Quaternions and Arithmetic

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Quaternions came from Hamilton after his really good work had been done; and, though beautifully ingenious, have been an unmixed evil to those who have touched them in any way, including Maxwell. – Lord Kelvin, 1892.

We beg to differ.

Hamilton's quaternions $\mathbb{H}$

$$\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k, \qquad i^2 = j^2 = -1,\ ij = k = -ji$$

For $x = a + bi + cj + dk$, we let

$$\mathrm{Norm}(x) = a^2 + b^2 + c^2 + d^2, \quad \mathrm{Tr}(x) = 2a.$$

This is a division algebra, $x^{-1} = (\mathrm{Tr}(x) - x)/\mathrm{Norm}(x)$. In fact, the normed division algebras over $\mathbb{R}$ are precisely

	dim	properties
$\mathbb{R}$	1	assoc., comm., ordered
$\mathbb{C}$	2	assoc., comm.
$\mathbb{H}$	4	assoc.
$\mathbb{O}$	8	

Classical motivation:

- Physics

  Generalization of the then new powerful complex numbers. Couples of real numbers to be replaced by triples (can't), quadruples (can). Today, subsumed by Clifford algebras.

- Topology

  $\{\text{Quaternions of norm } 1\} \cong S^3$, so S^3 is a topological group. The other div. alg. give top. groups S^0, S^1, S^7(H$-$space). No other spheres are top. groups $\Leftrightarrow$

  no other normed division algebras over $\mathbb{R}$.

- Euclidean geometry and engineering

  $\{$Trace zero, norm 1 quaternions$\} \cong S^2$. The quaternions of norm 1 act by $x * v = x^{-1}vx$. This gives a double cover $S^3 =$ Spin(3) $\to SO_3$. This is an efficient way to describe rotations. Used in spacecraft attitude control, etc.

- Arithmetic

  Lagrange: Every natural number is a sum of 4 squares.

  $$\text{Norm}(x) \cdot \text{Norm}(y) = \text{Norm}(xy) \quad \text{(Euler)}$$

  Apply to $x, y \in \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}k$ to reduce the proof to the case of prime numbers.

Bhargava-Conway-Schneeberger: a quadratic form represents all natural numbers if and only if it represents $1, 2, \ldots, 15$.

How often is a number a sum of squares?

A modular form of level $\Gamma_1(N)$ and weight k is a holomorphic function

$$f : \mathfrak{H} \to \mathbb{C}, \qquad f(\gamma\tau) = (c\tau + d)^k f(\tau),$$
$$\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}), \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{N}$$

Since $f(\tau+1) = f\left(\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}\tau\right) = f(\tau)$, the modular form f has q-expansion

$$f(\tau) = \sum_{n \in \mathbb{Z}} a_n q^n, \qquad q = \exp(2\pi i\tau).$$

In fact, such Fourier expansions can be carried at other "cusps" and we require that in all of them $a_n = 0$ for $n < 0$. If also $a_0 = 0$ we call f a cusp form.

Eisenstein series

$$E_{2k}(\tau) = c \cdot \sum_{(n,m)\in\mathbb{Z}^2-\{(0,0)\}} \frac{1}{(m\tau+n)^{2k}}$$

$$= \zeta(1-2k) + \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n,$$

$\sigma_r(n) = \sum_{d|n} d^r$. This is a modular form on $\mathrm{SL}_2(\mathbb{Z})$ of weight $2k$.

Theta series of a quadratic form

$$q(x_1, \ldots, x_r) = \frac{1}{2} x^t A x,$$

where A is integral symmetric positive definite with even entries on the diagonal. The level $N(A)$ of A is defined as the minimal integer N such that NA^{-1} is integral.

Theorem. The theta series

$$\sum_{n=0}^{\infty} a_q(n) \cdot q^n, \qquad a_q(n) = \sharp\{(x_1, \ldots, x_r) \in \mathbb{Z}^n : q(x_1, \ldots, x_r) = n\}$$

is a modular form of weight $r/2$ and level $N(A)$.

In particular, if

$$q(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + x_3^2 + x_4^2 = \frac{1}{2} x^t \begin{pmatrix} 2 & & & \\ & 2 & & \\ & & 2 & \\ & & & 2 \end{pmatrix} x$$

we get a modular form of level 2. It is obviously not a cusp form.

Two options

• Particular quadratic form: identify the modular form (for fixed level and weight this is a finite dimensional vector space). Find explicit answer. One gets $a(n) = \begin{cases} 4\sum_{d|n} d & n \text{ odd} \\ 24\sum_{d|n, d \text{ odd}} d & n \text{ even.} \end{cases}$

• General quadratic form: estimate coefficients.

1) Coeff. of " basic" Eisenstein series of weight k grow like n^{k-1}. Show little cancelation in the Eisenstein part.

2) Deligne (Ramanujan's conjecture): The coefficients of cusp forms of weight k grow like $\sigma_0(n) \cdot n^{(k-1)/2}$.

Using this we see that $a_q(n) = O(n) \to \infty$ for 4 squares.

$\boxed{\text{Deuring's quaternions } B_{p,\infty}}$

K = field, char$(K) \neq 2$.
The quaternion algebra $\left(\frac{a,b}{K}\right)$ is the central simple algebra

$$K \oplus Ki \oplus Kj \oplus Kk, \quad i^2 = a,\ j^2 = b,\ ij = -ji = k.$$

Example, $K = \mathbb{R}$. Then $\mathbb{H} \cong \left(\frac{-1,-1}{K}\right)$ and $M_2(\mathbb{R}) \cong \left(\frac{1,1}{K}\right)$. No others!

Example, $K = \mathbb{Q}_p$. Then there are again only two quaternion algebras, one of which is $M_2(\mathbb{Q}_p)$ and the other is a division algebra.

Theorem. Let B be a quaternion algebra over $\mathbb{Q}$. B is uniquely determined by $\{B \otimes_{\mathbb{Q}} \mathbb{Q}_p : p \leq \infty\}$. For a (finite) even number of $p \leq \infty$ we have $B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ ramified, i.e. $B \otimes_{\mathbb{Q}} \mathbb{Q}_p \not\cong M_2(\mathbb{Q}_p)$.

An order in a quaternion algebra over $\mathbb{Q}$ is a subring, of rank 4 over $\mathbb{Z}$. Every order is contained in a maximal order.

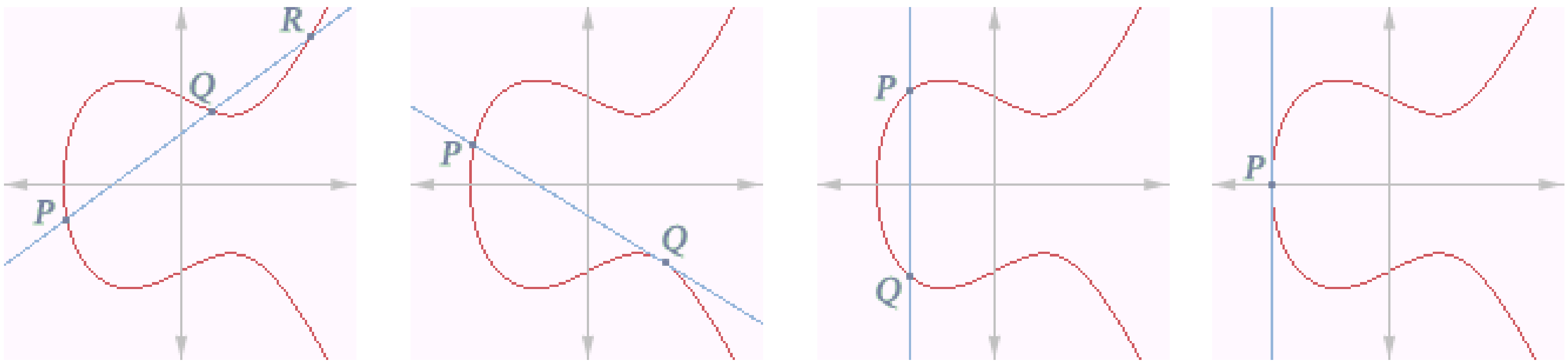
Example: in the rational Hamilton quaternions $\left(\frac{-1,-1}{\mathbb{Q}}\right)$ the order $\mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}k$ is not maximal. A maximal order is obtained by adding $\frac{1+i+j+k}{2}$.

Elliptic curves and Deuring's quaternions

Elliptic curve: homogeneous non-singular cubic $f(x, y, z) = 0$ in $\mathbb{P}^2$, with a chosen point.

An elliptic curve is a commutative algebraic group (addition given by the secant method).

End(E) is a ring with no zero divisors and for any elliptic curve E', Hom(E, E') is a right module.



$P + Q + R = 0$ $P + Q + Q = 0$ $P + Q + 0 = 0$ $P + P + 0 = 0$

Classification:

- if $char(K) = 0$ then $\mathrm{End}(E) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} \\ \mathbb{Q}(\sqrt{-d}) \end{cases}$

- if $char(K) = p$ then $\mathrm{End}(E) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} \\ \mathbb{Q}(\sqrt{-d}) \\ B_{p,\infty} \end{cases}$

An elliptic curve with $\mathrm{End}(E) \otimes \mathbb{Q} \cong B_{p,\infty}$ is called supersingular. It is known that $\mathrm{End}(E)$ is a maximal order in $B_{p,\infty}$. There are finitely many such elliptic curves up to isomorphism. Fix one, say E.

Deuring: there is a canonical bijection between supersingular elliptic curves and right projective rank 1 modules for $\mathrm{End}(E)$. One sends E' to $\mathrm{Hom}(E, E')$.

In this manner, quaternion algebras provide new information on elliptic curves.

Singular moduli

Let E_s (resp. E'_t) be the finitely many elliptic curves over $\mathbb{C}$ such that $\mathrm{End}(E_s)$ (resp. $\mathrm{End}(E'_t)$) has endomorphism ring which is the maximal order R_d (resp. $R_{d'}$) of $\mathbb{Q}(\sqrt{-d})$ (resp. $\mathbb{Q}(\sqrt{-d'})$).

Each elliptic curve is isomorphic to $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$, where $\tau \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H}$ is uniquely determined. There is a modular form of weight 0, namely a modular function

$$j : \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{H} \xrightarrow{\cong} \mathbb{C}, \qquad j(q) = \frac{1}{q} + 744 + 196884q + \dots$$

Gross-Zagier. There is an explicit formula for the integer

$$\prod_{s,t}(j(E_s) - j(E'_t)).$$

The numbers $j(E_i)$, called singular moduli, are of central importance in number theory, because they classify elliptic curves and allow generation of abelian extensions of $\mathbb{Q}(\sqrt{-d})$. (Hilbert's 12$^{\text{th}}$ problem).

Relation to quaternion algebras: If p divides $\prod_{s,t}(j(E_s) - j(E'_t))$ then it means that some E_s and E'_t become isomorphic modulo (a prime above) p. This implies that their reduction is a supersingular elliptic curve. The problem becomes algebraic: into which maximal orders of $B_{p,\infty}$ can one embed simultaneously R_d and $R_{d'}$.

Supersingular graphs (Lubotzky-Philips-Sarnak, Pizer, Mestre, Osterlé, Serre, . . .)

Pick a prime $\ell \neq p$ and construct the (directed) supersingular graph $\mathscr{G}^p(\ell)$.

• Vertices: supersingular elliptic curves.

• Edges: E is connected to E' if there is an isogeny $f : E \to E'$ of degree ℓ. (But we really only care about the kernel of f).

This graph has degree $\ell + 1$ and is essentially symmetric.

Ramanujan graphs

Expanders. Let $\mathscr{G}$ be a k-regular connected graph with n vertices and with adjacency matrix A and combinatorial Laplacian

$$\Delta = kI_n - A,$$

whose eigenvalues are $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \leq 2k$.
$\frac{1}{k}\Delta(f)(v)$ is $f(v)$ minus the average of f on the neighbors of v.

The expansion coefficient is

$$h(\mathscr{G}) = \min\left\{\frac{|\partial S|}{|S|} : |S| \leq n/2\right\} \leq 1 \quad \text{or} \quad \frac{n+1}{n-1}.$$

One is interested in getting a large $h(\mathscr{G})$.

Tanner, Alon-Milman: $\frac{2\lambda_1}{k+2\lambda_1} \leq h(\mathscr{G}) \leq \sqrt{2k\lambda_1}$.

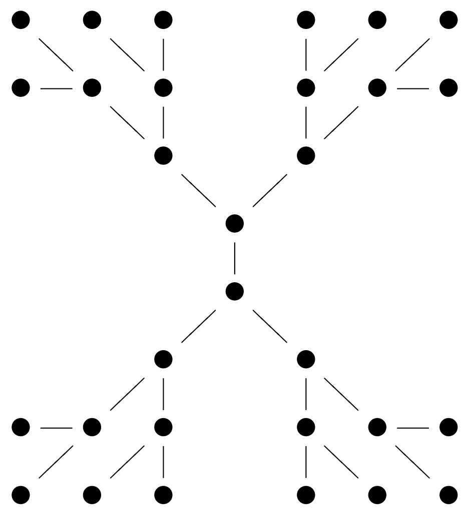
To have a graph in which information spreads rapidly/ random walk converges quickly, *one looks for a graph with a large* λ_1. Those have many technological and mathematical applications.

Alon-Boppana: $\liminf \mu_1(G) \geq 2\sqrt{k-1}$, where $k - \mu_1 = \lambda_1$ is the second largest eigenvalue of A, and where the limit is over all k-regular graphs of size growing to infinity.

Thus, asymptotically, the best family of expanding graphs of a fixed degree d will satisfy the Alon-Boppana bound.

A graph G is called a Ramanujan graph if $\mu_1(G) \leq 2\sqrt{k-1}$.

Trees



(3-regular tree)

A k-regular infinite tree $\mathscr{T}$ is the ideal expander. One can show that $h(\mathscr{T}) = k - 1$. The idea now is to find subgroups Γ of the automorphism group of a tree that does not identify vertices that are "very close" to each other. Arithmetic enters first in finding such subgroups Γ.

- Two distinct primes $p \neq \ell$.

- An $\ell + 1$ regular tree $\mathscr{T}$ could be viewed as the Bruhat-Tits tree for the group $\mathrm{GL}_2(\mathbb{Q}_\ell)$ and in particular, we have

$$\mathrm{PGL}_2(\mathbb{Q}_\ell) \subseteq \mathrm{Aut}(\mathscr{T}).$$

- $\mathcal{O}$ = maximal order of $B_{p,\infty}$. Then the group of units of norm 1 of $\mathcal{O}[\ell^{-1}]^\times$ maps into $B_{p,\infty} \otimes \mathbb{Q}_\ell = M_2(\mathbb{Q}_\ell)$ and gives a subgroup Γ of $\mathrm{Aut}(\mathscr{T})$ of the kind we want. In fact,

$$\Gamma \backslash \mathscr{T} \cong \mathscr{G}^p(\ell).$$

The Ramanujan property.

$\Gamma\backslash\mathscr{T}$ = moduli space of supersingular elliptic curves		$\Gamma_0(p)\backslash\mathfrak{H}$ = moduli space for elliptic curves + additional data
quaternionic modular forms = sections of line bundles = functions		modular forms = sections of line bundles
Hecke operators $T_\ell \sim$ averaging operators $\sim$ Adjacency matrices $\mathscr{G}^p(\ell)$		Hecke operators $T_\ell \sim$ averaging operators
system of eignevalues of T_ℓ acting on functions with integral zero	$\overset{\text{J.-L.}}{=}$	system of eignevalues for T_ℓ acting on cusp forms; given by the coeff. a_ℓ in q-exp.

The bound on the eigenvalues of the adjacency matrix of $\mathscr{G}^p(\ell)$ is thus given by the Ramanujan bound on the ℓ-th Fourier coefficient of elliptic modular forms.

Generalization: Quaternion algebras over totally real fields

• J. Cogdell - P. Sarnak - I. I. Piatetski-Shapiro. Bounds on Eisenstein series and cusp forms, mostly of half-integral weight.

• M.-H. Nicole. (McGill thesis, 2005) Generalizes Deuring theory for certain quaternion algebras over totally real fields.

• Bruinier - Yang. (2004) , G.-Lauter (2004, 2005). Certain generalizations of Gross-Zagier to totally real fields.

• B. Jordan - R. Livne (2000) , D. Charles - G. - K. Lauter (2005). Construction of Ramanujan graphs from quaternion algebras over totally real fields and superspecial graphs.

A. Cayley compared the quaternions to a pocket map "... which contained everything but had to be unfolded into another form before it could be understood."