Computing Hilbert modular forms

John Voight
University of Vermont

Second Montreal-Toronto Workshop in Number Theory
Fields Institute
9 April 2011

Main algorithm

Theorem (Dembélé, Dembélé-Donnelly, Greenberg-V, V)

There exists an algorithm which, given a

totally real field F,
a weight $k \in (\mathbb{Z}_{\geq 2})^{[F:\mathbb{Q}]}$, and
a nonzero ideal $\mathfrak{N} \subseteq \mathbb{Z}_F$,

computes the space $S_k(\mathfrak{N})$ of Hilbert cusp forms of weight k and level $\mathfrak{N}$ over F as a Hecke module.

In other words, there exists an explicit finite procedure which takes as input the field F, the weight k, and the ideal $\mathfrak{N}$ encoded in bits (in the usual way), and outputs: a finite set of sequences $(a_\mathfrak{p}(f))_\mathfrak{p}$ encoding the Hecke eigenvalues for each cusp form constituent f in $S_k(\mathfrak{N})$, where $a_\mathfrak{p}(f) \in E_f \subseteq \overline{\mathbb{Q}}$.

Example

Let $F = \mathbb{Q}(\sqrt{5})$, with $w = (1+\sqrt{5})/2$. Let $k = (2,2)$ and write simply $S_2(\mathfrak{N}) = S_{2,2}(\mathfrak{N})$.

For ideals $\mathfrak{N} \subset \mathbb{Z}_F = \mathbb{Z} \oplus \mathbb{Z}w$ with $N(\mathfrak{N}) \leq 30$ we have $\dim S_2(\mathfrak{N}) = 0$.

Let $\mathfrak{N} = (2w-7)$ with $N(\mathfrak{N}) = 31$. Then $\dim S_2(\mathfrak{N}) = 1$.

π	2	$w+2$	3	$w+3$	$w-4$	$\dots$	$2w+5$	$2w-7$
$N\mathfrak{p}$	4	5	9	11	11	$\dots$	31	31
$a_{\mathfrak{p}}$	-3	-2	2	4	-4	$\dots$	8	-1

Here, $\mathfrak{p} = (\pi)$.

The numbers $a_{\mathfrak{p}}$ satisfy $a_{\mathfrak{p}} = N\mathfrak{p} + 1 - \#A(\mathbb{F}_{\mathfrak{p}})$ where

$$A : y^2 + xy + wy = x^3 + (w+1)x^2 + wx$$

and $\mathbb{F}_{\mathfrak{p}}$ denotes the residue class field of $\mathfrak{p}$.

Geometry

In these lectures, for simplicity we restrict to forms of parallel weight $k = (2, \ldots, 2)$.

To compute with the space $S_2(N)$ of classical (elliptic) cusp forms of level N, one approach is to use the geometry of the modular curve $X_0(N) = \Gamma_0(N)\backslash\mathcal{H}^*$, where $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$ denotes the completed upper half-plane.

A cusp form $f \in S_2(N)$ corresponds to a holomorphic differential form $f(z)\,dz$ on $X_0(N)$ and so by the theorem of Eichler-Shimura arises naturally in the space $H^1(X_0(N), \mathbb{C})$.

In a similar way, a Hilbert cusp form $f \in S_2(\mathfrak{N})$ gives rise to a holomorphic differential n-form $f(z_1, \ldots, z_n)\,dz_1 \ldots dz_n$ on the *Hilbert modular variety* $X_0(\mathfrak{N})$. But now $X_0(\mathfrak{N})$ has dimension n and f arises in $H^n(X_0(\mathfrak{N}), \mathbb{C})$. Yikes!

Computing with higher dimensional varieties (and higher degree cohomology groups) is not an easy task.

General strategy

Langlands functoriality predicts that $S_2(\mathfrak{N})$ as a Hecke module occurs in the cohomology of other "modular" varieties. We use a principle called the *Jacquet-Langlands correspondence*, which allows us to work with varieties of complex dimension 0 or 1 by considering twisted forms of GL_2 over F.

Let B be a quaternion algebra over F with discriminant $\mathfrak{D}$ and let $\mathfrak{N} \subseteq \mathbb{Z}_F$ be coprime to $\mathfrak{D}$.

The Jacquet-Langlands correspondence implies the isomorphism of Hecke modules

$$S_2^B(\mathfrak{N}) \hookrightarrow S_2(\mathfrak{D}\mathfrak{N})$$

where $S_2^B(\mathfrak{N})$ denotes the space of quaternionic cusp forms for B (of weight 2) and level $\mathfrak{N}$. The image consists exactly of those forms which are new at all primes $\mathfrak{p} \mid \mathfrak{D}$.

Quaternionic modular forms: Notation

Quaternionic modular forms are, roughly speaking, analytic functions on the ideles of B with a certain left- and right-invariance.

Let $v_1, \ldots, v_n$ be the real places of F, and suppose that B is split at $v_1, \ldots, v_r$ and ramified at $v_{r+1}, \ldots, v_n$, i.e.

$$\iota_\infty : B \hookrightarrow B_\infty = B \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} \mathrm{M}_2(\mathbb{R})^r \times \mathbb{H}^{n-r}$$

where $\mathbb{H}$ denotes the division ring of real Hamiltonians. Let

$$K_\infty = (\mathbb{R}^\times \mathrm{SO}_2(\mathbb{R}))^r \times (\mathbb{H}^\times)^{n-r} \subseteq B_\infty$$

be the stabilizer of $(\sqrt{-1}, \ldots, \sqrt{-1}) \in \mathcal{H}^r$.

Let $\mathcal{O}_0(1) \subseteq B$ be a maximal order and let $\mathcal{O} = \mathcal{O}_0(\mathfrak{N}) \subset \mathcal{O}_0(1)$ be an Eichler order of level $\mathfrak{N}$.

Let $\widehat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z} = {\prod_p}' \mathbb{Z}_p$ and let $\widehat{\ }$ denote tensor with $\widehat{\mathbb{Z}}$.

Quaternionic modular forms: Definition

Modular forms on B are analytic functions on $B_\infty^\times \times \widehat{B}^\times$ which are invariant on the left by $B^\times$ and transform on the right by $K_\infty \times \widehat{\mathcal{O}}^\times$ on the right in a specified way.

A *(quaternionic) modular form* of parallel weight 2 and level $\mathfrak{N}$ for B is an analytic function

$$\phi : B_\infty^\times \times \widehat{B}^\times \to \mathbb{C}$$

such that for all $(g, \widehat{\alpha}) \in B_\infty^\times \times \widehat{B}^\times$, we have:

(i) $\phi(g, \widehat{\alpha}\widehat{u}) = \phi(g, \widehat{\alpha})$ for all $\widehat{u} \in \widehat{\mathcal{O}}^\times$;

(ii) $\phi(g\kappa, \widehat{\alpha}) = \left(\prod_{i=1}^{r} \frac{j(\kappa_i, \sqrt{-1})^2}{\det \kappa_i}\right) \phi(g, \widehat{\alpha})$ for all $\kappa \in K_\infty$; and

(iii) $\phi(\gamma g, \gamma\widehat{\alpha}) = \phi(g, \widehat{\alpha})$ for all $\gamma \in B^\times$.

Let $M_2^B(\mathfrak{N})$ denote the space of such forms.

Quaternionic modular forms: Upper and lower half-planes

Modular forms on B are analytic functions on $B_\infty^\times \times \widehat{B}^\times$ which are invariant on the left by $B^\times$ and transform by $K_\infty \times \widehat{\mathcal{O}}^\times$ in a specified way. Such a function uniquely defines a function on the quotient

$$B_\infty^\times / K_\infty \times \widehat{B}^\times / \widehat{\mathcal{O}}^\times .$$

We identify $B_\infty^\times / K_\infty \to (\mathcal{H}^\pm)^r = (\mathbb{C} \setminus \mathbb{R})^r$ by $g \mapsto z = g(\sqrt{-1}, \ldots, \sqrt{-1})$.

Thus, a modular form is equivalently a function

$$f : (\mathcal{H}^\pm)^r \times \widehat{B}^\times / \widehat{\mathcal{O}}^\times \to \mathbb{C}$$

which is holomorphic in the first variable and locally constant in the second one and such that

$$f(\gamma z, \gamma \widehat{\alpha} \widehat{\mathcal{O}}^\times) = \left(\prod_{i=1}^{r} \frac{j(\gamma_i, z_i)^2}{\det \kappa_i} \right) f(z, \widehat{\alpha} \widehat{\mathcal{O}}^\times)$$

for all $\gamma \in B^\times$ and $(z, \widehat{\alpha} \widehat{\mathcal{O}}^\times) \in (\mathcal{H}^\pm)^r \times \widehat{B}^\times / \widehat{\mathcal{O}}^\times$.

Quaternionic Shimura variety: Upper half-plane

Now we include the invariance on the right. Let

$$X_0^B(\mathfrak{N})(\mathbb{C}) = B^\times\backslash(B_\infty^\times/K_\infty \times \widehat{B}^\times/\widehat{\mathcal{O}}^\times) = B^\times\backslash((\mathcal{H}^\pm)^r \times \widehat{B}^\times/\widehat{\mathcal{O}}^\times).$$

By Eichler's theorem of norms, we have

$$\mathrm{nrd}(B^\times) = F_{(+)}^\times = \{a \in F^\times : v_i(a) > 0 \text{ for } i = r+1, \ldots, n\}.$$

In particular, $B^\times/B_+^\times \cong (\mathbb{Z}/2\mathbb{Z})^r$, where

$$B_+^\times = \{\gamma \in B : \mathrm{nrd}(\gamma) \in F_+^\times\}.$$

The group $B_+^\times$ acts on $\mathcal{H}^r$, therefore we may identify

$$X_0^B(\mathfrak{N})(\mathbb{C}) = B_+^\times\backslash(\mathcal{H}^r \times \widehat{B}^\times/\widehat{\mathcal{O}}^\times)$$

and a modular form on $(\mathcal{H}^\pm)^r \times \widehat{B}^\times/\widehat{\mathcal{O}}^\times$ can be uniquely recovered from its restriction to $\mathcal{H}^r \times \widehat{B}^\times/\widehat{\mathcal{O}}^\times$.

Quaternionic Shimura variety: Components

Now we have a natural (continuous) projection map

$$X_0^B(\mathfrak{N})(\mathbb{C}) = B_+^\times \backslash (\mathcal{H}^r \times \widehat{B}^\times / \widehat{\mathcal{O}}^\times) \to B_+^\times \backslash \widehat{B}^\times / \widehat{\mathcal{O}}^\times.$$

The reduced norm gives a surjective map

$$\mathrm{nrd} : B_+^\times \backslash \widehat{B}^\times / \widehat{\mathcal{O}}^\times \to F_+^\times \backslash \widehat{F}^\times / \widehat{\mathbb{Z}}_F^\times \cong \mathrm{Cl}^+ \mathbb{Z}_F$$

where $\mathrm{Cl}^+ \mathbb{Z}_F$ denotes the strict class group of $\mathbb{Z}_F$, i.e. the ray class group of $\mathbb{Z}_F$ with modulus equal to the product of all real (infinite) places of F. Strong approximation implies that this map is a bijection if B is indefinite (but in general it is not if B is indefinite). Accordingly, our description will depend on if B is definite or indefinite.

Quaternionic Shimura variety: Indefinite case

First, suppose that B is indefinite. Then the space $X_0^B(\mathfrak{N})(\mathbb{C})$ is the disjoint union of connected Riemannian manifolds of dimension r indexed by $\mathrm{Cl}^+\mathbb{Z}_F$.

Let the ideals $\mathfrak{a} \subseteq \mathbb{Z}_F$ form a set of representatives for $\mathrm{Cl}^+\mathbb{Z}_F$, and let $\widehat{a} \in \widehat{\mathbb{Z}}_F$ be such that $\widehat{a}\,\widehat{\mathbb{Z}}_F \cap \mathbb{Z}_F = \mathfrak{a}$. (For the trivial class $\mathfrak{a} = \mathbb{Z}_F$, we choose $\widehat{a} = \widehat{1}$). There exists $\widehat{\alpha} \in \widehat{B}^\times$ such that $\mathrm{nrd}(\widehat{\alpha}) = \widehat{a}$. We let $\mathcal{O}_\mathfrak{a} = \widehat{\alpha}\widehat{\mathcal{O}}\widehat{\alpha}^{-1} \cap B$ so that $\mathcal{O}_{(1)} = \mathcal{O}$, and we put $\Gamma_\mathfrak{a} = \mathcal{O}_{\mathfrak{a},+}^\times = \widehat{\mathcal{O}}_\mathfrak{a}^\times \cap B_+^\times$. Then we have

$$X_0^B(\mathfrak{N})(\mathbb{C}) = \bigsqcup_{[\mathfrak{a}] \in \mathrm{Cl}^+(\mathbb{Z}_F)} B_+^\times(\mathcal{H}^r \times \widehat{\alpha}\widehat{\mathcal{O}}^\times) \xrightarrow{\sim} \bigsqcup_{[\mathfrak{a}] \in \mathrm{Cl}^+(\mathbb{Z}_F)} \Gamma_\mathfrak{a} \backslash \mathcal{H}^r,$$

where the last identification is obtained via the bijection

$$\begin{aligned} B_+^\times \setminus (\mathcal{H}^r \times \widehat{\alpha}\widehat{\mathcal{O}}^\times) &\xrightarrow{\sim} \Gamma_\mathfrak{a} \backslash \mathcal{H}^r \\ B_+^\times(z, \widehat{\alpha}\widehat{\mathcal{O}}^\times) &\mapsto z. \end{aligned}$$

Shimura curves

Then the space $X(\mathbb{C}) = X_0^B(\mathfrak{N})(\mathbb{C})$ is the disjoint union of Riemannian manifolds of dimension r indexed by $\mathrm{Cl}^+ \mathbb{Z}_F$.

Let $r = 1$. Then the space

$$X(\mathbb{C}) = \coprod_{[\mathfrak{a}]\in \mathrm{Cl}^+(\mathbb{Z}_F)} \Gamma_{\mathfrak{a}} \backslash \mathcal{H} = \coprod_{[\mathfrak{a}]\in \mathrm{Cl}^+(\mathbb{Z}_F)} X_{\mathfrak{a}}(\mathbb{C})$$

is the disjoint union of Riemann surfaces indexed by $\mathrm{Cl}^+ \mathbb{Z}_F$, where $\mathcal{O}_{\mathfrak{a}} = \widehat{\alpha}\widehat{\mathcal{O}}\widehat{\alpha}^{-1} \cap B$ and $\Gamma_{\mathfrak{a}} = \mathcal{O}_{\mathfrak{a},+}^{\times}$.

Therefore, a modular form of parallel weight 2 and level $\mathfrak{N}$ is a tuple $(f_{\mathfrak{a}})$ of functions $f_{\mathfrak{a}} : \mathcal{H} \to \mathbb{C}$, indexed by $[\mathfrak{a}] \in \mathrm{Cl}^+ \mathbb{Z}_F$, such that for all $\mathfrak{a}$, we have

$$f_{\mathfrak{a}}(\gamma z) = (cz + d)^2 f_{\mathfrak{a}}(z)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{\mathfrak{a}}$ and all $z \in \mathcal{H}$.

Shimura curves: Example

Let $F = \mathbb{Q}(w)$ be the (totally real) cubic field of prime discriminant 257, with $w^3 - w^2 - 4w + 3 = 0$. Then F has Galois group S_3 and $\mathbb{Z}_F = \mathbb{Z}[w]$. The field F has class number 1 but strict class number 2: the unit $(w-1)(w-2)$ generates the group $\mathbb{Z}_{F,+}^\times / \mathbb{Z}_F^{\times 2}$ of totally positive units modulo squares.

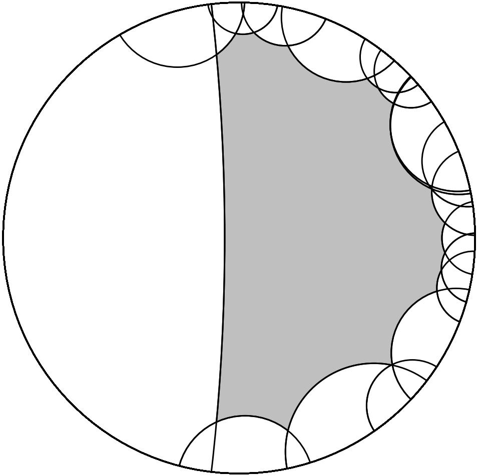
Let $B = \left(\dfrac{-1, w-1}{F}\right)$ be the quaternion algebra with $i^2 = -1$, $j^2 = w-1$, and $ji = -ij$. Then B has discriminant $\mathfrak{D} = (1)$ and is ramified at two of the three real places and unramified at the place with $w \mapsto 2.19869\ldots$, corresponding to $\iota_\infty : B \hookrightarrow \mathrm{M}_2(\mathbb{R})$. The order $\mathcal{O}$ with $\mathbb{Z}_F$-basis

$$1,\ (w^2+w-3)i,\ \frac{(w^2+w) - 8i + j}{2},\ \frac{(w^2+w-2)i + ij}{2}$$

is an Eichler order of level $\mathfrak{N} = (w)^2$ where $N(w) = 3$.

Shimura curves: Example

A fundamental domain for the action of Γ on $\mathcal{H}$ is as follows.



Shimura curves: Example

The ideals (1) and $\mathfrak{a} = (w+1)\mathbb{Z}_F$ represent the classes in the strict class group $\mathrm{Cl}^+ \mathbb{Z}_F$.

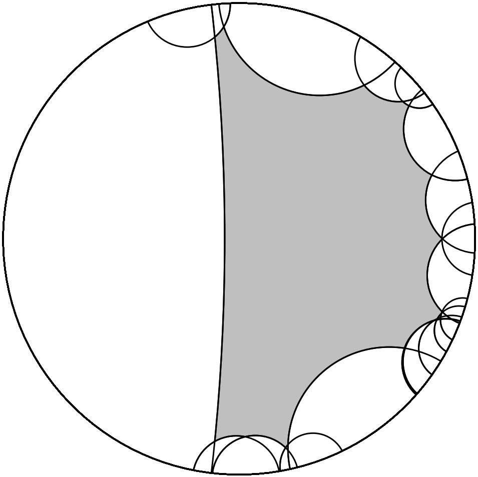
The ideal $J_{\mathfrak{a}} = 2\mathcal{O} + \left((w^2+w+2)/2 - 4i + (1/2)j\right)\mathcal{O}$ has $\mathrm{nrd}(J_{\mathfrak{a}}) = \mathfrak{a}$.

The left order of $J_{\mathfrak{a}}$ is $\mathcal{O}_L(J_{\mathfrak{a}}) = \mathcal{O}_{\mathfrak{a}}$ with basis

$$1,\ (w^2-2w-3)i,\ \frac{(w^2+w)-8i+j}{2},$$
$$\frac{(174w^2-343w-348)i+(w^2-2w-2)j+(-w^2+2w+2)ij}{10}.$$

Shimura curves: Example

A fundamental domain for the action of $\Gamma_{\mathfrak{a}}$ on $\mathcal{H}$ is as follows.



Shimura curves: Example

The orders $\mathcal{O}$ and $\mathcal{O}_{\mathfrak{a}}$ are not isomorphic. So the groups Γ and $\Gamma_{\mathfrak{a}}$ are not conjugate as subgroups of $\mathrm{PSL}_2(\mathbb{R})$ but nevertheless are isomorphic as abstract groups: they both have signature $(1; 2, 2, 2, 2)$, so that

$$\Gamma \cong \Gamma_{\mathfrak{a}} \cong \langle \gamma, \gamma', \delta_1, \ldots, \delta_4 : \delta_1^2 = \cdots = \delta_4^2 = [\gamma, \gamma']\delta_1 \cdots \delta_4 = 1 \rangle.$$

In particular, both $X_{(1)}(\mathbb{C})$ and $X_{\mathfrak{a}}(\mathbb{C})$ have genus 1, so

$$\dim H^1(X(\mathbb{C}), \mathbb{C}) = \dim H^1(X_{(1)}(\mathbb{C}), \mathbb{C}) + \dim H^1(X_{\mathfrak{a}}(\mathbb{C}), \mathbb{C}) = 4.$$

We choose a basis of characteristic functions on γ, γ' as a basis for $H^1(X_{(1)}(\mathbb{C}), \mathbb{C})$ and similarly for $H^1(X_{\mathfrak{a}}(\mathbb{C}), \mathbb{C})$.

The theorem of Eichler-Shimura applied to each component yields an isomorphism

$$S_2(\mathfrak{N}) \oplus \overline{S_2(\mathfrak{N})} \xrightarrow{\sim} H^1(X(\mathbb{C}), \mathbb{C})$$

so $\dim S_2(\mathfrak{N}) = 2$.

Shimura curves: Example

We now compute Hecke operators (a black box). Let $H = H^1(X(\mathbb{C}), \mathbb{C})$. Complex conjugation acts on H by

$$H \mid W_\infty = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Now consider the Hecke operator $T_\mathfrak{p}$ where $\mathfrak{p} = (2w-1)$ and $N(\mathfrak{p}) = 7$. Then $\mathfrak{p}$ represents the nontrivial class in $\mathrm{Cl}^+ \mathbb{Z}_F$, and

$$H \mid T_\mathfrak{p} = \begin{pmatrix} 0 & 0 & -3 & 2 \\ 0 & 0 & -2 & -4 \\ -4 & -2 & 0 & 0 \\ 2 & -3 & 0 & 0 \end{pmatrix} \quad \text{and} \quad H^+ \mid T_\mathfrak{p} = \begin{pmatrix} 0 & -2 \\ -8 & 0 \end{pmatrix}.$$

Therefore there are two eigenspaces for $T_\mathfrak{p}$ with eigenvalues $4, -4$. By contrast, the Hecke operator $T_{(2)}$ acts as the scalar 3 on H.

Shimura curves: Example

Continuing in this way, we find the following table of eigenvalues:

$N\mathfrak{p}$	3	7	8	9	19	25	37	41	43	47	49	53	61	61	61
$a_\mathfrak{p}(f)$	-1	4	3	-4	-4	-8	4	-6	-8	0	4	12	-8	2	4
$a_\mathfrak{p}(g)$	-1	-4	3	4	-4	8	-4	-6	8	0	-4	-12	8	2	-4

The form g is the quadratic twist of the form f by the nontrivial character of the strict class group $\mathrm{Gal}(F^+/F)$, where F^+ is the strict class field of F. Note also that these forms do not arise from base change from $\mathbb{Q}$, since $a_\mathfrak{p}$ has three different values for the primes $\mathfrak{p}$ of norm 61.

Shimura curves: Example

We are then led to search for elliptic curves of conductor $\mathfrak{N} = (w)^2$, and we find two:

$$E_f : y^2 + (w^2+1)xy = x^3 - x^2 + (-36w^2 + 51w - 18)x + (-158w^2 + 557w - 317)$$

$$E_g : y^2 + (w^2+w+1)xy + y = x^3 + (w^2 - w - 1)x^2 + (4w^2 + 11w - 11)x + (4w^2 + w - 3)$$

Each of these curves have nontrivial $\mathbb{Z}/2\mathbb{Z}$-torsion over F, and so can be proven to be modular. We match Hecke eigenvalues to find that E_f corresponds to f and E_g corresponds to g.

By Deligne's theory of canonical models, we know that $X(\mathbb{C}) = X_{(1)}(\mathbb{C}) \sqcup X_{\mathfrak{a}}(\mathbb{C})$ has a model X_F over F, but the curves themselves are not defined over F: they are interchanged by the action of $\mathrm{Gal}(F^+/F)$. Nevertheless, the Jacobian of X_F is an abelian variety of dimension 2 defined over F which is isogenous to $E_f \times E_g$.

Definite quaternionic Shimura varieties

Now suppose that B is definite. Then the Shimura variety is simply

$$X_0^B(\mathfrak{N})(\mathbb{C}) = B^\times \backslash \widehat{B}^\times / \widehat{\mathcal{O}}^\times = \mathrm{Cl}\,\mathcal{O}$$

and so is canonically identified with the set of right ideal classes of $\mathcal{O}$. The reduced norm map here is the map $\mathrm{nrd} : \mathrm{Cl}\,\mathcal{O} \to \mathrm{Cl}^+ \mathbb{Z}_F$ which is surjective but not a bijection, in general.

A modular form f on B of parallel weight 2 is thus completely determined by its values on a set of representatives of $\mathrm{Cl}\,\mathcal{O}$. Therefore, there is an isomorphism of complex vector spaces given by

$$\begin{aligned} M_2^B(\mathfrak{N}) &\to \bigoplus_{\substack{[I] \in \mathrm{Cl}(\mathcal{O}) \\ I = \widehat{\alpha}\widehat{\mathcal{O}} \cap B}} \mathbb{C} \\ f &\mapsto \quad (f(\widehat{\alpha})). \end{aligned}$$

Definite quaternionic Shimura varieties: Example

Consider the totally real quartic field $F = \mathbb{Q}(w)$ where $w^4 - 5w^2 - 2w + 1 = 0$. Then F has discriminant $5744 = 2^4 359$ and Galois group S_4. We have $\mathrm{Cl}^+ \mathbb{Z}_F = 2$ (but $\mathrm{Cl}\,\mathbb{Z}_F = 1$).

The quaternion algebra $B = \left(\dfrac{-1,-1}{F}\right)$ is unramified at all finite places (and ramified at all real places). We compute a maximal order $\mathcal{O}$ and find that $\#\,\mathrm{Cl}\,\mathcal{O} = 4$.

We now compute the action of the Hecke operators: we identify the isomorphism classes of the $N\mathfrak{p} + 1$ right ideals of norm $\mathfrak{p}$ inside each right ideal I in a set of representatives for $\mathrm{Cl}\,\mathcal{O}$. We compute

$$T_{(w^3-4w-1)} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 4 & 4 \\ 2 & 2 & 0 & 0 \\ 3 & 3 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T_{(w^2-w-4)} = \begin{pmatrix} 6 & 2 & 0 & 0 \\ 8 & 12 & 0 & 0 \\ 0 & 0 & 8 & 4 \\ 0 & 0 & 6 & 10 \end{pmatrix}$$

representing the nontrivial and trivial classes, respectively.

Definite quaternionic Shimura varieties: Example

In this case, the space $E_2(1)$ of functions that factor through the reduced norm has dimension $\dim E_2(1) = 2$, so $\dim S_2(1) = 2$, and we find that this space is irreducible as a Hecke module and so has a unique constituent f.

We obtain the following table of Hecke eigenvalues:

π	$w^3 - 4w - 1$	$w - 1$	$w^2 - w - 2$	$w^2 - 3$
$N\mathfrak{p}$	4	5	7	13
$a_\mathfrak{p}(f)$	0	t	$-2t$	$-t$

Here t satisfies the polynomial $t^2 - 6 = 0$. We therefore predict the existence of an abelian variety over F with real multiplication by $\mathbb{Q}(\sqrt{6})$ and everywhere good reduction.