ASSIGNMENT 1 - NUMBER THEORY, WINTER 2009

Submit by Monday, January 19, 16:00 (use the designated mailbox in Burnside Hall, 10^{th} floor).

Solve the following questions:

(1) Let f_n be the n-th Fibonacci number. Thus, $f_0 = 0, f_1 = 1$ and $f_{n+2} = f_{n+1} + f_n$. The first Fibonacci numbers are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

Prove that f_{n+1} is the sum of the numbers on the n-th diagonal of Pascal's triangle, written in automaton form. For example: $f_6 = 8 = 1 + 4 + 3, f_8 = 21 = 1 + 6 + 10 + 4$.

(2) Let $\alpha = \frac{1+\sqrt{5}}{2}$ be the golden ratio and $\bar{\alpha} = 1 - \alpha = \frac{1-\sqrt{5}}{2}$. Prove by induction that

$$f_n = \frac{\alpha^n - \bar{\alpha}^n}{\sqrt{5}}.$$

Consider the rectangles formed by fitting together more and more squares whose sides are Fibonacci numbers, as explained in class. Prove that the ratio of edges of these rectangles approaches α.

(3) Show that $1/\sqrt{2}$ and $\sqrt{2} + \sqrt{3}$ are algebraic numbers.

(4) Provide upper and lower bound on $1^1 \cdot 2^2 \cdot 3^3 \cdots n^n$ by comparing to a suitable integral.

(5) Prove the congruence for $n \geq 3$

$$\binom{n}{3} = \begin{cases} 1 \pmod 2 & n \equiv 3 \pmod 4 \\ 0 & \text{otherwise.} \end{cases}$$

(This congruence explains the periodicity of the 3-rd diagonal of the Pascal triangle modulo 2 (the 3-rd column in the automaton form). Recall that our number of diagonal/columns begins with 0.)

Formulate a congruence for the 5-th diagonal of the Pascal triangle modulo 2 (the 5-th column in the automaton form) explaining its periodicity and prove it. The 5-th diagonal is

$$1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, \ldots$$

The honors students need also to submit the following problems.

A. Prove that the cardinality of the set of real algebraic numbers is $\aleph_0$. You may find it useful to prove the following facts, but if you have a proof that doesn't require these statements then you do need not to prove them. (i) If $A_1, A_2, \ldots$ are sets of cardinality $\aleph_0$ then so is $\cup_{i=1}^{\infty} A_i$. (ii) If A is a set of cardinality $\aleph_0$ then so is $A^n = A \times A \times \cdots \times A$ (n-times) for any integer $n \geq 1$.

B. Explain the self-similarity of the Pascal triangle modulo 2.

Suggestion: Formulate 2 properties of binomial coefficients modulo 2 that will explain the self-similarity of the Pascal triangle modulo 2. The first property should just state that certain binomial coefficients are even and explain the inverted central triangles of zeros. The second property should be a congruence between binomial coefficients modulo 2. Prove these statements and therefore establish the self similarity of the Pascal triangle modulo 2.