SPECTRAL SEQUENCES FOR SURFACES

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Let X and $\widetilde{X}$ be surfaces over $\Bbbk$, and suppose $f : \widetilde{X} \to X$ is a birational morphism. Suppose also that there is a finite set of closed points $x_1, \ldots, x_n$ such that f is an isomorphism from $\widetilde{X} \setminus f^{-1}(\{x_1, \ldots, x_n\})$ to $X \setminus \{x_1, \ldots, x_n\}$. For example, X could be a singular surface, and $\widetilde{X}$ could be a resolved model of X.

Suppose $\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_{\widetilde{X}}$-modules on $\widetilde{X}$. Then $f_*(\mathcal{F})$ is a coherent sheaf of $\mathcal{O}_X$-modules on X. One might well want to ask how $H^i(\widetilde{X}, \mathcal{F})$ and $H^i(X, f_*\mathcal{F})$ are related, especially when $i = 0$.

1. Spectral sequences

We can apply the Leray spectral sequence as discussed in [God58, II.4.17]. There the author constructs a sheaf $\mathcal{H}^q(\mathcal{F})$ on X as the sheaf associated to the presheaf

$$U \mapsto H^q(f^{-1}(U), \mathcal{F}|_U).$$

By [Har77, Prop. III.8.1], this is just $R^q f_*\mathcal{F}$.

Then the Leray spectral sequence tells us that

$$H^p(X, \mathcal{H}^q(\mathcal{F})) \Rightarrow H^{p+q}(\widetilde{X}, \mathcal{F}).$$

Before we look at what this actually tells us, let us examine the groups involved. First of all, since we are dealing with surfaces, all the nonzero groups will have $0 \le p \le 2$. We know $\mathcal{H}^0(\mathcal{F}) = R^0 f_*\mathcal{F} = f_*\mathcal{F}$. But what is $\mathcal{H}^i(\mathcal{F})$? Let $V = X \setminus \{x_1, \ldots, x_n\}$. Then we have seen already that $\mathcal{H}^i(\mathcal{F}) = R^i f_*\mathcal{F}$, and by [Har77, Cor. III.8.2] we have $R^i f_*\mathcal{F}|_U = R^i f'_*(\mathcal{F}|_{f^{-1}(U)})$, where $f' : f^{-1}(U) \to U$ is the restriction of f. But by assumption f' is an isomorphism, so f_* is exact and $R^i f'_*\mathcal{F} = 0$. In particular, if $x \in U$, the stalk $R^i f_*\mathcal{F}_x$ is zero. So $\mathcal{H}^1(\mathcal{F})$ is supported on a set of dimension zero. Thus $H^p(X, \mathcal{H}^i(\mathcal{F})) = 0$ for $p > 0$ (to see this, take a Čech cover consisting of V and an affine neighborhood of each x_i). Finally, we know the fibers of f have dimension at most one, so since $\mathcal{F}$ is coherent, by [Har77, Cor. III.11.2] all the nonzero terms will have $0 \le q \le 1$. Thus the nonzero terms of the spectral sequence look like:

$$\begin{tikzcd}
H^0(X, \mathcal{H}^1(\mathcal{F})) \arrow[drr, "d"] & & \\
H^0(X, f_*\mathcal{F}) & H^1(X, f_*\mathcal{F}) & H^2(X, f_*\mathcal{F})
\end{tikzcd}$$

Since X and $\widetilde{X}$ are varieties over $\Bbbk$, all these objects are naturally $\Bbbk$-vector spaces, which will be finite-dimensional since $\mathcal{F}$ is coherent ([Har77, Thm. III.5.2]). Write $H^0(X, \mathcal{H}^1(\mathcal{F}))$ as $M(\mathcal{F})$, for convenience.

Remark 1.1. Let X be a surface and let $\widetilde{X}$ be a resolved model of X. Then, by the definition of rational singularities, $R^1 f_*\mathcal{O}_{\widetilde{X}} = 0$ if and only if all the singularities of

X are rational. Since $R^1 f_* \mathcal{O}_{\widetilde{X}}$ is supported on a finite set of points, this occurs if and only if $H^0(X, R^1 f_* \mathcal{O}_{\widetilde{X}}) = M(\mathcal{O}_{\widetilde{X}}) = 0$.

As we can see from the map d, this spectral sequence has not converged yet, but it will if we extract cohomology one more time. The fully converged object looks like:

$$\begin{array}{ccc} \ker d & & \\ & & \\ H^0(X, f_*\mathcal{F}) & H^1(X, f_*\mathcal{F}) & H^2(X, f_*\mathcal{F})/\operatorname{im} d \end{array}$$

Now, as is usual with spectral sequences, we get formulas relating the fully converged terms and the values to which the spectral sequence converges. In particular, $H^0(X, f_*\mathcal{F}) \cong H^0(\widetilde{X}, \mathcal{F})$, $H^2(X, f_*\mathcal{F})/\operatorname{im} d \cong H^2(\widetilde{X}, \mathcal{F})$, and we have an injection $\ker d \hookrightarrow H^1(\widetilde{X}, \mathcal{F})$, which gives $H^1(X, f_*\mathcal{F}) \cong H^1(\widetilde{X}, \mathcal{F})/\ker d$.

Proposition 1.2. *Let $m = \dim \operatorname{im} d$, so that $0 \leq m \leq \dim M(\mathcal{F})$. Then we have shown:*

$$\begin{aligned} \dim H^0(X, f_*\mathcal{F}) &= \dim H^0(\widetilde{X}, \mathcal{F}) \\ \dim H^1(X, f_*\mathcal{F}) &= \dim H^1(\widetilde{X}, \mathcal{F}) + m - \dim M(\mathcal{F}) \\ \dim H^2(X, f_*\mathcal{F}) &= \dim H^2(\widetilde{X}, \mathcal{F}) - m \end{aligned}$$

and

$$\chi(f_*\mathcal{F}) = \chi(\mathcal{F}) + \dim M(\mathcal{F}).$$

We will obtain a description of $M(\mathcal{F})$ in Section 2.

Corollary 1.3. *Recall that p_a is the arithmetic genus of a scheme. We have:*

$$p_a(X) = p_a(\widetilde{X}) + \dim M(\mathcal{O}_{\widetilde{X}}).$$

Proof. Consider this last formula for $\mathcal{F} = \mathcal{O}_{\widetilde{X}}$. We know $f_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$, so we get $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{\widetilde{X}}) + \dim M(\mathcal{O}_{\widetilde{X}})$. Recalling that the arithmetic genus of a variety Y is defined by $p_a(X) = (-1)^{\dim Y}(\chi(\mathcal{O}_Y) - 1)$, we see that

$$p_a(X) = p_a(\widetilde{X}) + \dim M(\mathcal{O}_{\widetilde{X}}). \qquad \square$$

2. Understanding $M(\mathcal{F})$

It is clear that an understanding of $M(\mathcal{F})$ would greatly aid calculations comparing cohomology groups.

Proposition 2.1. *Let $\widehat{\widetilde{X}}_i$ denote the formal scheme obtained by completing $\widetilde{X}$ along $f^{-1}(x_i)$, and let $\widehat{\mathcal{F}}_i$ denote $v^*\mathcal{F}$, where $v\colon \widehat{\widetilde{X}}_i \to X$ is the canonical map. Similarly, let $\widehat{X}$ be the formal completion of X along $f^{-1}(\{x_1, \ldots, x_n\})$ and $\widehat{\mathcal{F}}$ be the corresponding sheaf. Then*

$$M(\mathcal{F}) = \oplus_i H^1(\widehat{\widetilde{X}}_i, \widehat{\mathcal{F}}_i) = H^1(\widehat{\widetilde{X}}_i, \widehat{\mathcal{F}}).$$

Remark 2.2. Let $\mathcal{G}$ be a sheaf of abelian groups on a topological space T whose stalks are all zero except for the stalk at t, which is G. Then, recalling the description of a sheaf in terms of its espace étalé ([Har77, Ex. I.1.13], we see that

$$\mathcal{G}(U) = \begin{cases} G & \text{if } t \in U \\ 0 & \text{otherwise.} \end{cases}$$

Observe that if t were not a closed point, then for $t_0 \neq t$ in its closure,

$$\mathcal{G}_{t_0} = \varprojlim_{U \ni t_0} \mathcal{G}(U) = \varprojlim_{U \ni t_0} G = G,$$

which contradicts our assumption that $\mathcal{G}_t$ was the only nonzero stalk.

Proof of proposition 2.1. Recall $M(\mathcal{F})$is defined to be $H^0(X, \mathcal{H}^1(\mathcal{F}))$. But $H^0(X, \cdot) = \Gamma(X, \cdot)$, the global sections functor. Also, we had that $\mathcal{H}^1(\mathcal{F})$ was the sheaf associated to the presheaf

$$U \mapsto H^q(f^{-1}(U), \mathcal{F}|_U),$$

and its stalk was zero everywhere except (possibly) for $\{x_1, \dots, x_n\}$. Since $\mathcal{H}^1(\mathcal{F}) = R^1 f_* \mathcal{F}$ we see from [Har77, Cor. III.8.6] that if $\mathcal{F}$ is quasi-coherent, so is $R^1 f_* \mathcal{F}$.

Let $\operatorname{Spec} A$ be any affine scheme, and let M be an A-module, so that $\widetilde{M}$ (the sheaf associated to M) is a quasi-coherent sheaf on $\operatorname{Spec} A$. Suppose that $\widetilde{M}$ has exactly one nonzero stalk, at a maximal ideal $\mathfrak{m}$. Then, as an abelian group, $\widetilde{M}(U) = M_{\mathfrak{m}}$ for every U containing $\mathfrak{m}$. Choosing $U = \operatorname{Spec} A$, we see $M_{\mathfrak{m}} = \widetilde{M}(\operatorname{Spec} A) = M$.

Let $m \in \mathfrak{m}$. Then $U = \operatorname{Spec} A \setminus \{\mathfrak{p} \ni m\}$ is an open set in $\operatorname{Spec} A$ which is isomorphic to $\operatorname{Spec} A_m$. By our choice of m, $\mathfrak{m} \notin U$ so $0 = \widetilde{M}(U) = M_m$ and for every $\mu \in M$, $m^n \mu = 0$ for some n.

Now, let us look again at the sheaf $R^i f_* \mathcal{F}$. Write $R^i f_* \mathcal{F} = \oplus \mathcal{M}_i$, where $\mathcal{M}_i$ is a sheaf whose only nonzero stalk is at x_i. Each $\mathcal{M}_i$ is then completely described by an $\mathcal{O}_{X,x_i}$-module M_i.

Can we describe these modules M_i in more detail? We can apply [Har77, Thm. III.11.1], which states that

$$\varprojlim R^1 f_* \mathcal{F} \otimes \mathcal{O}_{X,x_i} / \mathfrak{m}_{X,x_i}^n = H^1(\widehat{\widehat{X}}_i, \widehat{F}),$$

where $\widehat{\widehat{X}}_i$ is the formal scheme constructed by completing X along $f^{-1}(x_i)$.

First consider the sheaves on $\operatorname{Spec} \mathcal{O}_{X,x_i}$ given by $R^1 f_* \mathcal{F} \otimes \mathcal{O}_{X,x_i} / \mathfrak{m}_{X,x_i}^n$. We see that $R^1 f_* \mathcal{F} \otimes \mathcal{O}_{X,x_i} / \mathfrak{m}_{X,x_i}^n = \mathcal{M}_i / \mathfrak{m}_{X,x_i} \mathcal{M}$. Thus we get

$$\begin{aligned} \Gamma(\operatorname{Spec} \mathcal{O}_{X,x_i} / \mathfrak{m}_{X,x_i}^n, R^1 f_* \mathcal{F} \otimes \mathcal{O}_{X,x_i} / \mathfrak{m}_{X,x_i}^n) &= \Gamma(\operatorname{Spec} \mathcal{O}_{X,x_i} / \mathfrak{m}_{X,x_i}^n, \mathcal{M}_i / \mathfrak{m}_{X,x_i}^n \mathcal{M}) \\ &= M_i / \mathfrak{m}_{X,x_i}^n M_i \end{aligned}$$

But since every element μ of M_i is annihilated by a high enough power of every element of $\mathfrak{m}_{X,x_i}$ and all our rings are Noetherian, we see that

$$\varprojlim R^1 f_* \mathcal{F} \otimes \mathcal{O}_{X,x_i} / \mathfrak{m}_{X,x}^n = M_i.$$

So we see that

$$M(\mathcal{F}) = H^0(X, R^1 f_*(\mathcal{F})) = \oplus_i H^1(\widehat{\widehat{X}}_i, \widehat{F}).$$

The final equality we observe by noticing that the preimages of the x_i are disjoint closed sets. $\square$

3. Understanding d

It would certainly be nice to understand the kernel and image of d better, or at least know their dimensions.

Proposition 3.1. *Let X and $\widetilde{X}$ be local complete intersections, and suppose $\mathcal{F}$ is locally free. Then $d = 0$ and we have*

$$\begin{aligned} H^0(\widetilde{X}, \mathcal{F}) &\cong H^0(\widetilde{X}, f_*\mathcal{F}) \\ H^1(\widetilde{X}, \mathcal{F})/M(\mathcal{F}) &\cong H^1(\widetilde{X}, f_*\mathcal{F}) \\ H^2(\widetilde{X}, \mathcal{F}) &\cong H^2(\widetilde{X}, f_*\mathcal{F}). \end{aligned}$$

Proof. We have an exact sequence due to the fact that we're dealing with a spectral sequence:

$$0 \to H^1(X, f_*\mathcal{F}) \to H^1(\widetilde{X}, \mathcal{F}) \to M(\mathcal{F}) \xrightarrow{d} H^2(X, f_*\mathcal{F}) \to H^2(\widetilde{X}, \mathcal{F}) \to 0$$

We can append the final zero (which is normally not present) because in our case we know $H^2(X, f_*\mathcal{F})/\operatorname{im} d \cong H^2(\widetilde{X}, \mathcal{F})$. We see that $\operatorname{im} d$ is the kernel of the map induced by f_* on the second cohomology groups, and so $\dim \operatorname{im} d = \dim H^2(X, f_*\mathcal{F}) - \dim H^2(\widetilde{X}, \mathcal{F})$.

Now, by assumption X and $\widetilde{X}$ are local complete intersections. Then for $\widetilde{X}$ there is a dualizing sheaf $\omega_{\widetilde{X}}$, and $f_*\omega_{\widetilde{X}} = \omega_X$ is a dualizing sheaf for X ([Arc03]).

We have also assumed that $f_*\mathcal{F}$ is locally free. Then clearly $\mathcal{F}$ is locally free.[1] Using Serre duality ([Har77, Sec. III.7]) we get

$$\begin{aligned} \dim H^2(\widetilde{X}, \mathcal{F}) &= \dim H^0(\widetilde{X}, \mathcal{F}^\vee \otimes \omega_{\widetilde{X}}) \\ &= \dim H^0\big(X, f_*\big(\mathcal{F}^\vee \otimes \omega_{\widetilde{X}}\big)\big) \\ &= \dim H^0\big(X, f_*(\mathcal{F})^\vee \otimes f_*\big(\omega_{\widetilde{X}}\big)\big) \\ &= \dim H^0(X, f_*(\mathcal{F})^\vee \otimes \omega_X) \\ &= \dim H^2(X, f_*(\mathcal{F})). \end{aligned}$$

Together, this implies that $d = 0$ for every such sheaf $\mathcal{F}$. □

A good example of such a sheaf $\mathcal{F}$ is $f^*\mathcal{G}$ for some invertible sheaf $\mathcal{G}$ on X. Unfortunately, for the purposes of intersection theory one is often interested in sheaves on X that are not locally principal — these correspond to divisors that are not linearly equivalent to any divisor missing the singularities.

References

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[1] The converse is certainly not true: if we take a divisor on $\widetilde{X}$ passing through the exceptional fiber, its push-forward may well yield a Weil divisor that is not locally principal.