HILBERT MODULAR FORMS: MOD P AND P-ADIC ASPECTS

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This paper is concerned with developing the theory of Hilbert modular forms along the lines of the theory of elliptic modular forms. Our main interests in this paper are:

1. to determine the ideal of congruences between Hilbert modular forms in characteristic p and to find conditions on the existence of congruences over artinian local rings. This allows us to derive explicit congruences between special values of zeta functions of totally real fields, to establish the existence of filtration for Hilbert modular forms, to establish the existence of p-adic weights for p-adic modular forms (defined as p-adic uniform limit of classical modular forms), and more;
2. to construct operators U, V, Θ_ψ (one for each suitable weight ψ) on modular forms in characteristic p and to study the variation of the filtration under these operators. This allows us to prove that every ordinary eigenform has filtration in a prescribed box of weights.

Our approach to modular forms is emphatically geometric. Our goal is to develop systematically the geometric and arithmetic aspects of Hilbert modular varieties. As to be expected in such a project, we use extensively the ideas of N. Katz [10], [11], [12], [13] and J.-P. Serre [18], of the founders of the theory in the case of elliptic modular forms, and we have benefited much from B. Gross' paper [9]. In regard to previous work on the subject, we mention that some of the methods in this paper were introduced by the second named author, in the unramified case [6], [7], and the congruences we list for values of Dedekind zeta functions may be derived from the work of Deligne-Ribet [3]. For this reason we restrict our discussion to zeta functions, though the same reasoning applies to a wide class of L-functions. For completeness we summarize in the second part of this introduction our results concerning the geometry of Hilbert modular varieties.

We now describe in more detail the main results of this paper.

Let L be a totally real field of degree g over $\mathbb{Q}$ and ring of integers $\mathcal{O}_L$. Let K be a normal closure of L. Let $N \geq 4$ be an integer prime to p. Let $\mathfrak{M}(S, \mu_N)$ be the fine moduli scheme parameterizing polarized abelian schemes over S-schemes with RM by $\mathcal{O}_L$ and μ_N-level structure (§??). A Hilbert modular form defined over an $\mathcal{O}_K$-scheme S has a weight ψ belonging to $\mathbb{X}_S$, where $\mathbb{X}_S$ is the group of characters of the algebraic group $\mathcal{G}_S = \mathbf{Res}_{\mathcal{O}_L/\mathbb{Z}}\mathbb{G}_m \times_{\mathrm{Spec}(\mathbb{Z})} S$ (§??). We shall mostly be concerned with weights obtained from the characters $\mathbb{X}$ of $\mathcal{G}_{\mathcal{O}_K} = \mathbf{Res}_{\mathcal{O}_L/\mathbb{Z}}\mathbb{G}_m \times_{\mathrm{Spec}(\mathbb{Z})} \mathcal{O}_K$; we shall use the notation $\mathbb{X}_S^U$ to the denote the group of characters of $\mathcal{G}_S$ induced from $\mathbb{X}$ by base change.

The group $\mathbb{X}$ is a free abelian group of rank g and has a positive cone $\mathbb{X}^+$ generated by the characters coming from the embeddings $\sigma_1, \dots, \sigma_g : L \longrightarrow K$. Indeed, the map $\mathcal{O}_L \otimes K \cong \oplus_{i=1}^g K$ induces a splitting of the torus $\mathcal{G}_K$, and hence canonical generators of $\mathbb{X}$ that we denote accordingly by $\chi_1, \dots, \chi_g$, and call the *fundamental characters*. A complex Hilbert modular form of weight $\chi_1^{a_1} \dots \chi_g^{a_g}$ is of weight $(a_1, \dots, a_g)$ in classical terminology.

It is important to note that $\mathbb{X}_S^U$ depends very much on S. For example, assume that L is Galois and $S = \mathrm{Spec}(\mathcal{O}_L/p)$, p an inert prime in L, then $\mathbb{X} \cong \mathbb{X}_S^U$, while if p is totally ramified in L, say $p = \mathfrak{p}^g$, then $\mathbb{X}_S^U \cong \mathbb{Z}$; in this case letting Ψ denote the reduction of any fundamental character χ_i we obtain that an integral Hilbert modular form of weight $\chi_1^{a_1} \dots \chi_g^{a_g}$ reduces to a modular form of weight $\Psi^{a_1+\cdots+a_g}$.

We denote the Hilbert modular forms defined over a scheme S, of level μ_N and weight $\psi \in \mathbb{X}_S$ by $\mathbf{M}(S, \mu_N, \psi)$.

Let p be a rational prime, let k be a finite field of characteristic p, which is an $\mathcal{O}_K$-algebra. Let $\mathbb{X}_k(1)$ be the subgroup of characters of $\mathbb{X}$ that are trivial on $(\mathcal{O}_L/(p))^\times$ under the map $(\mathcal{O}_L/(p))^\times \longrightarrow \mathcal{G}(k) = (\mathcal{O}_L \otimes k)^\times \longrightarrow k^\times$. The map $\mathbb{X} \longrightarrow \mathbb{X}_k$ is surjective. This allows us to define a positive cone $\mathbb{X}_k^+$ in $\mathbb{X}_k$ as follows. For every i there exists $1 \leq \tau(i) \leq g$ such that the image of $\chi_i^p \chi_{\tau(i)}^{-1}$ in X_k^U is in $\mathbb{X}_k(1)$. The character $\chi_i^p \chi_{\tau(i)}^{-1}$ in X_k^U does not depend on the choice of $\tau(i)$. The positive cone in $\mathbb{X}_k$ is the one induced by these generators. The positive cone induces an order $\leq_k$ on $\mathbb{X}_k$; we say that $\tau_1 \leq_k \tau_2$ if $\tau_1^{-1}\tau_2$ belongs to the positive cone. Note that we have provided $\mathbb{X}_k(1)^+ := \mathbb{X}_k(1) \cap \mathbb{X}_k^+$ with a canonical set of generators.

For every character $\psi \in \mathbb{X}_k(1)^+$ we construct a holomorphic modular form h_ψ over k. It has the property that all its q-expansions are 1. Moreover, the ideal $\mathcal{I}$ of congruences

$$\mathcal{I} := \mathrm{Ker}\{ \bigoplus_{\chi \in \mathbb{X}_k} \mathbf{M}(k, \mu_N, \chi) \stackrel{q-\exp}{\longrightarrow} k[[q^\nu]]_{\nu \in \mathfrak{d}} \}$$

($\mathfrak{d}$ is a suitable $\mathcal{O}_L$-module depending on the cusp used to get the q-expansion) is given by

$$(h_\psi - 1 : \psi \in \mathbb{X}_k(1)^+).$$

It is a finitely generated ideal and a canonical set of generators is obtained by letting ψ range over the generators for $\mathbb{X}_k(1)^+$ specified above. (see Theorem ??; Cf. [7]).

Again, it may beneficial to provide two examples. Assume that L is Galois. If p is inert in L than we may order $\sigma_1, \dots, \sigma_g$ cyclically with respect to Frobenius: $\sigma \circ \sigma_i = \sigma_{i+1}$. Let $k = \mathcal{O}_L/(p)$ then $\mathbb{X}_k(1)$ and $\mathbb{X}_k(1)^+$ are generated by $\chi_1^p\chi_2^{-1}, \dots, \chi_i^p\chi_{i+1}^{-1}, \dots, \chi_g^p\chi_1^{-1}$. Note that this positive cone is different from the one obtained from $\mathbb{X}^+$ via the reduction map. The kernel of the q-expansion is generated by g relations $h_1 - 1, \dots, h_g - 1$, where $h_i = h_{\chi_{i-1}^p \chi_i^{-1}}$ is a modular form of weight $\chi_{i-1}^p\chi_i^{-1}$. On the other hand, when $p = \mathfrak{p}^g$ is totally ramified, $k = \mathcal{O}_L/\mathfrak{p}$, we find that $\mathbb{X}_k(1)$ is generated by the characters $\chi_1^{p-1}, \dots, \chi_g^{p-1}$ that are all the same character in $\mathbb{X}_k^U$ and the q-expansion kernel is generated by a single relation $h_{\Psi^{p-1}} - 1$ where h is a modular form of weight Ψ^{p-1}.

We offer two constructions of the modular forms h_ψ (§§??). One is related to a compactification of $\mathfrak{M}(\mathbb{F}, \mu_{Np})$ and the other allows us to prove that the divisor of h_ψ, for ψ one of the canonical generators of $\mathbb{X}_k(1)^+$, is a reduced divisor.

The proof of the theorem on the ideal of congruences is based on the isomorphism between the ring $\oplus_{\chi \in \mathbb{X}_k} \mathbf{M}(k, \chi, \mu_N)/\mathcal{I}$ of *modular forms modulo q-expansions* and the ring of regular *functions* on the quasi-affine scheme $\mathfrak{M}(k, \mu_{Np})$. The latter scheme can be compactified by adding suitable roots of certain of the sections h_ψ. This isomorphism creates a *dictionary* between modular forms of level μ_N and functions on the compactification of $\mathfrak{M}(k, \mu_{Np})$ with poles supported the non-ordinary locus. Under this dictionary the weight of a modular form, a character in $\mathbb{X}$, is mapped to an element of $\mathbb{X}_k/\mathbb{X}_k(1)$, the $k^\times$-valued characters of the Galois group $(\mathcal{O}_L/(p))^\times$ of the cover $\mathfrak{M}(k, \mu_{Np}) \longrightarrow \mathfrak{M}(k, \mu_N)^{ord}$. The exact behavior of the poles is related to a minimal weight (with respect to the order $\leq_k$ on $\mathbb{X}_k$), called *filtration*, from which a q-expansion may arise, and here the explicit description of the compactification is invaluable in studying the properties of the filtration.

This dictionary also allows us to define operators that clearly depend only on q-expansions, like U, V and suitable Θ_ψ operators, first as operators on functions on $\mathfrak{M}(k, \mu_{Np})$, and then as operators on modular forms (§§??). This enables us to read some of the finer properties of these operators from the corresponding properties on $\mathfrak{M}(k, \mu_{Np})$. Our main results are the following:

1. There exists a notion of filtration for Hilbert modular forms: A q-expansion arising from a modular form f arises from a modular form of minimal weight $\Phi(f)$ with respect to $\leq_k$. This weight satisfies $0 \leq_k \Phi(f)$. See §§??.
2. There exists a linear operator $V : \mathbf{M}(k, \mu_N, \chi) \longrightarrow \mathbf{M}(k, \mu_N, \chi^{(p)})$ whose effect on q-expansions is $\sum a(\nu)q^\nu \mapsto \sum a(\nu)q^{p\nu}$. The character $\chi^{(p)}$ is the character induced from χ by composing with Frobenius. (Concretely, in the inert case $\chi_i^{(p)} = \chi^p_{\sigma^{-1}(i)}$, and in the totally ramified case $\Psi^{(p)} = \Psi^p$.) We have $\Phi(Vf) = \Phi(f)^{(p)}$. Moreover, this operator "comes" from the Frobenius morphism of $\mathfrak{M}(\mathbb{F}_p, \mu_{Np})$. See §§??.
3. For every $\psi \in \mathbb{X}^+$ there exists a linear operator Θ_ψ taking holomorphic modular forms of weight χ to holomorphic modular forms of weight $\chi\psi^{(p)}\psi$. Its effect on q-expansions at a suitable cusp is given by $\sum a(\nu)q^\nu \mapsto \sum \psi(\nu)a(\nu)q^\nu$.

   The behavior of filtration under such operator involves too much notation to include here and we refer the reader to §??. In the inert case, $\Phi(f) = \chi_1^{a_1} \cdots \chi_g^{a_g}$, we have $\Phi(\Theta_{\chi_i} f) \leq_k \Phi(f)\chi_{i-1}^p\chi_i$, and $\Phi(\Theta_{\chi_i} f) \leq_k \Phi(f)\chi_i^2$ if $p|a_i$ (note: $\chi_i^2 = (\chi_{i-1}^p\chi_i)/(\chi_{i-1}^p\chi_i^{-1})$). If p is completely ramified the result resembles the elliptic case: if $\Phi(f) = \Psi^a$ then $\Phi(\Theta_\Psi f) \leq_k \Psi^{a+p+1}$ and $\Phi(\Theta_\Psi f) \leq_k \Psi^{a+2}$ if $p|a$. See §§??.
4. There exists a linear operator U taking holomorphic modular forms of weight χ to meromorphic modular forms of weight χ. For every $\psi \in \mathbb{X}_p(1) \cap \mathbb{X}^+$ we have identity on q-expansions

   $$VUf(q) = (I - \Theta_\psi)f(q),$$

   provided p is unramified (see §?? for the general formula) and the effect on q-expansion is $\sum a(\nu)q^\nu \mapsto \sum_{(p,\nu)=1} a(\nu)q^\nu$. If χ is "positive enough" (see §?? for a precise statement) then U takes holomorphic modular forms of weight χ to holomorphic modular forms of weight χ and agree with the complex U operator on modular forms obtained by reduction from characteristic zero. Every ordinary modular form has filtration in an explicit box of weights (§??), equal to $[0, p+1]^g$ for p inert and to $[2, p+1]$ if p is totally ramified. See §§??.

♢

We now summarize our results concerning the geometry of Hilbert modular varieties. These results will appear in a subsequent paper.[1]

Many aspects of the geometry of the modular varieties $\mathfrak{M}(\mathbb{F}, \mu_N)$ are obtained via local deformation theory that "factorizes" according to the decomposition of p in $\mathcal{O}_L$. The unramified case was considered in [8] (see also [6]). Given that, for all practical matters, one may restrict one's attention to the case $p = \mathfrak{p}^e$ in $\mathcal{O}_L$. We discuss here only the case $e = g$, *i.e.*, p is totally ramified in L.

The ramified case was treated first by Deligne and Pappas in [2] (the case $g = 2$ was considered in [1]). We recall some of their results under the assumption that p is totally ramified. Let A/k be a polarized abelian variety with RM, defined over a field k of characteristic p. Fix an isomorphism $\mathcal{O}_L \otimes_{\mathbb{Z}} k \cong k[T]/(T^g)$. One knows that $H^1_{dR}(A)$ is a free $k[T]/(T^g)$-module of rank 2. The elementary divisors theorem furnishes us with $k[T]/(T^g)$-generators α, β for $H^1_{dR}(A)$, such that

$$H^1(A, \mathcal{O}_A) = (T^i)\alpha + (T^j)\beta, \quad i \geq j,\ i + j = g.$$

The index $j = j(A)$ gives a stratification $\mathcal{S}_j$ of the moduli space $\mathfrak{M}(\mathbb{F}_p, \mu_N)$; the j-th stratum parameterizes abelian varieties A with $j(A) \geq j$. We call this stratification the *singularity stratification*, and we call $j(A)$ the *singularity index* of A.

Using comparison of local moduli with a suitable Grassmannian variety, Deligne and Pappas gave local equations for a point in the stratum $\mathcal{S}_j$ inside the stratum $\mathcal{S}_{j'}$ for any $j' \leq j$. Their results imply that the $\mathcal{S}_j$ stratum, if non-empty, has dimension $g - 2j$. They did not prove that each of the strata $\mathcal{S}_j$ are non-empty.

To have better understanding of the moduli space $\mathfrak{M}(\mathbb{F}_p, \mu_N)$ one would like to refine the singularity stratification. After recent works by Oort [16] and others [5], [8], [15], [20], one idea that comes to mind is to stratify the moduli space according to the isomorphism type of the p-torsion as a polarized group scheme with $\mathcal{O}_L$-action. It turns out that this is not desirable:

1. Let k be an algebraically closed field of characteristic p. Already for $g = 2$ there infinitely many non-isomorphic polarized group schemes with $\mathcal{O}_L$-action arising as p-torsion of polarized abelian surfaces with RM by $\mathcal{O}_L$ (where L is a quadratic field in which p ramifies). This yields infinitely many different "strata".
2. Again, already for $g = 2$, there is a one dimensional non-isotrivial flat family of polarized commutative superspecial group schemes of order p^4 with RM, while the superspecial locus is zero dimensional. This shows that contrary to the unramified case, the deformation theory of p-divisible groups with $\mathcal{O}_L$-action does not surject onto the deformation theory of the corresponding truncated at one Barsotti-Tate groups.

Still, one would like to refine the singularity stratification and study its relation to the Newton stratification and to arithmetic. To this end we introduce another invariant. Given a polarized abelian variety A with RM, defined over a field k of characteristic p, we define its *slope* $n = n(A)$

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by the equation

$$j(A) + n(A) = a(A),$$

where $a(A)$ is the a-number of the abelian variety (for us, $a(A)$ is equal to the rank of the kernel of the Hasse-Witt matrix of A). One proves that

$$j \leq n \leq i.$$

We prove that there exists a locally closed subset of $\mathfrak{M}(\mathbb{F}_p, \mu_N)$, denoted by $W_{(j,n)}$, that parameterizes abelian varieties with singularity index j and slope n. We prove that $W_{(j,n)}$ is a non-empty set and is a quasi-affine non-singular variety of dimension

$$\dim(W_{(j,n)}) = g - (j+n).$$

The proof that $W_{(j,n)}$ is non-empty uses a construction of Moret-Bailly families and studies the variation of (j,n) along these families. The quasi-affine property follows from a "Raynaud trick" argument, while the non-singularity and dimension follow from study of local deformation theory via displays.

The reason for calling n the slope is the following. Let β_r denote the Newton polygon with the two slopes r/g and $(g-r)/g$, each of multiplicity g, and if g is odd let $\beta_{(g+1)/2}$ denote the Newton polygon with unique slope $1/2$ of multiplicity $2g$. In our case, where p is totally ramified, the polygons $\beta_0, \beta_1, \ldots, \beta_{[(g+1)/2]}$ are precisely the Newton polygons that appear on $\mathfrak{M}(\mathbb{F}_p, \mu_N)$. Let us also define

$$\lambda(n) = \min\left\{n, \left[\frac{g+1}{2}\right]\right\}.$$

Then the Newton polygon on $W_{(j,n)}$ is *constant*, equal to $\beta_{\lambda(n)}$. We write

$$\mathcal{N}(W_{(j,n)}) = \beta_{\lambda(n)}.$$

The proof of this result is based on classification of Dieudonné modules over $\mathcal{O}_L \otimes W(k)[F,V]$.

One surprising consequence of these results is that for $n < g/2$ not only do j and $j+n$ go up under specialization but *both* j and n go up under specialization. Thus, for $n < g/2$ we have

$$\overline{W_{(j,n)}} \subseteq \bigcup_{j \leq j',\, n \leq n'} W_{(j',n')}.$$

On the other hand, using local deformation theory, we show that

$$W_{(j,n)} \subset \overline{W_{(j-1,n)}}, \quad W_{(j,n)} \subset \overline{W_{(j,n-1)}}$$

(if $n-1 \geq j$). This gives

$$\overline{W_{(j,n)}} = \bigcup_{j \leq j',\, n \leq n'} W_{(j',n')}, \quad n < g/2.$$

For $n \geq g/2$ our results are not yet complete; nonetheless, we show that

$$\bigcup_{j \leq j',\, n \leq n'} W_{(j',n')} \subseteq \overline{W_{(j,n)}} \subseteq \bigcup_{j \leq j',\, j+n \leq j'+n'} W_{(j',n')}.$$

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