Zeta function of Grassmann Varieties

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Topics Covered :

I will present some simple calculations about Zeta function of Grassmann Varieties and Lagrangian Grassmann Varieties. The main topics covered are:

1. Introduction to Grassmann Varieties.
2. Zeta function of Grassmann Varieties.
3. Lagrangian Grassmannian and its Zeta function.
4. A bit of Schubert Calculus...
5. Understanding cohomology of Grassmannians in characteristic zero.

1 Grassmann Varieties

The Grassmannian $\mathrm{G(d,n)}$: Let V be a vector space of dimension $n \geq 2$ over field k. Let $1 \leq d \leq n$ be any integer. Then the Grassmannian $\mathrm{G(d,n)}$ is defined to be the set of all d-dimensional subspaces of V, i.e.

$$\mathrm{G(d,n)} = \{\ \mathrm{W} \mid \mathrm{W\,subspace\ of\ V\ of\ dim\ d}\}.$$

Alternately, it is the set of all $(d-1)$-dimensional linear subspaces of the projective space $\mathbb{P}^{n-1}(k)$. If we think of the grassmannian this way, we denote it by $\mathrm{G}^{\mathbb{P}}(\mathrm{d}-1,\mathrm{n}-1)$. The simplest example of the grassmannian could be $\mathrm{G(1,n)}$ which is the set of all 1 dimensional subspaces of the vector space V which is nothing but the projective space on V.
Plücker map: We can embed $\mathrm{G(d,n)}$ in the projective space $\mathbb{P}(\bigwedge^d V)$ via Plücker map P as follows: Let U be a d dimensional subspace of V having basis $\{u_1, .., u_d\}$. Define P(U) as the point of $\mathbb{P}(\bigwedge^d V)$ which is determined by $u_1 \wedge ... \wedge u_d$. It can be shown that P is a well defined injective map. Thus we may consider $\mathrm{G(d,n)}$ as a subset of $\mathbb{P}(\bigwedge^d V)$ via P.
Plücker Coordinates: Let $e_1, .., e_n$ be a basis for V then the canonical basis for $\Lambda^d V$ is given by :

$$\{e_{i_1} \wedge ... \wedge e_{i_d} | 1 \leq i_1 < ... < i_d \leq n\}.$$

Let U be a d-dimensional subspace of V having basis $\{u_1, ..., u_d\}$. Let $u_j = \sum_{i=1}^{n} a_{ij} e_i$. Then the coordinates of $\mathrm{P(U) = u_1 \wedge .. \wedge u_d}$ are called the Plücker coordinates. These are nothing but the $\binom{n}{d}$ maximal minors of the matrix $(a_{ij})_{1 \le i \le n,\, 1 \le j \le d}$.

Grassmannian as an algebraic variety: It can be shown that $\mathrm{G(d, n)}$ is a projective algebraic variety defined by quadratic polynomials called Plücker relations. The grassmannian $\mathrm{G(d, n)}$ can be covered by open sets isomorphic to the affine space $\mathbb{A}^{d(n-d)}$ and so we have

$$\mathrm{dim(G(d, n)) = d(n - d)}.$$

1.1 To find the number of points of $\mathrm{G(d, n)(\mathbb{F}_q)}$.

In order to calculate the Zeta function of $\mathrm{G(d, n)}$ we first need to calculate the number of points of $\mathrm{G(d, n)}$ over any finite field. To calculate this, we consider the action of $\mathrm{Gal(\bar{k}/k)}$ on $\mathrm{G(d, n)(\bar{k})}$. Let k be a perfect field. We see that the Galois Group $\Gamma = \mathrm{Gal(\bar{k}/k)}$ acts on $\mathbb{P}^n(\bar{k})$ as follows:
For $\sigma \in \Gamma$ and $(a_0 : a_1 : ... : a_n) \in \mathbb{P}^n(\bar{k})$ we define

$$\sigma(a_0 : ... : a_n) = (\sigma(a_0) : ..... : \sigma(a_n)).$$

The action is well defined since $\forall \lambda \in k^*$ we have:

$$\sigma(\lambda a_0 : ... : \lambda a_n) = (\sigma(\lambda(a_0)) : ...... : \sigma(\lambda(a_n))) = \sigma(\lambda)(\sigma(a_0) : .... : \sigma(a_n)) = \sigma(a_0 : ... : a_n).$$

Moreover we have,

1. $Id(a_0 : .... : a_n) = (a_0 : .... : a_n)$.
2. $\sigma_1\sigma_2(a_0 : ... : a_n) = \sigma_1(\sigma_2(a_0 : ... : a_n))$.

One can prove the following lemma:

Lemma 1.1.1 *The Galois group* $\Gamma = \mathrm{Gal(\bar{k}/k)}$ *acts on* $\mathbb{P}^n(\bar{k})$ *and the fixed points are precisely the points in* $\mathbb{P}^n(k)$ *, i.e.*

$$\{u = (a_0 : ... : a_n) \in \mathbb{P}^n(\bar{k}) |\ \sigma(u) = u \forall\ \sigma \in \Gamma\} = \mathbb{P}^n(k).$$

We will now consider the action of the Galois group $\Gamma = \mathrm{Gal(\bar{k}/k)}$ on the grassmannian $\mathrm{G(d, n)}$ and use that to calculate the number of points of $\mathrm{G(d, n)(\mathbb{F}_q)}$

1.1.1 Action of the Galois group $\Gamma = \mathrm{Gal(\bar{k}/k)}$ on $\mathrm{G(d, n)}$:

Without loss of generality suppose that the n dimensional vector space V is $(\bar{k})^n$. $\mathrm{G(d, n)}$ is the collection of all d dimensional subspaces of $(\bar{k})^n$ and Γ acts on it as follows:
For $U \in \mathrm{G(d, n)}$ and $\sigma \in \Gamma$, define:

$$\sigma(U) = \{\sigma(x_1, x_2, \dots x_n) \,|\, (x_1, \dots, x_n) \in U\} \ \textit{where,}$$

$$\sigma(x_1, x_2, \ldots, x_n) = (\sigma(x_1), \ldots, \sigma(x_n)).$$

Then it is easy to verify that if U is spanned by $v_1, v_2, \ldots, v_d$ then, $\sigma(U)$ is again a d dimensional subspace of $(\bar{k})^n$ spanned by $\sigma(v_1), \ldots, \sigma(v_d)$. We can also think of $\mathrm{G(d,n)}$ as embedded in the projective space $\mathbb{P}^N = \mathbb{P}(\Lambda^d V)$ via the Plücker map $\mathrm{P} : \mathrm{G(d,n)} \to \mathbb{P}^{\mathrm{N}}$ and we may consider the action of Γ on it as induced by the action on the projective space. Note that the two actions of Γ on $\mathrm{G(d,n)}$ are Γ equivalent.

We say that $U \in \mathrm{G(d,n)}$ is Γ invariant if $\sigma(U) = U \ \forall \sigma \in \Gamma$. And one has the following lemma

Lemma 1.1.2 $U \in \mathrm{G(d,n)}$ *is* Γ *invariant iff* U *has a basis* $\{\mathrm{w_1, w_2, \ldots, w_d}\}$ *with each* $\mathrm{w_i \in k^n}$.

Proof : Clearly, if U has a basis $\{\mathrm{v_1, v_2, \ldots, v_d}\}$ with each $\mathrm{v_i \in k^n}$, then U is Γ invariant. Now let U be a d dimensional subspace of V spanned by vectors $\mathrm{v_1, v_2, \ldots, v_d}$. Let $\sigma(U) = U \ , \forall \ \sigma \in \Gamma = \mathrm{Gal(\bar{k}/k)}$. We prove that $\exists$ a basis $\{\mathrm{w_1, w_2, \ldots, w_d}\}$ of U such that

$$\forall \sigma \in \Gamma \,,\ \sigma(\mathrm{w_i}) = \mathrm{w_i}, \mathrm{i} = 1, 2, \ldots, \mathrm{d}.$$

As $\sigma(U) = U$, $\exists\, A(\sigma) \in \mathrm{GL(d, \bar{k})}$ such that

$$\sigma \begin{pmatrix} \mathbf{v_1} \\ \mathbf{v_2} \\ . \\ . \\ . \\ \mathbf{v_d} \end{pmatrix} = A(\sigma) \begin{pmatrix} \mathbf{v_1} \\ \mathbf{v_2} \\ . \\ . \\ . \\ \mathbf{v_d} \end{pmatrix}.$$

Then, $A(\sigma\,\tau) = \sigma A(\tau)\, A(\sigma)$. $[A(\sigma\tau)]^{-1} = (A(\sigma))^{-1}\,\sigma\, A(\tau)^{-1}$.

So $\{(A(\sigma))^{-1}\}$ is a 1-cocycle and using the result that $H^1(\mathrm{GL_n})$ is identity, we get that the 1-cocycle $\{(A(\sigma))^{-1}\}$ splits i.e. $\exists\ B \in \mathrm{GL(d, \bar{k})}$ such that $(A(\sigma))^{-1} = B^{-1}\,\sigma\, B$, *i.e.* $B = (\sigma B)\, A(\sigma)$. Now let

$$\begin{pmatrix} \mathbf{w_1} \\ \mathbf{w_2} \\ . \\ . \\ . \\ \mathbf{w_d} \end{pmatrix} = B \begin{pmatrix} \mathbf{v_1} \\ \mathbf{v_2} \\ . \\ . \\ . \\ \mathbf{v_d} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{w_1} \\ \mathbf{w_2} \\ . \\ . \\ . \\ \mathbf{w_d} \end{pmatrix} = B \begin{pmatrix} \mathbf{v_1} \\ \mathbf{v_2} \\ . \\ . \\ . \\ \mathbf{v_d} \end{pmatrix} = (\sigma B)\ A(\sigma) \begin{pmatrix} \mathbf{v_1} \\ \mathbf{v_2} \\ . \\ . \\ . \\ \mathbf{v_d} \end{pmatrix} = (\sigma B)\ \sigma \begin{pmatrix} \mathbf{v_1} \\ \mathbf{v_2} \\ . \\ . \\ . \\ \mathbf{v_d} \end{pmatrix} = \sigma \left[B \begin{pmatrix} \mathbf{v_1} \\ \mathbf{v_2} \\ . \\ . \\ . \\ \mathbf{v_d} \end{pmatrix} \right] = \sigma \begin{pmatrix} \mathbf{w_1} \\ \mathbf{w_2} \\ . \\ . \\ . \\ \mathbf{w_d} \end{pmatrix}.$$

So, $\forall\sigma \in \Gamma,\ \sigma(\mathrm{w_i}) = \mathrm{w_i},\ \mathrm{i} = 1, 2, \dots \mathrm{d}$ which implies that U has a basis $\{\mathrm{w_1}, \mathrm{w_2}, \dots \mathrm{w_d}\}$ with $\mathrm{w_i} \in \mathrm{k^n}$ (As $(\bar{k})^\Gamma = k$).

Now, let $k = \mathbb{F}_q$. Then we have,

$$\mid \mathrm{G(d,n)(k)} \mid = \mid [\mathrm{G(d,n)}(\overline{\mathrm{k}}^{\mathrm{n}})]^{\Gamma} \mid$$

which is the number of d dimensional subspaces of $(\bar{k})^n$ which are Γ invariant. Let J denote the collection of all bases $\{v_1, v_2, \dots, v_d\}$ with each $v_i \in k^n$. Then J defines an open subset of $(k^n)^d$. So, the number of bases $\{v_1, v_2, \dots, v_d\}$ with each $v_i \in k^n$ equals the cardinality of J. We first find $|\mathrm{J}|$.
The general linear group $\mathrm{GL(n,k) = Aut(k^n)}$ acts naturally on J and the action is transitive. The stabilizer of $X = \{e_1, ..., e_d\}$ has the block matrix of the form:

$$\begin{pmatrix} \mathbf{I} & * \\ \mathbf{0} & \mathbf{GL(n-d)} \end{pmatrix}$$

$$|\mathrm{J}| = \frac{|\mathrm{GL(n,k)}|}{|\mathrm{Stabilizer(X)}|} = \frac{|\mathrm{GL(n,k)}|}{|\mathrm{GL(n-d,k)}|.\mathrm{q}^{\mathrm{d}\cdot(\mathrm{n-d})}}.$$

By the lemma it follows that computing the number of subspaces which are Γ-invariant is same as computing elements of J, however, one has to be more careful as one may have different bases giving rise to the same element of $\mathrm{G(d,n)}$. The number of bases $\{v_1, v_2, \dots, v_d\}$ with each $v_i \in k^n$ is same as

$$\frac{\text{number of points of J}}{\text{number of bases for each U}}$$

The number of bases for each U is $|\mathrm{GL(d,k)}|$. So,

$$|\mathrm{G(d,n)}(\mathbb{F}_\mathrm{q})| = \frac{|\mathrm{GL(n)}(\mathbb{F}_\mathrm{q})|}{|\mathrm{GL(d)}(\mathbb{F}_\mathrm{q})|.|\mathrm{GL(n-d)}(\mathbb{F}_\mathrm{q})|.\mathrm{q}^{\mathrm{d(n-d)}}} = \frac{\mathrm{f(n)}}{\mathrm{f(d).f(n-d).q^{d(n-d)}}}.$$

where $f(n) = (q^n - 1)(q^n - q)...(q^n - q^{n-1})$.

1.2 Zeta function of Grassmannians

As seen before, the Grassmann variety $\mathrm{G(d,n)}$ can be embedded into projective space $\mathbb{P}(\Lambda^d V)$ by Plücker map. Also $\mathrm{G(d,n)}$ can be covered by open affine spaces of dimension $d(n-d)$. So it is a smooth projective variety of dimension $d(n-d)$ which we may consider over any finite field $\mathbb{F}_q$. We now calculate the Zeta function of some grassmannians over $\mathbb{F}_q$. We will also verify the rationality of Zeta function and the functional equation. First of all recall the definition of Zeta function of a smooth projective variety X over $k = \mathbb{F}_q$. Then the Zeta function is given by

$$Z(X,t) := exp\left(\sum_{r=1}^{\infty} N_r.\frac{t^r}{r}\right) \in \mathbb{Q}[[t]].$$

where N_r is the number of points of X defined over $\mathbb{F}_{q^r}$.

Example 1.2.1 *Projective space* $\mathbb{P}^n(\mathbb{F}_q)$.
One has, $|\mathbb{P}^n(\mathbb{F}_q)| = 1 + q + q^2 + \ldots + q^n$.
$N_r = |\mathbb{P}^n(\mathbb{F}_{q^r})| = 1 + q^r + q^{2r} + ... + q^{nr}$.

$$Z(t) = exp\left(\sum_{r=1}^{\infty}(1 + q^r + ... + q^{nr})\frac{t^r}{r}\right).$$

Taking logarithm on both sides we get,

$$ln[Z(t)] = \sum_{r=1}^{\infty}(1 + q^r + ... + q^{nr})\frac{t^r}{r}.$$

We use the formula : $ln(1-t) = -t - t^2/2 - t^3/3 - \ldots$.

$$\begin{aligned} ln[Z(t)] &= -ln(1-t) - ln(1-qt) - ... - ln(1-q^n t). \\ &= -ln[(1-t).........(1-q^n t)]. \end{aligned}$$

$$ln[Z(t)(1-t).........(1-q^n t)] = 0$$

$$Z(t) = \frac{1}{(1-t)(1-qt).........(1-q^n t)}.$$

We see that $P_i(t) = 1$ *for all odd* i *and* $P_0(t) = 1 - t, P_{2i}(t) = 1 - q^i t$ *for* $i = 1, 2, \ldots, n$. *Degree of* $P_i(t)$ *is zero for* i *odd and 1 for* i *even So, all odd Betti numbers are zero and the even Betti numbers equal to 1.*
$E = \sum b_i = n + 1$. *We now verify the functional equation:*

$$\begin{aligned} Z\left(\frac{1}{q^n t}\right) &= \frac{1}{(1 - 1/q^n t)(1 - q/q^n t)\ldots(1 - q^n/q^n t)}. \\ &= \frac{q^n t.q^{n-1}t\ldots qt.t}{(1-t)(1-qt)\ldots(1-q^n t)}. \\ &= q^{n(n+1)/2}.t^{n+1}. \\ &= q^{n.E/2}.t^E.Z(t). \end{aligned}$$

So, the functional equation is verified. Also the numbers $b_0, b_1, \ldots b_n$ *match with the Betti numbers of the complex projective space* $\mathbb{P}^n(\mathbb{C})$ *and the number* $E = n + 1$ *matches with Euler characteristic of* $\mathbb{P}^n(\mathbb{C})$.

Example 1.2.2 $\mathrm{G}(2,4)$
$\dim \mathrm{G}(2,4) = 2(4-2) = 4$. *First calculate* N_r. *We have,*

$$\begin{aligned} |\mathrm{G}(2,4)(\mathbb{F}_q)| &= \frac{(q^4-1)(q^4-q)(q^4-q^2)(q^4-q^3)}{(q^2-1)^2(q^2-q)^2q^4}. \\ &= (q^2+1)(q^2+q+1) = q^4 + q^3 + 2q^2 + q + 1. \end{aligned}$$

$$N_r = q^{4r} + q^{3r} + 2q^{2r} + q^r + 1.$$

$$\begin{aligned}
Z(t) &= exp\left(\sum_{r=1}^{\infty}(1 + q^r + 2q^{2r} + q^{3r} + q^{4r})\frac{t^r}{r}\right). \\
ln[Z(t)] &= -ln[(1-t)(1-qt)(1-q^2t)^2(1-q^3t)(1-q^4t). \\
Z(t) &= \frac{1}{(1-t)(1-qt)(1-q^2t)^2(1-q^3t)(1-q^4t)}.
\end{aligned}$$

We see that $Z(t)$ is a rational function in t. $P_i(t) = 1$ for all odd i. $P_0(t) = 1-t, P_2(t) = 1-qt, P_4(t) = (1-q^2t)^2, P_6(t) = 1-q^3t, P_8(t) = 1-q^4t$. The Betti numbers b_i are zero for all odd i and $b_0 = 1, b_2 = 1, b_4 = 2, b_6 = 1, b_8 = 1$. $E = \sum b_i = 6$.
We now verify the functional equation:

$$\begin{aligned}
Z\left(\frac{1}{q^4t}\right) &= \frac{1}{(1-1/q^4t)(1-q/q^4t)(1-q^2/q^4t)^2(1-q^3/q^4t)(1-q^4/q^4t)}. \\
&= q^4t.q^3t.(q^2t)^2.qt.t.Z(t). \\
&= q^{12}.t^6.Z(t). \\
&= q^{4.6/2}.t^6.Z(t). \\
&= q^{nE/2}t^E.Z(t).
\end{aligned}$$

and the functional equation is verified.

Example 1.2.3 $\mathrm{G}(2,5)(\mathbb{F}_q)$

$$\begin{aligned}
|\mathrm{G}(2,5)(\mathbb{F}_q)| &= \frac{(q^5-1)(q^5-q)(q^5-q^2)(q^5-q^3)(q^5-q^4)}{(q^2-1)(q^2-q)(q^3-1)(q^3-q)(q^3-q^2)q^6}. \\
&= 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6.
\end{aligned}$$

$$N_r = 1 + q^r + 2q^{2r} + 2q^{3r} + 2q^{4r} + q^{5r} + q^{6r}.$$

$$Z(t) = exp\left(\sum_{r=1}^{\infty}(1 + q^r + 2q^{2r} + 2q^{3r} + 2q^{4r} + q^{5r} + q^{6r})\frac{t^r}{r}\right).$$

and by similar calculations we get,

$$Z(t) = \frac{1}{(1-t)(1-qt)(1-q^2t)^2(1-q^3t)^2(1-q^4t)^2(1-q^5t)(1-q^6t)}.$$

Example 1.2.4 $\mathrm{G}(3,6)(\mathbb{F}_q)$

$$\begin{aligned}
|\mathrm{G}(3,6)(\mathbb{F}_q)| &= \frac{(q^6-1)(q^6-q)\dots(q^6-q^5)}{(q^3-1)^2(q^3-q)^2(q^3-q^2)^2q^9}. \\
&= (q^3+1)(q^2+1)(q^4+q^3+q^2+q+1). \\
&= q^9 + q^8 + 2q^7 + 3q^6 + 3q^5 + 3q^4 + 3q^3 + 2q^2 + q + 1. \\
N_r &= q^{9r} + q^{8r} + 2q^{7r} + 3q^{6r} + 3q^{5r} + 3q^{4r} + 3q^{3r} + 2q^{2r} + q^r + 1.
\end{aligned}$$

$$Z(t) = exp\left(\sum_{r=1}^{\infty}(q^{9r}+q^{8r}+2q^{7r}+3q^{6r}+3q^{5r}+3q^{4r}+3q^{3r}+2q^{2r}+q^{r}+1)\frac{t^r}{r}\right).$$

Taking logarithm on both sides and simplifying we get,

$$Z(t) = \frac{1}{(1-t)(1-qt)(1-q^(2t)^2(1-q^3t)^3(1-q^4t)^3(1-q^5t)^3(1-q^6t)^3(1-q^7t)^2(1-q^8t)(1-q^9t)}.$$

The functional eqation can be easily verified in a similar way as we did for $G(2,4)$.

The general case $\mathrm{G(d,n)(\mathbb{F}_q)}$:
As seen before,

$$N_r = |\mathrm{G(d,n)(\mathbb{F}_{q^r})}| = \frac{\mathrm{(q^{nr}-1)(q^{nr}-q^r)\dots(q^{nr}-q^{(n-1)r})}}{\mathrm{(q^{dr}-1)\dots(q^{dr}-q^{(d-1)r}).(q^{(n-d)r}-1)\dots(q^{(n-d)r}-q^{(n-d-1)r}).q^{rd(n-d)}}}.$$

For simplicity set $q^r = l$. So we have

$$N_r = \frac{(l^n-1)(l^n-l)\dots(l^n-l^{n-1})}{(l^d-1)\dots(l^d-l^{d-1}).(l^{n-d}-1)\dots(l^{n-d}-l^{n-d-1}).l^{d(n-d)}}.$$

Multiplying and dividing by $l^{d(n-d)}$ and simplifying we get,

$$N_r = \frac{(l^n-1)(l^{n-1}-1)\dots(l^{n-d+1}-1)}{(l^d-1)(l^{d-1}-1)\dots(l-1)}.$$

This is the usual Gaussian Binomial coefficient $\binom{n}{d}_l$ and it can be interpreted as a polynomial in l. To be more precise,

$$\binom{n}{d}_l = \sum_{i=0}^{d(n-d)} b_i l^i.$$

where the coefficient b_k of l^k in this polynomial is the number of distinct partitions of k elements that fit inside a rectangle of size $d \times (n-d)$. We illustrate this with examples.

Example 1.2.5 *Find the Gaussian binomial coefficient* $\binom{4}{2}_l$.
Suppose $\binom{4}{2}_l = b_0 + b_1 l + b_2 l^2 + b_3 l^3 + b_4 l^4$.
We summarize the number of partitions of k for $k = 0, 1, 2, 3, 4$ *in the following table:*

k	*Partitions of k*	b_k *= number of allowed partitions*
0	$\{\}$	1
1	$\{1\}$	1
2	$\{\{2\},\{1,1\}\}$	2
3	$\{\{3\},\{2,1\},\{1,1,1\}\}$	1
4	$\{\{4\},\{3,1\},\{2,2\},\{2,1,1\},\{1,1,1,1\}\}$	1

Hence we see that:

$$\binom{4}{2}_l = 1 + l + 2l^2 + l^3 + l^4.$$

i.e. $N_r = 1 + q^r + 2q^{2r} + q^{3r} + q^{4r}$. *Note that this calculation matches with the calculation done before while calculating Zeta function for* $\mathrm{G}(2,4)(\mathbb{F}_q)$.

Example 1.2.6 *Find the Gaussian binomial coefficient* $\binom{5}{2}_l$.
Suppose $\binom{5}{2}_l = b_0 + b_1 l + b_2 l^2 + b_3 l^3 + b_4 l^4 + b_5 l^5$.
We summarize the number of allowed partitions of k *for* $k = 0, 1, 2, 3, 4, 5, 6$ *in the following table:*

k	*Allowed partitions of* k	b_k = *number of allowed partitions*
0	$\{\}$	1
1	$\{1\}$	1
2	$\{\{2\},\{1,1\}\}$	2
3	$\{\{2,1\},\{1,1,1\}\}$	1
4	$\{\{2,2\},\{2,1,1\}\}$	1
5	$\{\{2,2,1\}\}$	1
6	$\{\{2,2,2\}\}$	1

Hence we see that:

$$\binom{5}{2}_l = 1 + l + 2l^2 + 2l^3 + 2l^4 + l^5 + l^6.$$

i.e. $N_r = 1 + q^r + 2q^{2r} + 2q^{3r} + 2q^{4r} + q^{5r} + q^{6r}$. *Again this calculation matches with the calculation done before while calculating Zeta function for* $\mathrm{G}(2,5)(\mathbb{F}_q)$.

Example 1.2.7 *Find the Gaussian binomial coefficient* $\binom{6}{3}_l$.
Here $d(n-d) = 3.3 = 9$. *Suppose* $\binom{6}{3}_l = b_0 + b_1 l + b_2 l^2 + b_3 l^3 + b_4 l^4 + b_5 l^5 + b_6 l^6 + b_7 l^7 + b_8 l^8 + b_9 l^9$.
We summarize the number of allowed partitions of k *for* $k = 0, 1, \ldots, 9$ *in the following table:*

k	*Allowed partitions of k*	b_k *= number of allowed partitions*
0	$\{\}$	1
1	$\{1\}$	1
2	$\{\{2\},\{1,1\}\}$	2
3	$\{\{3\},\{2,1\},\{1,1,1\}\}$	3
4	$\{\{3,1\},\{2,2\},\{2,1,1\}\}$	3
5	$\{\{2,2,1\},\{3,2\},\{3,1,1\}\}$	3
6	$\{\{2,2,2\},\{3,2,1\},\{3,3\}\}$	3
7	$\{\{3,2,2\},\{3,3,1\}\}$	2
8	$\{\{3,3,2\}\}$	1
9	$\{\{3,3,3\}\}$	1

Hence we see that:

$$\binom{6}{3}_l = 1 + l + 2l^2 + 3l^3 + 3l^4 + 3l^5 + 3l^6 + 2l^7 + l^8 + l^9.$$

i.e. $N_r = 1 + q^r + 2q^{2r} + 3q^{3r} + 3q^{4r} + 3q^{5r} + 3q^{6r} + 2q^{7r} + q^{8r} + q^{9r}.$

We now consider the general case. Regarding l as a formal variable, it is possible to express the coefficient N_r for any grassmannian $\mathrm{G(d,n)(\mathbb{F}_q)}$ as

$$N_r = \sum_{i=0}^{d(n-d)} b_i l^i$$

where b_i can be found as explained before and the Zeta function of the grassmannian $\mathrm{G(d,n)}$ then comes out to be :

$$Z(t) = \frac{1}{(1-t)^{b_0}(1-qt)^{b_1}\dots(1-q^{d(n-d)}t)^{b_{d(n-d)}}}.$$

From this we see that all the odd Betti mubers of of the grassmannians are zero. The numbers b_i here are the even topological Betti numbers of the complex Grassmannian $\mathrm{X(\mathbb{C}) = G(d,n)(\mathbb{C})}$ i.e. $\mathrm{b_i = dim\ H_{2i}(X(\mathbb{C}),\mathbb{Z})}$(The odd Betti numbers of $\mathrm{X(\mathbb{C})}$ are zero).

2 Lagrangian Grassmannian

Let V be a vector space over field k of dimension $2n, n \geq 1$. Consider the set of all n dimensional subspaces of V i.e. the grassmannian $\mathrm{G(n,2n)}$. We are interested in a subvariety of $\mathrm{G(n,2n)}$. We define a pairing on V. For $x, y \in V, x = (x_1, x_2, ..., x_{2n}), y = (y_1, y_2, ..., y_{2n})$ define:

$$< x, y >= \sum_{i=1}^{n} [(x_i.y_{2n+1-i}) - (x_{2n+1-i}.y_i)].$$

This is a non-degenerate alternating pairing on V. We say that $U \in \mathrm{G(n,2n)}$ is isotropic iff $< x, y >= 0 \ \forall x, y \in U$.

Definition 2.0.8 *In the above notations, the Lagrangian Grassmannian* $\mathrm{L(n,2n)}$ *is defined by :* $\mathrm{L(n,2n)} = \{\mathrm{U} \in \mathrm{G(n,2n)} | \mathrm{U}$ *is isotropic*$\}$.

It can be shown that $\mathrm{L(n,2n)}$ is a projective subvariety of $\mathrm{G(n,2n)}$ of dimension $\frac{n(n+1)}{2}$.

2.1 To calculate the number of points of $\mathrm{L(n,2n)(\mathbb{F}_q)}$.

The symplectic group $\mathrm{Sp(2n)(\mathbb{F}_q)}$ acts transitively on the set of all isotropic subspaces of $\mathrm{G(n,2n)(\mathbb{F}_q)}$, i.e. on the Lagrangian grassmannian. So we have,

$$|\mathrm{L(n,2n)(\mathbb{F}_q)}| = \frac{|\mathrm{Sp(2n)(\mathbb{F}_q)}|}{|\mathrm{Stabilizer\,of\,X}|}, \mathrm{X} \in \mathrm{L(n,2n)}.$$

To find $|\mathrm{Sp(2n)(\mathbb{F}_q)}|$ we use the following result from the linear algebra.

Lemma 2.1.1 *If f is a non-degenerate alternating paring on a $2n$ dimensional vector space V over a field of q elements then the number of pairs $\{u, v\}$ s.t. $f(u, v) =< u, v >= 1$ is $(q^{2n} - 1)q^{2n-1}$.*

Now, given f-non degenerate, alternating pairing on vector space V of dimension $2n$ by standard results, there exists a symplectic basis
$\{v_1, v_2, ..., v_{2n}\}$ for V such that

$$< v_i, v_{i+n} >= 1, i = 1, ..., n; < v_i, v_j >= 0, |i - j| \neq n.$$

If $\{v_i\}$ is a symplectic basis of V then, $\theta \in Sp(2n)$ iff $\theta.v_i$ is also a symplectic basis for V. i.e.

$$< \theta v_i, \theta v_{i+n} >= 1, i = 1, ..., n; < \theta v_i, \theta v_j >= 0, |i - j| \neq n.$$

The number of pairs such that $< \theta v_1, \theta v_{1+n} >= 1$ is $(q^{2n} - 1)q^{2n-1}$. Once we choose $\{\theta v_1, \theta v_{1+n}\}$ for $\{\theta v_i\}$ to be a symplectic basis the number of pairs $\{\theta v_2, \theta v_{2+n}\}$ such that $< \theta v_2, \theta v_{2+n} >= 1$ is $q^{(2n-2)-1}(q^{2n-2} - 1)$; and so on ... Finally, the number of pairs $\{\theta v_n, \theta v_{2n}\}$ such that $< \theta v_n, \theta v_{2n} >= 1$ is $q(q^2 - 1)$. And so,

$$|\mathrm{Sp(2n)(\mathbb{F}_q)}| = \prod_{\mathrm{i=1}}^{\mathrm{n}} (\mathrm{q^{2i}} - 1)\mathrm{q^{2i-1}} = \mathrm{q^{n^2}} \prod_{\mathrm{i=1}}^{\mathrm{n}} (\mathrm{q^{2i}} - 1) = \mathrm{q^{n^2}} \prod_{\mathrm{i=1}}^{\mathrm{n}} (\mathrm{q^i} - 1)(\mathrm{q^i} + 1).$$

To find the stabilizer of $\mathrm{X} \in \mathrm{L(n,2n)}$:
Notation : We denote the transpose of a matrix A by A^t. Let

$$J = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}$$

$$\mathrm{Sp(2n) = \{A \in GL(2n) | A^t J A = J\}}$$

Let $\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{C} \end{pmatrix} \in \mathrm{StabX}$. If it has to be in Sp(2n) we must have,

$$\begin{aligned}
\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{C} \end{pmatrix}^t \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{C} \end{pmatrix} &= \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix} \\
\begin{pmatrix} \mathbf{A^t} & \mathbf{0} \\ \mathbf{B^t} & \mathbf{C^t} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{C} \end{pmatrix} &= \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix} \\
\begin{pmatrix} \mathbf{A^t} & \mathbf{0} \\ \mathbf{B^t} & \mathbf{C^t} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{C} \\ -\mathbf{A} & -\mathbf{B} \end{pmatrix} &= \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix} \\
\begin{pmatrix} \mathbf{0} & \mathbf{A^t C} \\ -\mathbf{C^t A} & \mathbf{B^t C - c^t B} \end{pmatrix} &= \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}
\end{aligned}$$

$C = (A^{-1})^t$ and $B^tC = C^tB$ i.e. C^tB is a symmetric matrix. So, if

$$M = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{C} \end{pmatrix} \in \mathrm{StabilizerX}$$

Then it is of the form :

$$M = \begin{pmatrix} \mathbf{A} & \mathbf{(C^t)^{-1}S} \\ \mathbf{0} & \mathbf{(A^{-1})^t} \end{pmatrix} = \begin{pmatrix} \mathbf{A} & \mathbf{AS} \\ \mathbf{0} & \mathbf{(A^{-1})^t} \end{pmatrix}$$

for some symmetric $n \times n$ matrix S. One can see that the StabilizerX is the semidirect product of GL(n) the general linear $n \times n$ group and S(n); the group of symmetric $n \times n$ matrices.

$$\begin{aligned}
|\mathrm{Stab\,(X)(\mathbb{F}_q)}| &= |\mathrm{S(n)(\mathbb{F}_q)}|.|\mathrm{GL(n)(\mathbb{F}_q)}|. \\
&= q^{\frac{n(n+1)}{2}} \prod_{i=0}^{n-1} (q^n - q^i). \\
&= q^{\frac{n(n+1)}{2}} q^{\frac{n(n-1)}{2}} \prod_{i=1}^{n} (q^i - 1)
\end{aligned}$$

$$\begin{aligned}
|\mathrm{L(n, 2n)(\mathbb{F}_q)}| &= \frac{|\mathrm{Sp(2n)(\mathbb{F}_q)}|}{|\mathrm{GL(n)(\mathbb{F}_q)}|.|\mathrm{S(n)(\mathbb{F}_q)}|} \\
&= \frac{q^{n^2} \prod_{i=1}^{n} (q^i - 1)(q^i + 1)}{q^{\frac{n(n+1)}{2}} q^{\frac{n(n-1)}{2}} \prod_{i=1}^{n} (q^i - 1)}. \\
&= \prod_{i=1}^{n} (1 + q^i).
\end{aligned}$$

2.2 Zeta function for Lagrangian Grassmannians:

The Lagrangian Grassmannian $\mathrm{L(n,2n)}$ is a smooth projective subvariety of the grassmannian $\mathrm{G(n,2n)}$ and we may consider it over any finite field $\mathbb{F}_q$. The number of points in $\mathrm{L(n,2n)(\mathbb{F}_q)}$ is given by:

$$|\mathrm{L(n,2n)(\mathbb{F}_q)}| = \prod_{i=1}^{n}(1+q^i).$$

As there are no terms in the denominator, N_r is a polynomial in powers of q^r and the Zeta function of such grassmannians are easy to calculate.

Example 2.2.1 $\mathrm{L(2,4)(\mathbb{F}_q)}$

$$\begin{aligned} |\mathrm{L(2,4)(\mathbb{F}_q)}| &= (1+q)(1+q^2). \\ &= 1+q+q^2+q^3 = 1+q+q^2+q^3. \end{aligned}$$

$$N_r = 1+q^r+q^{2r}+q^{3r} = 1+q^r+q^{2r}+q^{3r}.$$

$$Z(t) = \frac{1}{(1-t)(1-qt)(1-q^2t)(1-q^3t)}.$$

Example 2.2.2 $\mathrm{L(3,6)(\mathbb{F}_q)}$

$$\begin{aligned} |\mathrm{L(3,6)(\mathbb{F}_q)}| &= (1+q)(1+q^2)(1+q^3). \\ &= 1+q+q^2+2q^3+q^4+q^5+q^6. \end{aligned}$$

$$N_r = 1+q^r+q^{2r}+2q^{3r}+q^{4r}+q^{5r}+q^{6r}.$$

$$Z(t) = \frac{1}{(1-t)(1-qt)(1-q^2t)(1-q^3t)^2(1-q^4t)(1-q^5t)(1-q^6t)}.$$

Example 2.2.3 $\mathrm{L(4,8)(\mathbb{F}_q)}$

$$\begin{aligned} N_r &= (1+q^r)(1+q^{2r})(1+q^{3r})(1+q^{4r}). \\ &= 1+q^r+q^{2r}+2q^{3r}+2q^{4r}+2q^{5r}+2q^{6r}+2q^{7r}+q^{8r}+q^{9r}+q^{10r}. \end{aligned}$$

$$Z(t) = \frac{1}{(1-t)(1-qt)(1-q^2t)(1-q^3t)^2(1-q^4t)^2(1-q^5t)^2(1-q^6t)^2(1-q^7t)^2(1-q^8t)(1-q^9t)(1-q^{10}t)}$$

General Case: We have:

$$\left|\mathrm{L(n,2n)(\mathbb{F}_q^r)}\right| = \prod_{\mathrm{i=1}}^{\mathrm{n}}(1+\mathrm{q^{ir}}).$$

For simplicity set $q^r = l$.

$$N_r = \left|L(n,2n)(\mathbb{F}_q^r)\right| = \prod_{i=1}^{n}(1+l^i) = (1+l)(1+l^2)\dots(1+l^n) = 1+b_1l+b_2l^2+\dots+b_ml^m.$$

where the coefficient b_i is equal to the number of strict partitions of i whose parts do not exceed n, $m = n(n+1)/2$. So, the coeficcients b_i can be calculated precisely and one observes that the Zeta function in general case is

$$Z(t) = \frac{1}{(1-t)(1-qt)^{b_1}(1-q^2t)^{b_2}\dots(1-q^mt)^{b_m}}.$$

where, b_i and m are described as above.

3 Schubert Calculus

Schubert Calculus provides us the machinery necessary to describe the cohomology ring of $\mathrm{G}^{\mathbb{P}}(\mathrm{d},\mathrm{n})$ with integer coefficients when the base field is $\mathbb{C}$. We now define some important notions in schubert calculus.

(1) Schubert conditions and Schubert varieties : We are interested in finding a necessary and sufficient condition for a d-plane in $\mathbb{P}^n$ to intersect a given sequence of linear spaces in $\mathbb{P}^n$ in a prescribed way. Let $\underline{A} : A_0 \subset A_1 \subset \dots \subset A_d$ be a strictly increasing sequence of $d+1$ linear spaces of $\mathbb{P}^n$. Such a sequence is called a **flag**. A d-plane L in $\mathbb{P}^n$ is said to satisfy the **Schubert condition** defined by a flag $\underline{A}$ if, $\mathrm{dim}(\mathrm{A_i} \bigcap \mathrm{L}) \geq \mathrm{i} \quad \forall \mathrm{i} = 0, 1, \dots, \mathrm{d}$. i.e. A d-plane satisfying the Schubert conditions with respect to a flag $\underline{A}$ intersects A_0 at least in a point, A_1 at least in a line,... etc and it lies in A_d. One can show that the condition $\dim(A_i \bigcap L) \geq i$ for $i = 0, \dots, d$ is satisfied iff the Plücker coordinates of d-plane L satisfy certain linear relations in addition to the quadratic plücker relations. Hence, the collection of all such d-planes in $\mathrm{G}^{\mathbb{P}}(\mathrm{d},\mathrm{n})$ satisfying the Schubert condition with respect to a given flag $\underline{A}$ defines a projective variety. It is known as **Schubert variety** $\Omega(\underline{A})$ corresponding to the flag $\underline{A}$. In fact, this variety is the intersection of a linear subspace of $\mathbb{P}^n$ with $\mathrm{G}^{\mathbb{P}}(\mathrm{d},\mathrm{n})$. The dimension of Schubert variety $\Omega(\underline{A})$ with $\underline{A}$ as above is $\sum_{i=0}^{d}(a_i - i)$.

(2)Schubert cycle : The Schubert variety $\Omega(\underline{A})$ defines a cohomology class in the cohomology ring $\mathrm{H}^*(\mathrm{G}^{\mathbb{P}}(\mathrm{d},\mathrm{n});\mathbb{Z})$. The cohomology class of $\Omega(\underline{A})$ in $\mathrm{H}^*(\mathrm{G}^{\mathbb{P}}(\mathrm{d},\mathrm{n});\mathbb{Z})$ is called a Scubert cycle. Although the variety $\Omega(\underline{A})$ depends on the choice of the flag $\underline{A}$, the cohomology class of $\Omega(\underline{A})$ depends only on the integers $a_i = \mathrm{dim} A_i$. So, we denote the class of $\Omega(\underline{A})$ by $\Omega(\underline{a})$ where, $\underline{a}$ is defined by integers $a_i = dim A_i$, $0 \leq a_0 < a_1 < \dots < a_d \leq n$.

We now state the fundamental theorem of Schubert Calculus which asserts that the Schubert cycles completely determine the cohomology of $\mathrm{G}^{\mathbb{P}}(\mathrm{d},\mathrm{n})$.

Theorem 3.0.4 *The Basis Theorem : Considered additively,* $\mathrm{H}^*(\mathrm{G}^{\mathbb{P}}(\mathrm{d},\mathrm{n});\mathbb{Z})$ *is a free abelian group and the Schubert cycles* $\Omega(a_0 \dots a_d)$ *form a basis. Each integral cohomology group* $\mathrm{H}^{2\mathrm{p}}(\mathrm{G}^{\mathbb{P}}(\mathrm{d},\mathrm{n});\mathbb{Z})$ *is a free abelian group and the Schubert* *cycles* $\Omega(\underline{a})$ *with* $[(d+1)(n-d) - \sum_{i=0}^{d}(a_i - i)] = p$ *form a basis. Each cohomology group* $\mathrm{H^r}(\mathrm{G^{\mathbb{P}}}(\mathrm{d,n});\mathbb{Z})$*, with* r *odd, is zero.*

This theorem determines the additive structure of the cohomology ring $\mathrm{H}^*(\mathrm{G^{\mathbb{P}}}(\mathrm{d,n});\mathbb{Z})$. Since each odd cohomology group is zero we observe that the cup product is commutative and the ring $\mathrm{H}^*(\mathrm{G^{\mathbb{P}}}(\mathrm{d,n});\mathbb{Z})$ is a commutative ring.

We now calculate the cohomology groups of some grassmannians and find their dimensions.

Example 3.0.5 *Projective space* $\mathbb{P}^n$=G(1,n+1)= $\mathrm{G^{\mathbb{P}}}(0,\mathrm{n})$. $\dim(\mathbb{P}^{\mathrm{n}}) = \mathrm{n}$.
Using the basis theorem, for $p = 0, 1, \ldots, n$, $H^{2p}(\mathbb{P}^n;\mathbb{Z})$ is one dimensional generated by the cycle $\Omega(a_0)$ with $n - a_0 = p$. $H^r(\mathbb{P}^n;\mathbb{Z})$ is 0 for r odd. So all odd Betti numbers are zero and the even Betti numbers are equal to 1.

Example 3.0.6 $\mathrm{G}(2,4) = \mathrm{G^{\mathbb{P}}}(1,3)$.
$\dim(\mathrm{G}(2,4)) = 2.2 = 4$. *For* $0 \le p \le 4$, $\mathrm{H^{2p}}(\mathrm{G^{\mathbb{P}}}(1,3);\mathbb{Z})$*is generated by cycles* $\Omega(a_0.a_1)$ *with* $4 - [a_0 + (a_1 - 1)] = p$. *i.e.* $a_0 + a_1 = 5 - p$. *For* $p = 0$*, the only integer solution to* $a_0 + a_1 = 5$ *with* a_0 *and* a_1 *as in Schubert conditions is* $a_0 = 2$ *and* $a_1 = 3$. *Hence,* $\mathrm{H^0}(\mathrm{G^{\mathbb{P}}}(1,3);\mathbb{Z})$ *is generated by the cycle* $\Omega(2.3)$ *and has dimension 1. We do similar calculations and form the following table:*

p	$\dim(\mathrm{H^{2p}}(\mathrm{G^{\mathbb{P}}}(1,3);\mathbb{Z}))$	Generators
0	1	$\Omega(2.3)$
1	1	$\Omega(1.3)$
2	2	$\Omega(0.3), \Omega(1.2)$
3	1	$\Omega(0.2)$
4	1	$\Omega(0.1)$

Example 3.0.7 $\mathrm{G}(2,5) = \mathrm{G^{\mathbb{P}}}(1,4)$.
$\dim(G(2,5)) = 2.3 = 6$. For $0 \le p \le 6$, $\mathrm{H^{2p}}(\mathrm{G^{\mathbb{P}}}(1,4);\mathbb{Z})$ is generated by the cycles $\Omega(a_0.a_1)$ with $6 - [a_0 + (a_1 - 1)] = p$. i.e. $a_0 + a_1 = 7 - p$. For $p = 0$, the only integer solution to $a_0 + a_1 = 7$ with a_0 and a_1 as in Schubert conditions is $a_0 = 3$ and $a_1 = 4$. We summarize the calculation for other cohomology groups in the following table:

p	$\dim(\mathrm{H^{2p}}(\mathrm{G^{\mathbb{P}}}(1,4);\mathbb{Z}))$	Generators
0	1	$\Omega(3.4)$
1	1	$\Omega(2.4)$
2	2	$\Omega(1.4), \Omega(2.3)$
3	2	$\Omega(0.4), \Omega(1.3)$
4	2	$\Omega(0.3), \Omega(1.2)$
5	1	$\Omega(0.2)$
6	1	$\Omega(0.1)$

Connections to the cohomology in characteristic zero:

If $X \to Spec\ \mathbb{Z}_{(p)}$ is a smooth and proper morphism of schemes then, the cohomology of $X \otimes \overline{\mathbb{Q}}$ with the Galois action gives the information about the cohomology of $X \otimes \overline{\mathbb{F}}_p$ with its Galois action. Let $\mathcal{O}$ be the ring of integers of $\overline{\mathbb{Q}}$. Suppose p is a prime and μ is a maximal ideal containing p. Then $\mathbb{O}_\mu$ is a local ring with unique maximal ideal $\mu\mathcal{O}_\mu$. The residue field $k = \mathcal{O}_\mu/\mu\mathcal{O}_\mu \cong \overline{\mathbb{F}}_p$. Let $\tilde{X} = X \otimes \mathcal{O}_\mu$. If $X \otimes \mathcal{O}_\mu \to Spec\ \mathcal{O}_\mu$ is a smooth and proper morphism of schemes then the cohomology of $\tilde{X} \otimes \overline{\mathbb{Q}}$ with Galois action gives the cohomology of $\tilde{X} \otimes k$ with its Galois action. Now let $X = \mathcal{G}$ be the grassmannn variety $\mathrm{G(d,n)}$. Let $m = \dim\mathbb{G} = \mathrm{d(n-d)}$. The equations defining $\mathcal{G}$ i.e. the Plücker relations are relations with integer coefficients. So, we can consider $\mathcal{G}$ over fields of characteristic zero namely $\mathbb{Q}$, over $\mathbb{C}$ and also over finite field $\mathbb{F}_q$. Let $\mathcal{G} \otimes_{\mathcal{O}_\mu} \overline{\mathbb{Q}}$ denote the grassmann variety $\mathrm{G(d,n)}$ over $\overline{\mathbb{Q}}$ and let $\mathcal{G} \otimes_{\mathcal{O}_\mu} k$ denote the grassmann variety $\mathrm{G(d,n)}$ over $\overline{\mathbb{F}}_p$. Since over any algebraically closed field L, $\mathcal{G}$ is smooth and proper, the morphism $\mathcal{G} \to Spec(\mathcal{O}_\mu)$ is smooth and proper. Let l be a prime other than p. We have an isomorphism

$$f:\ H^i_{et}(\mathcal{G} \otimes \overline{\mathbb{Q}}; \mathbb{Q}_l) \to H^i_{et}(\mathcal{G} \otimes k; \mathbb{Q}_l)$$

(see Milne Lecture notes 20.4) which is Galois equivariant. The Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$ contains the decomposition group D_μ and the inertia group I_μ as its subgroups. We have $I_\mu \subset D_\mu \subset \mathrm{Gal}(\overline{\mathbb{Q}}\,/\,\mathbb{Q})$. To say f is Galois equivariant means that, if $\tau \in D_\mu$ then, $\overline{\tau} \in \mathrm{Gal}(\overline{\mathbb{F}_p}\,/\,\mathbb{F}_p)$ and for a class $c \in H^{2i}_{et}(\mathcal{G} \otimes \overline{\mathbb{Q}}; \mathbb{Q}_l)$ one has

$$f(\tau c) = \overline{\tau}.f(c).$$

This implies that the inertia group I_μ acts trivially on $H^i_{et}(\mathcal{G} \otimes \overline{\mathbb{Q}}; \mathbb{Q}_l)$. The Frobenius morphism $\mathrm{F}: \mathcal{G} \otimes \mathbb{F}_p \to \mathcal{G} \otimes \mathbb{F}_p$ induces linear map F^* on cohomology. Let $\alpha \in \mathrm{Gal}(\overline{\mathbb{F}}_p\,/\,\mathbb{F}_p)$ be the geometric Frobenius $x \mapsto x^{1/p}$. Let us denote by α also the induced linear map on cohomology. Then $\alpha = \mathrm{F}^*$. Also there exists $\beta \in D_\mu$ such that $\overline{\beta} = \alpha$. We now use all this to simplify the expression of Zeta function of $\mathcal{G}$. We have $Z(\mathcal{G}, t)$ as (see Hartshorne appendix C)

$$\begin{aligned} Z(\mathcal{G}, t) &= \prod_{i=0}^{2m} \det[1 - \mathrm{tF}^* \mid \mathrm{H}^{\mathrm{i}}_{\mathrm{et}}(\mathcal{G} \otimes \overline{\mathbb{F}}_{\mathrm{p}}; \mathbb{Q}_{\mathrm{l}})]^{(-1)^{\mathrm{i}+1}}. \\ &= \prod_{i=0}^{2m} \det[1 - \mathrm{t}\alpha \mid \mathrm{H}^{\mathrm{i}}_{\mathrm{et}}(\mathcal{G} \otimes \overline{\mathbb{F}}_{\mathrm{p}}; \mathbb{Q}_{\mathrm{l}})]^{(-1)^{\mathrm{i}+1}}. \\ &= \prod_{i=0}^{2m} \det[1 - \mathrm{t}\beta \mid \mathrm{H}^{\mathrm{i}}_{\mathrm{et}}(\mathcal{G} \otimes \overline{\mathbb{Q}}; \mathbb{Q}_{\mathrm{l}})]^{(-1)^{\mathrm{i}+1}}. \end{aligned}$$

We use 3.6 and 3.7 of *Hartshorne appendix C* and get,

$$H^i_{et}(\mathcal{G} \otimes \overline{\mathbb{Q}}; \mathbb{Q}_l) \cong H^i_{et}(\mathcal{G} \otimes \mathbb{C}; \mathbb{Q}_l) \cong H^i_{betti}(\mathcal{G} \otimes \mathbb{C}; \mathbb{Q}_l) \cong H^i_{betti}(\mathcal{G} \otimes \mathbb{C}; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{Q}_l.$$

As seen by the basis theorem in Schubert Calculus, we see that the Schubert cycles generate $H^*(\mathcal{G} \otimes \mathbb{C}; \mathbb{Z})$. Now, if Y is a subvariety of $\mathcal{G}$ of codimension i, it gives a class $[Y] \in H^{2i}_{et}(\mathcal{G} \otimes \overline{\mathbb{Q}}; \mathbb{Q}_l)$ on which β acts by $\beta[Y] = p^i[\beta(Y)]$. So, we have a simpler formula for the Zeta function for $\mathrm{G(d,n)}$ as :

$$Z(\mathcal{G}, t) = \frac{1}{\prod_{i=0}^{m}(1 - p^i t)^{b_{2i}}}.$$

where b_{2i} denotes the rank of $H^{2i}(\mathcal{G}; \mathbb{Z})$ over $\mathbb{Z}$. So the Zeta function for grassmann variety $\mathrm{G(d,n)}$ of dimension m is given by :

$$Z(\mathcal{G}, t) = \frac{1}{(1-t)(1-pt)^{b_2}(1-p^2t)^{b_4} \ldots (1-p^m t)^{b_{2m}}}.$$

which matches with the calculation done before. One observes that knowing the cohomology in characteristic p, we can know the cohomology in characteristic zero and vice versa.

References

[1] Joe Harris, *Algebraic Geometry, A First Course, Vol. 133 January 1994.*

[2] Robin Hartshorne, *Algebraic Geometry, Graduate Texts in mathematics, Springer Verlag, New-York, 1977, Appendix C : The Weil Conjectures.*

[3] S.L.Kleiman and Laksov, *Schubert Calculus, The American Mathematical Monthly, vol 79, no. 10, Dec 1972.*