta = q/\alpha$. Note also that since α is roots of a quadratic polynomial we have $\mathrm{val}_q(\alpha) \in \frac{1}{2}\mathbb{Z}$ and the same for β. Write

$$P_1(t) = 1 - (\alpha+\beta)t + qt^2.$$

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weight 3 (resp. 2). Let X^+ be the projective closure of X_{aff}. The Gysin map [9, Thm. 2.1] applied to X^+ with the hyperplane section at infinity (which is just X) gives a surjective map $H^1(X, \mathbb{Q}_\ell(-2)) \longrightarrow H^3(X^+, \mathbb{Q}_\ell)$ and that shows that $H^3(X^+, \mathbb{Q}_\ell)$ is pure and of dimension at most 2.
In general, if $U \subset T$ is open with complement Z then we have a long exact sequence (cf. [14]):

$$\ldots \longrightarrow H^i_c(U \otimes_k \bar{k}, \mathbb{Z}_\ell) \longrightarrow H^i(T \otimes_k \bar{k}, \mathbb{Z}_\ell) \longrightarrow H^i(Z \otimes_k \bar{k}, \mathbb{Z}_\ell) \longrightarrow H^{i+1}_c(U \otimes_k \bar{k}, \mathbb{Z}_\ell) \longrightarrow \ldots$$

Apply that to $U = X_{\text{aff}}, T = X^+, Z = X$. A similar calculation of the zeta function for X^+ gives that it is $\frac{(1-q\alpha t)(1-q\beta t)}{(1-q^2t)(1-qt)(1-t)}$. One deduces that $H^3(X^+, \mathbb{Q}_\ell)$ is pure and of dimension 2, that $H^1(X^+, \mathbb{Q}_\ell) = 0$ and that $H^2(X^+, \mathbb{Q}_\ell) = \mathbb{Q}_\ell$; cf. [9, 7.5]. Writing the long exact sequence in cohomology the only possibility for non-pure $H^i(X_{\text{aff}}, \mathbb{Q}_\ell)$, $i = 2, 3$ is if the factor $(1-qt)$ appears in both H^2 and H^3. I couldn't rule this out so far.

We see that the Newton polygon is one of those appearing in Figure 3.2. The ordinary case happens iff one of α, β is a p-adic unit.

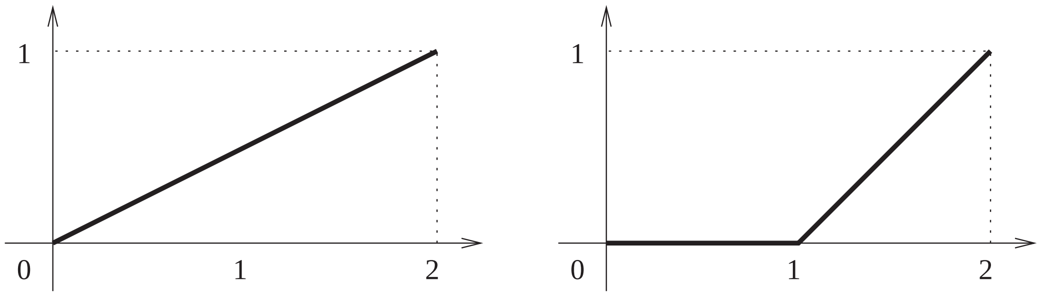

FIGURE 3.2. A supersingular (left) and ordinary (right) Newton polygon

3.1. **Newton polygons in families.** Given a family $f : X \longrightarrow S$ over a characteristic p base, f a smooth proper morphism with geometrically connected fibers, we would like to study the p-adic variation of $Z(X_s/k(s), t)$ where $s \in S$ is a closed point with residue field $k(s)$ and X_s is the fibre product. This makes sense a priori only for s with finite residue field $k(s)$. However, if our interest is only in the Newton polygon then via crystalline cohomology we can make sense of it for every geometric point $s : \mathrm{Spec}(k) \longrightarrow S$.

Let $Y = X_s$, a scheme over k; let $Y^{(p)}$ be its base change with respect to the absolute Frobenius morphism $F_{abs} : \mathrm{Spec}(k) \longrightarrow \mathrm{Spec}(k)$ (induced by the Frobenius homomorphism on k). Let $\mathrm{Fr}_Y : Y \longrightarrow Y^{(p)}$ be the Frobenius morphism. This is described by the following diagram:

$$\begin{tikzcd}
Y \arrow[dr, dashed, "\mathrm{Fr}_Y"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& Y^{(p)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d] \\
& \mathrm{Spec}(k) \arrow[r, "\mathrm{Fr}_{\mathrm{abs}}"] & \mathrm{Spec}(k)
\end{tikzcd}$$

Note that Fr_Y is a morphism of schemes over k. It induces by functoriality a $W(k)$-algebras homomorphism $\mathrm{Fr}^* : H^*_{\mathrm{crys}}(Y^{(p)}/W(k)) \longrightarrow H^*_{\mathrm{crys}}(Y/W(k))$. Since

$$H^i_{\mathrm{crys}}(Y^{(p)}/W(k)) = H^i_{\mathrm{crys}}(Y/W(k)) \otimes_{W(k),\sigma^i} W(k)$$

(σ is the Frobenius automorphism of $W(k)$) we obtain a σ^i-linear map

$$\mathrm{Fr} : H^i_{\mathrm{crys}}(Y/W(k)) \longrightarrow H^i_{\mathrm{crys}}(Y/W(k)), \quad \mathrm{Fr}(x) = \mathrm{Fr}^*(x \otimes 1).$$

Fix an index i. One can define a Newton polygon for

$$\mathrm{Fr} : H^i_{\mathrm{crys}}(Y/W(k)) \longrightarrow H^i_{\mathrm{crys}}(Y/W(k)).$$

This polygon agrees with the Newton polygon of P_i when Y is a variety over $\mathbb{F}_q$ and thus will generalize the notion of the Newton polygons coming from the zeta functions. The construction of the Newton polygon is given below as a limit of "abstract Hodge polygons". We first provide the main result due to Grothendieck and Katz [12].

Theorem 3.1.1. *For every geometric point s let $Newton^i(X_s)$ be the Newton polygon of the operator* $\mathrm{Fr} : H^i_{crys}(X_s/W(k)) \longrightarrow H^i_{crys}(X_s/W(k))$. *There is a partition $S = \coprod S_\beta$, β running over Newton polygons, such that each S_β is a locally closed set and such that for all geometric points $s : \mathrm{Spec}(k) \longrightarrow S$ such that $Newton^i(X_s) = \beta$ we have that s factors through S_β. If S_γ intersect that closure of S_β then γ lies over β.*

4. Abstract Hodge polygons and Newton polygons

Let k be a perfect field of characteristic p, $W(k)$ its Witt vectors and $\sigma : W(k) \longrightarrow W(k)$ the lift of the absolute Frobenius automorphism $k \longrightarrow k$. Let M be a free $W(k)$ module of finite rank r and let

$$F : M \longrightarrow M$$

be a σ-linear morphism with no kernel, that is $F : M \otimes \mathbb{Q} \longrightarrow M \otimes \mathbb{Q}$ is an isomorphism. One can show that

$$M \otimes \mathbb{Q} \cong \oplus_i [M(a_i, b_i) \otimes \mathbb{Q}]^{n_i}$$

where a_i, b_i are relatively prime positive integers, or $a_i = 1, b_i = 0$. Here $M(a_i, b_i)$ is a free $W(k)$ module of rank a_i, and on each $M(a_i, b_i)$ the map F satisfies on a certain basis $F^{a_i} = p^{b_i}$. One then associates to M the Newton polygon $\mathrm{Newton}(M)$ with slopes b_i/a_i with multiplicity $a_i n_i$. This polygon depends only on M. We define the Hodge numbers $h^i(M)$ to be the multiplicity of the elementary divisor p^i in $M/F(M)$.

On the other hand, assume M to be torsion free of rank r and consider $M/F(M)$ - a torsion $W(k)$-module. Let $p^{a_1}, p^{a_2}, \dots, p^{a_r}$ be the elementary divisors, $0 \leq a_1 \leq a_2 \leq \cdots \leq a_r$. One construct a lower convex polygon starting at $(0, 0)$ and having segments with slopes a_i. The same can be done for F^a and one can normalize the polygons to always have the same end point $(r, \sum a_i)$. We get a normalized polygon for every a, that we call the Hodge polygon of F^a and denote $\mathrm{Hodge}(a, M)$. The following theorem was proved by Katz [12]:

Theorem 4.0.2. *The abstract Hodge polygons Hodge(a, M) converge as a tends to infinity to the Newton polygon Newton(M) of M (in particular, they have the same end point).*[6] *Moreover, the Newton polygon always lies over the Hodge polygons Hodge(a, M).*

This is now applied to $M = H^i_{\text{crys}}(X/W(k))$ modulo torsion to define its Newton and Hodge polygons. As such, the slopes of the polygon $\text{Hodge}(a, H^i_{\text{crys}}(X/W(k))/\text{Torsion})$ serve as a lower bound to the p-adic valuations of the roots of the polynomial P_i – the characteristic polynomials of Fr acting on the i-th cohomology group $H^i_{\text{ét}}(\bar{X}, \mathbb{Q}_\ell)$.

5. Results of Deligne, Katz, Mazur and Ogus

Theorem 5.0.3. *(Mazur* [17], *Ogus* [3]*) Let X be a smooth proper variety over k - a perfect field of characteristic p. Let Hodge$^j(X)$ be the lower convex polygon starting at $(0,0)$ and having slope i with multiplicity $\dim H^{j-i}(X, \Omega^i_{X/k}) = h^{i,j-i}$.*

(1) *The Newton polygon of $H^j_{crys}(X/W(k))$ always lies above Hodge$^j(X)$.*
(2) *Assume that $H^j_{crys}(X/W(k))$ are torsion free for all j and that the Hodge to de Rham spectral sequence $E_1^{i,j-i} = H^{j-i}(X, \Omega^i_{X/k}) \Rightarrow H^j_{dR}(X/k)$ degenerates at E_1. Then the abstract Hodge number $h^i(H^j_{crys}(X/W(k)))$ is equal to the Hodge number $h^{i,j-i} = \dim H^{j-i}(X, \Omega^i_{X/k})$.*

An application of (1) to the number of point on X is provided by the following:

Corollary 5.0.4. *Assume that $X/\mathbb{F}_q$ is a smooth complete intersection in $\mathbb{P}^{n+j}_{\mathbb{F}_q}$ of dimension n and multi-degree $(a_1, \dots, a_j)$. Then*

$$N_r(X/\mathbb{F}_q) = N_r(\mathbb{P}^n_{\mathbb{F}_q}/\mathbb{F}_q) \pmod{q^{r\mu}},$$

where μ is the minimal non-negative integer such that $h^{\mu,n-\mu} - \delta_{\mu,n-\mu} \neq 0$ (Kronecker's δ). Moreover, if $\max a_i > 1$ (i.e., X is not linear) then one has $\mu = [(n+j+1) - \sum a_i]/\max a_i$ by [11, Prop. 2.7], [5].

To derive this corollary, one notes that the assumptions only control what happens in the middle cohomology $H^n_{\text{crys}}(X/W(\mathbb{F}_q))$. The other cohomologies of X agree with those of $\mathbb{P}^{n+j}$ in dimensions smaller than n (using a version of Weak Lefschetz, cf. [13]; this is also true for X a complete intersection, which is not necessarily smooth - see [22]) and are determined by duality (using smoothness) in dimensions higher than d (hard Lefschetz). This yields for a Weil cohomology:

$$H^*(X) = H^*(\mathbb{P}^n) + H^n(X)',$$

[6] We remark that the convergence need not be monotone. See [12].

where the $H^n(X)'$ are the so called primitive cohomology classes. The modified Hodge numbers $h^{\mu,n-\mu} - \delta_{\mu,n-\mu}$ are the Hodge numbers of the Hodge decomposition of the primitive part. There are closed formulas for the dimension of $H^n(X)'$. For example, if we have one equation only, say of degree a, then

$$\dim H^n(X)' = \frac{(a-1)^{n+2} + (-1)^n(a-1)}{a}.$$

On the other hand, the differences $N_r(X) - N_r(\mathbb{P}^n)$ have a generating series

$$\exp(\sum [N_r(X) - N_r(\mathbb{P}^n)]t^r/r) = Z(X/\mathbb{F}, t)/Z(\mathbb{P}^n/\mathbb{F}, t) = \det(1 - t\mathrm{Fr}|H^n(X)')^{(-1)^(n+1)}. \tag{5.1}$$

Thus, the point is to have a lower bound of the minimal slope of the Newton polygon for $H^n(X)'$ and such is derived from the Hodge polygon with the modified Hodge numbers.

In a similar vain one can get for a complete intersection that

$$|\sharp(X(\mathbb{F}_q)) - \sharp(\mathbb{P}^d(\mathbb{F}_q))| \le Cq^{m/2},$$

where $C = \dim H^n(X)' \le \dim H^n_{\text{ét}}(X, \mathbb{Q}_\ell)$. For elliptic curves we get $C = 2$. See also Milne's lecture notes pp. 99 ff., [11, 10, 1, 22, 9] for estimates and information on the zeta functions of not necessarily smooth complete intersections.

6. Some special cases

In some special cases the analysis of the Newton polygon and hence, to an extent, of the zeta function, reduces to the examination of a single cohomology group. We have seen above the case of complete intersections; we indicate two more.

6.1. **Abelian varieties.** Let $X/\mathbb{F}_q$ be an abelian variety. Then for every i we have

$$H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell) = \wedge^i H^1_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell),$$

as Galois modules. Thus, all the interesting information lies in $H^1_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$. In this case, one has the symmetry condition: a slope λ appears if and only if $1-\lambda$ appears and with the same multiplicity. The polygon has integral breaking points and ends at $(2g, g)$. It is known that every polygon β satisfying this condition appears for some principally polarized abelian variety.

The set of Newton polygons appearing for a g dimensional principally polarized abelian variety is a partially ordered set under the relation of "lying above". The minimal element is the ordinary Newton polygon N-ord and the maximal element is the supersingular polygon N-ss. Let $\coprod S_\beta$ be the stratification of the moduli space $\mathscr{A}_g$ of principally polarized abelian varieties in characteristic

p induced by the Newton polygons. The variety $\mathscr{A}_g$ is irreducible of dimension $g(g+1)/2$. The S_β are equi-dimensional and we have

$$\mathrm{codim}(S_\beta) = \mathrm{dist}(\beta, \text{N-ord}).$$

There is much known on this stratification: see [19].

6.2. **Curves.** Let $C/\mathbb{F}_q$ be a smooth curve. The étale cohomology group $H^0_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ (resp. $H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$) is one dimensional with Frobenius acting by multiplication by 1 (resp. p). Thus, the only interest is in the first cohomology. We have the canonical isomorphism ("unramified class field theory")

$$H^1_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell) = H^1_{\text{ét}}(\overline{Jac(X)}, \mathbb{Q}_\ell)$$

(the same with crystalline cohomology) and thus the Newton polygons satisfy the same restriction as those of g dimensional abelian varieties. Note, however, that the moduli space of curves is of dimension $3g-3$ for $g > 1$ while the number of Newton polygons that are possible is much larger (at least $g^2/4$). It is thus not clear whether to expect that every Newton polygon appears or not. Opinions among the expert differ.

There are very little results in this direction. It is known that the generic curve is ordinary [16]. It is also known that the generic hyperelliptic curve is ordinary.[7] Most of the results have to do with showing the existence of supersingular curves. I remark that even the study of the variation of the Newton polygon along the moduli space of hyperelliptic curves seems very difficult. A striking recent result of Scholten and Zhu [20] is that there are no supersingular hyperelliptic curves in characteristic 2 of genus of the form $2^n - 1$. On the other hand, there are supersingular curves of any genus in characteristic 2 [8].

[7]This implies that the generic curve is ordinary, because ordinary is an open condition and the moduli space of curves is irreducible. I could not find a reference for this fact. Rachel Pries (in a letter) explained to me how the theory of formal patchings, deformations of admissible covers allow one to show inductively on the genus g that for every $f \leq g$ there is a hyperelliptic curve having slope 0 with multiplicity f. The idea is to glue a suitable hyperelliptic of genus $g-1$ with an elliptic curve and deform that to a hyperelliptic curve.

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