Hilbert modular forms and cohomology

Shervin Shahrokhi Tehrani

University of Toronto

April 9th, 2011

Notations

- F is totally real quadratic field extension.
- $\mathcal{O}_{\mathcal{F}}$ is the ring of integers of F.
- $\mathfrak{a}$ is a fractional ideal of $\mathcal{O}_{\mathcal{F}}$.
- $Cl(F)$ is the ideal class group of F.
- $Cl(F)^+$ is the narrow ideal class group of F.
- $\mathbb{H}$ is upper half plane.
- $\mathbb{P}^1(F) = F \bigcup \{\infty\}$.
- $e(\omega) = e^{2\pi i \omega}$.
- If $M \subseteq F$ is a $\mathbb{Z}$-module of rank 2, then

$$M^\vee = \{\lambda \in F; tr(\mu\lambda) \in \mathbb{Z}, \forall \mu \in M\}$$

- The $\mathbb{A} = \mathbb{A}_\infty \mathbb{A}_f$ is adelic ring over F where $\mathbb{A}_f$ is finite part of $\mathbb{A}$.

Notations

- F is totally real quadratic field extension.
- $\mathcal{O}_{\mathcal{F}}$ is the ring of integers of F.
- $\mathfrak{a}$ is a fractional ideal of $\mathcal{O}_{\mathcal{F}}$.
- $Cl(F)$ is the ideal class group of F.
- $Cl(F)^+$ is the narrow ideal class group of F.
- $\mathbb{H}$ is upper half plane.
- $\mathbb{P}^1(F) = F \bigcup \{\infty\}$.
- $e(\omega) = e^{2\pi i \omega}$.
- If $M \subseteq F$ is a $\mathbb{Z}$-module of rank 2, then

$$M^\vee = \{\lambda \in F; tr(\mu\lambda) \in \mathbb{Z}, \forall \mu \in M\}$$

- The $\mathbb{A} = \mathbb{A}_\infty \mathbb{A}_f$ is adelic ring over F where $\mathbb{A}_f$ is finite part of $\mathbb{A}$.

Notations

- F is totally real quadratic field extension.
- $\mathcal{O}_{\mathcal{F}}$ is the ring of integers of F.
- $\mathfrak{a}$ is a fractional ideal of $\mathcal{O}_{\mathcal{F}}$.
- $Cl(F)$ is the ideal class group of F.
- $Cl(F)^{+}$ is the narrow ideal class group of F.
- $\mathbb{H}$ is upper half plane.
- $\mathbb{P}^1(F) = F \bigcup \{\infty\}$.
- $e(\omega) = e^{2\pi i \omega}$.
- If $M \subseteq F$ is a $\mathbb{Z}$-module of rank 2, then

$$M^{\vee} = \{\lambda \in F; tr(\mu\lambda) \in \mathbb{Z}, \forall \mu \in M\}$$

- The $\mathbb{A} = \mathbb{A}_{\infty}\mathbb{A}_f$ is adelic ring over F where $\mathbb{A}_f$ is finite part of $\mathbb{A}$.

Hilbert Modular Group

Let F be a real quadratic field.

$$SL_2(F) \hookrightarrow SL_2(\mathbb{R}) \times SL_2(\mathbb{R}).$$

$SL_2(F)$ acts on $\mathbb{H} \times \mathbb{H}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z_1, z_2) = \left(\frac{az_1 + b}{cz_1 + d}, \frac{a' z_2 + b'}{c' z_2 + d'} \right)$$

Definition

$$\Gamma_{(\mathcal{O}_\mathcal{F} \oplus \mathfrak{a})} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(F); a, d \in \mathcal{O}_\mathcal{F}, b \in \mathfrak{a}^{-1}, c \in \mathfrak{a} \right\}$$

Hilbert Modular Group

Let F be a real quadratic field.

$$SL_2(F) \hookrightarrow SL_2(\mathbb{R}) \times SL_2(\mathbb{R}).$$

$SL_2(F)$ acts on $\mathbb{H} \times \mathbb{H}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z_1, z_2) = \left(\frac{az_1 + b}{cz_1 + d}, \frac{a' z_2 + b'}{c' z_2 + d'} \right)$$

Definition

$$\Gamma_{(\mathcal{O}_\mathcal{F} \oplus \mathfrak{a})} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(F); a, d \in \mathcal{O}_\mathcal{F}, b \in \mathfrak{a}^{-1}, c \in \mathfrak{a} \right\}$$

Hilbert Modular Group

Let F be a real quadratic field.

$$SL_2(F) \hookrightarrow SL_2(\mathbb{R}) \times SL_2(\mathbb{R}).$$

$SL_2(F)$ acts on $\mathbb{H} \times \mathbb{H}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z_1, z_2) = \left(\frac{az_1 + b}{cz_1 + d}, \frac{a' z_2 + b'}{c' z_2 + d'} \right)$$

Definition

$$\Gamma_{(\mathcal{O}_{\mathcal{F}} \oplus \mathfrak{a})} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(F); a, d \in \mathcal{O}_{\mathcal{F}}, b \in \mathfrak{a}^{-1}, c \in \mathfrak{a} \right\}$$

Hilbert Modular Group

Definition

The Hilbert full modular group is

$$\Gamma_F = \Gamma(\mathcal{O}_\mathcal{F} \oplus \mathcal{O}_\mathcal{F}) = SL_2(\mathcal{O}_\mathcal{F})$$

Definition

Any subgroup of $SL_2(F)$ which is commensurable with Γ_F is called an arithmetic subgroup.

Let Γ be an arithmetic subgroup. It acts properly discontinuous on $\mathbb{H}^2$, i.e., if $W \subseteq \mathbb{H}^2$ is compact, then $\{\gamma \in \Gamma; \gamma W \cap W \neq \varnothing\}$ is finite.

Hilbert Modular Group

Definition

The Hilbert full modular group is

$$\Gamma_F = \Gamma(\mathcal{O}_{\mathcal{F}} \oplus \mathcal{O}_{\mathcal{F}}) = SL_2(\mathcal{O}_{\mathcal{F}})$$

Definition

Any subgroup of $SL_2(F)$ which is commensurable with Γ_F is called an arithmetic subgroup.

Let Γ be an arithmetic subgroup. It acts properly discontinuous on $\mathbb{H}^2$, i.e., if $W \subseteq \mathbb{H}^2$ is compact, then $\{\gamma \in \Gamma; \gamma W \cap W \neq \varnothing\}$ is finite.

Hilbert Modular Group

Definition

The Hilbert full modular group is

$$\Gamma_F = \Gamma(\mathcal{O}_\mathcal{F} \oplus \mathcal{O}_\mathcal{F}) = SL_2(\mathcal{O}_\mathcal{F})$$

Definition

Any subgroup of $SL_2(F)$ which is commensurable with Γ_F is called an arithmetic subgroup.

Let Γ be an arithmetic subgroup. It acts properly discontinuous on $\mathbb{H}^2$, i.e., if $W \subseteq \mathbb{H}^2$ is compact, then $\{\gamma \in \Gamma; \gamma W \cap W \neq \varnothing\}$ is finite.

Hilbert Modular Group

Definition

The Hilbert full modular group is

$$\Gamma_F = \Gamma(\mathcal{O}_{\mathcal{F}} \oplus \mathcal{O}_{\mathcal{F}}) = SL_2(\mathcal{O}_{\mathcal{F}})$$

Definition

Any subgroup of $SL_2(F)$ which is commensurable with Γ_F is called an arithmetic subgroup.

Let Γ be an arithmetic subgroup. It acts properly discontinuous on $\mathbb{H}^2$, i.e., if $W \subseteq \mathbb{H}^2$ is compact, then $\{\gamma \in \Gamma; \gamma W \cap W \neq \varnothing\}$ is finite.

Modular Surfaces

Definition

The space

$$X'_\Gamma = \Gamma \backslash \mathbb{H}^2$$

is the modular surface.

Elliptic fixed points

The stabilizer of $a \in \mathbb{H}^2$

$$\Gamma_a = \{\gamma \in \Gamma; \gamma a = a\}$$

is finite subgroup of Γ.

Definition

a is called elliptic fixed point if

$$\overline{\Gamma_a} = \Gamma_a/\{\pm 1\}$$

is not trivial.

Proposition

There are finite number of elliptic fixed points, and these are only singularities of X'_Γ.

Elliptic fixed points

The stabilizer of $a \in \mathbb{H}^2$

$$\Gamma_a = \{\gamma \in \Gamma; \gamma a = a\}$$

is finite subgroup of Γ.

Definition

a is called elliptic fixed point if

$$\overline{\Gamma_a} = \Gamma_a / \{\pm 1\}$$

is not trivial.

Proposition

There are finite number of elliptic fixed points, and these are only singularities of X'_Γ.

Elliptic fixed points

The stabilizer of $a \in \mathbb{H}^2$

$$\Gamma_a = \{\gamma \in \Gamma; \gamma a = a\}$$

is finite subgroup of Γ.

Definition

a is called elliptic fixed point if

$$\overline{\Gamma_a} = \Gamma_a / \{\pm 1\}$$

is not trivial.

Proposition

There are finite number of elliptic fixed points, and these are only singularities of X'_Γ.

Cusp points

The X'_Γ is not compact in general, therefore, there are points at infinity.

$SL_2(F)$ acts on $\mathbb{P}^1(F)$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\alpha, \beta) = \frac{a\alpha + b\beta}{c\alpha + d\beta}.$$

Definition

Γ-classes of $\mathbb{P}^1(F)$ are called cusp points of X'_Γ.

Cusp points

The X'_Γ is not compact in general, therefore, there are points at infinity.
$SL_2(F)$ acts on $\mathbb{P}^1(F)$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\alpha,\beta) = \frac{a\alpha + b\beta}{c\alpha + d\beta}.$$

Definition

Γ-classes of $\mathbb{P}^1(F)$ are called cusp points of X'_Γ.

Cusp points

The X'_Γ is not compact in general, therefore, there are points at infinity.
$SL_2(F)$ acts on $\mathbb{P}^1(F)$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\alpha, \beta) = \frac{a\alpha + b\beta}{c\alpha + d\beta}.$$

Definition

Γ-classes of $\mathbb{P}^1(F)$ are called cusp points of X'_Γ.

Cusp points

The X'_Γ is not compact in general, therefore, there are points at infinity.
$SL_2(F)$ acts on $\mathbb{P}^1(F)$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\alpha, \beta) = \frac{a\alpha + b\beta}{c\alpha + d\beta}.$$

Definition

Γ-classes of $\mathbb{P}^1(F)$ are called cusp points of X'_Γ.

Cusp points

Theorem

The map

$$\varphi : \Gamma_F \backslash \mathbb{P}^1(F) \longrightarrow Cl(F)$$

$$(\alpha : \beta) \longrightarrow \alpha\mathcal{O}_{\mathcal{F}} + \beta\mathcal{O}_{\mathcal{F}}$$

is bijective.

Corollary

The number of cusp points of X'_{Γ_F} is the class number of F.

Cusp points

Theorem

The map

$$\varphi : \Gamma_F \backslash \mathbb{P}^1(F) \longrightarrow Cl(F)$$

$$(\alpha : \beta) \longrightarrow \alpha\mathcal{O}_{\mathcal{F}} + \beta\mathcal{O}_{\mathcal{F}}$$

is bijective.

Corollary

The number of cusp points of X'_{Γ_F} is the class number of F.

Adelic version

Let $G = R_{F/\mathbb{Q}}GL_2(F)$ be reductive algebraic group over $\mathbb{Q}$.
Therefore,

$$G(\mathbb{R}) = GL_2(\mathbb{R})^2$$

$$K_\infty = SO(2).\mathbb{R}_{>0} \times SO(2).\mathbb{R}_{>0}$$

The quotient $G(\mathbb{R})/K_\infty$ is homeomorphic with $\mathbb{H}^\pm \times \mathbb{H}^\pm$.

Adelic version

Let $G = R_{F/\mathbb{Q}}GL_2(F)$ be reductive algebraic group over $\mathbb{Q}$.
Therefore,

$$G(\mathbb{R}) = GL_2(\mathbb{R})^2$$

$$K_\infty = SO(2).\mathbb{R}_{>0} \times SO(2).\mathbb{R}_{>0}$$

The quotient $G(\mathbb{R})/K_\infty$ is homeomorphic with $\mathbb{H}^\pm \times \mathbb{H}^\pm$.

Adelic version

Let $G = R_{F/\mathbb{Q}}GL_2(F)$ be reductive algebraic group over $\mathbb{Q}$. Therefore,

$$G(\mathbb{R}) = GL_2(\mathbb{R})^2$$

$$K_\infty = SO(2).\mathbb{R}_{>0} \times SO(2).\mathbb{R}_{>0}$$

The quotient $G(\mathbb{R})/K_\infty$ is homeomorphic with $\mathbb{H}^\pm \times \mathbb{H}^\pm$.

Adelic version

Let $G = R_{F/\mathbb{Q}} GL_2(F)$ be reductive algebraic group over $\mathbb{Q}$. Therefore,

$$G(\mathbb{R}) = GL_2(\mathbb{R})^2$$

$$K_\infty = SO(2).\mathbb{R}_{>0} \times SO(2).\mathbb{R}_{>0}$$

The quotient $G(\mathbb{R})/K_\infty$ is homeomorphic with $\mathbb{H}^\pm \times \mathbb{H}^\pm$.

Adelic version

Let $G = R_{F/\mathbb{Q}} GL_2(F)$ be reductive algebraic group over $\mathbb{Q}$.
Therefore,

$$G(\mathbb{R}) = GL_2(\mathbb{R})^2$$

$$K_\infty = SO(2).\mathbb{R}_{>0} \times SO(2).\mathbb{R}_{>0}$$

The quotient $G(\mathbb{R})/K_\infty$ is homeomorphic with $\mathbb{H}^\pm \times \mathbb{H}^\pm$.

Adelic version

Let K_f be compact open subgroup of $G_f = G(\mathbb{A}_f)$. Using Strong Approximation Theorem, we have

Theorem

There is an identification

$$G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_f = \bigcup_{j=1}^{m} \Gamma_j\backslash\mathbb{H}^2$$

with $\Gamma_j = g_j G(\mathbb{R})^0 K_f g_j^{-1} \bigcap G(\mathbb{Q})$.

Adelic version

Let K_f be compact open subgroup of $G_f = G(\mathbb{A}_f)$. Using Strong Approximation Theorem, we have

Theorem

There is an identification

$$G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_f = \bigcup_{j=1}^{m} \Gamma_j\backslash\mathbb{H}^2$$

with $\Gamma_j = g_j G(\mathbb{R})^0 K_f g_j^{-1} \bigcap G(\mathbb{Q})$.

Adelic version

Let K_f be compact open subgroup of $G_f = G(\mathbb{A}_f)$. Using Strong Approximation Theorem, we have

Theorem

There is an identification

$$G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_f = \bigcup_{j=1}^{m} \Gamma_j\backslash\mathbb{H}^2$$

with $\Gamma_j = g_j G(\mathbb{R})^0 K_f g_j^{-1} \bigcap G(\mathbb{Q})$.

Adelic version

If $K_0 = \prod_{\nu \in S_f} GL_2(\mathcal{O}_{\mathcal{F}_\nu})$, then

Corollary

$G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_0$ can be identified with $\bigcup_{\mathfrak{a}} \Gamma(\mathcal{O}_{\mathcal{F}} \oplus \mathfrak{a})\backslash \mathbb{H}^2$, where $\mathfrak{a}$ runs over a complete set of representatives of $Cl(F)^+$.

Adelic version

If $K_0 = \prod_{\nu \in S_f} GL_2(\mathcal{O}_{\mathcal{F}_\nu})$, then

Corollary

$G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_0$ can be identified with $\bigcup_{\mathfrak{a}} \Gamma(\mathcal{O}_{\mathcal{F}} \oplus \mathfrak{a})\backslash \mathbb{H}^2$, where $\mathfrak{a}$ runs over a complete set of representatives of $Cl(F)^+$.

Further properties

- There is fundamental domain for action of Γ on $\mathbb{H}^2$ in terms of Siegel domains.
- The form $\omega = \omega_1 \wedge \omega_2$ where

$$\omega_1 = \frac{1}{2\pi}\frac{dx_1 \wedge dy_1}{y_1^2}, \omega_2 = \frac{1}{2\pi}\frac{dx_2 \wedge dy_2}{y_2^2}$$

is volume form on X_Γ'.

Further properties

- There is fundamental domain for action of Γ on $\mathbb{H}^2$ in terms of Siegel domains.
- The form $\omega = \omega_1 \wedge \omega_2$ where

$$\omega_1 = \frac{1}{2\pi}\frac{dx_1 \wedge dy_1}{y_1^2}, \omega_2 = \frac{1}{2\pi}\frac{dx_2 \wedge dy_2}{y_2^2}$$

is volume form on X_Γ'.

Compactification

We have

$$\mathbb{P}^1(F) \hookrightarrow \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$$

Let

$$(\mathbb{H}^2)^* = \mathbb{H}^2 \cup \mathbb{P}^1(F)$$

The group Γ acts on $(\mathbb{H}^2)^*$. let

$$X_\Gamma = \Gamma \backslash (\mathbb{H}^2)^*$$

then we have

Theorem (Baily-Borel)

On $(\mathbb{H}^2)^*$ there is unique topology such that the $\Gamma \backslash (\mathbb{H}^2)^*$ with quotient topology is a compact Hausdorff space. Moreover, there is a sheaf of functions $\mathcal{O}_{X_\Gamma}$ on X_Γ such that $(X_\Gamma, \mathcal{O}_{X_\Gamma})$ is complex normal space.

Compactification

We have

$$\mathbb{P}^1(F) \hookrightarrow \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$$

Let

$$(\mathbb{H}^2)^* = \mathbb{H}^2 \cup \mathbb{P}^1(F)$$

The group Γ acts on $(\mathbb{H}^2)^*$. let

$$X_\Gamma = \Gamma \backslash (\mathbb{H}^2)^*$$

then we have

Theorem (Baily-Borel)

On $(\mathbb{H}^2)^*$ there is unique topology such that the $\Gamma \backslash (\mathbb{H}^2)^*$ with quotient topology is a compact Hausdorff space. Moreover, there is a sheaf of functions $\mathcal{O}_{X_\Gamma}$ on X_Γ such that $(X_\Gamma, \mathcal{O}_{X_\Gamma})$ is complex normal space.

Compactification

We have

$$\mathbb{P}^1(F) \hookrightarrow \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$$

Let

$$(\mathbb{H}^2)^* = \mathbb{H}^2 \cup \mathbb{P}^1(F)$$

The group Γ acts on $(\mathbb{H}^2)^*$. let

$$X_\Gamma = \Gamma \backslash (\mathbb{H}^2)^*$$

then we have

Theorem (Baily-Borel)

On $(\mathbb{H}^2)^*$ there is unique topology such that the $\Gamma\backslash(\mathbb{H}^2)^*$ with quotient topology is a compact Hausdorff space. Moreover, there is a sheaf of functions $\mathcal{O}_{X_\Gamma}$ on X_Γ such that $(X_\Gamma, \mathcal{O}_{X_\Gamma})$ is complex normal space.

Compactification

We have

$$\mathbb{P}^1(F) \hookrightarrow \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$$

Let

$$(\mathbb{H}^2)^* = \mathbb{H}^2 \cup \mathbb{P}^1(F)$$

The group Γ acts on $(\mathbb{H}^2)^*$. let

$$X_\Gamma = \Gamma \backslash (\mathbb{H}^2)^*$$

then we have

Theorem (Baily-Borel)

On $(\mathbb{H}^2)^*$ there is unique topology such that the $\Gamma \backslash (\mathbb{H}^2)^*$ with quotient topology is a compact Hausdorff space. Moreover, there is a sheaf of functions $\mathcal{O}_{X_\Gamma}$ on X_Γ such that $(X_\Gamma, \mathcal{O}_{X_\Gamma})$ is complex normal space.

Compactification

We have

$$\mathbb{P}^1(F) \hookrightarrow \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$$

Let

$$(\mathbb{H}^2)^* = \mathbb{H}^2 \cup \mathbb{P}^1(F)$$

The group Γ acts on $(\mathbb{H}^2)^*$. let

$$X_\Gamma = \Gamma \backslash (\mathbb{H}^2)^*$$

then we have

Theorem (Baily-Borel)

On $(\mathbb{H}^2)^*$ there is unique topology such that the $\Gamma \backslash (\mathbb{H}^2)^*$ with quotient topology is a compact Hausdorff space. Moreover, there is a sheaf of functions $\mathcal{O}_{X_\Gamma}$ on X_Γ such that $(X_\Gamma, \mathcal{O}_{X_\Gamma})$ is complex normal space.

Compactification

- **Remark**: Using the line bundle of modular forms (in sufficiently large weights) on X_Γ, gives an embedding into projective space, therefore, X_Γ is projective algebraic variety and X'_Γ is quasi-projective.

Toroidal compactification and de singularization

- There is smooth compactification of X'_{Γ} using **Toroidal Theory**. Therefore, we can resolve the singularities at boundary of **Baily-Borel** compactification.
- Also, by using the theory of **Hironaka**, we are able to resolve the singularities caused by Elliptic fixed points.

We are going to use the adelic version and fix following spaces:

- $X'_{K_f} = G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}})/K_f K_\infty$
- X_{K_f} is its Baily-Borel compactification.
- Z_{K_f} be the minimal resolution of the cusps.
- Y_{K_f} be the minimal resolution of all singularities.

Toroidal compactification and de singularization

- There is smooth compactification of X'_Γ using **Toroidal Theory**. Therefore, we can resolve the singularities at boundary of **Baily-Borel** compactification.
- Also, by using the theory of **Hironaka**, we are able to resolve the singularities caused by Elliptic fixed points.

We are going to use the adelic version and fix following spaces:

- $X'_{K_f} = G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})/K_f K_\infty$
- X_{K_f} is its Baily-Borel compactification.
- Z_{K_f} be the minimal resolution of the cusps.
- Y_{K_f} be the minimal resolution of all singularities.

Toroidal compactification and de singularization

- There is smooth compactification of X'_Γ using **Toroidal Theory**. Therefore, we can resolve the singularities at boundary of **Baily-Borel** compactification.
- Also, by using the theory of **Hironaka**, we are able to resolve the singularities caused by Elliptic fixed points.

We are going to use the adelic version and fix following spaces:

- $X'_{K_f} = G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})/K_f K_\infty$
- X_{K_f} is its Baily-Borel compactification.
- Z_{K_f} be the minimal resolution of the cusps.
- Y_{K_f} be the minimal resolution of all singularities.

Toroidal compactification and de singularization

- There is smooth compactification of X'_Γ using **Toroidal Theory**. Therefore, we can resolve the singularities at boundary of **Baily-Borel** compactification.
- Also, by using the theory of **Hironaka**, we are able to resolve the singularities caused by Elliptic fixed points.

We are going to use the adelic version and fix following spaces:

- $X'_{K_f} = G(\mathbb{Q})\backslash G(\mathbb{A}_\mathbb{Q})/K_f K_\infty$
- X_{K_f} is its Baily-Borel compactification.
- Z_{K_f} be the minimal resolution of the cusps.
- Y_{K_f} be the minimal resolution of all singularities.

Hilbert modular forms

Definition

A holomorphic function $f : \mathbb{H}^2 \longrightarrow \mathbb{C}$ is called **Hilbert modular forms** of weight $k = (k_1, k_2) \in \mathbb{Z}^2$ on Γ if for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ one has

$$f(\gamma z) = (cz_1 + d)^{k_1}(c^{'}z_2 + d^{'})^{k_2} f(z).$$

- If $k = k_1 = k_2$ then k is called the weight of f.

Hilbert modular forms

Definition

A holomorphic function $f : \mathbb{H}^2 \longrightarrow \mathbb{C}$ is called **Hilbert modular forms** of weight $k = (k_1, k_2) \in \mathbb{Z}^2$ on Γ if for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ one has

$$f(\gamma z) = (cz_1 + d)^{k_1}(c^{'} z_2 + d^{'})^{k_2} f(z).$$

- If $k = k_1 = k_2$ then k is called the weight of f.

Fourier expansion

If f is Hilbert modular form then it has Fourier expansion at the cusp ∞ as:

There is $M \subset F$, $\mathbb{Z}$-module of rank 2, such that

$$f(z+\mu) = f(z) \qquad \forall \mu \in M, \qquad and$$

$$f = \sum_{\nu \in M^\vee} a_\nu e(tr(\nu z)) \qquad where$$

$$a_\nu = \frac{1}{vol(\mathbb{R}/M)} \int_{\mathbb{R}^2/M} f(z) e(-tr(\nu z)) dx_1 dx_2.$$

In contrast to the one dimensional case, we have

Fourier expansion

If f is Hilbert modular form then it has Fourier expansion at the cusp ∞ as:
There is $M \subset F$, $\mathbb{Z}$-module of rank 2, such that

$$f(z+\mu) = f(z) \qquad \forall \mu \in M, \qquad and$$

$$f = \sum_{\nu \in M^\vee} a_\nu e(tr(\nu z)) \qquad where$$

$$a_\nu = \frac{1}{vol(\mathbb{R}/M)} \int_{\mathbb{R}^2/M} f(z) e(-tr(\nu z)) dx_1 dx_2.$$

In contrast to the one dimensional case, we have

Fourier expansion

If f is Hilbert modular form then it has Fourier expansion at the cusp ∞ as:
There is $M \subset F$, $\mathbb{Z}$-module of rank 2, such that

$$f(z+\mu) = f(z) \qquad \forall \mu \in M, \qquad \textit{and}$$

$$f = \sum_{\nu \in M^\vee} a_\nu e(tr(\nu z)) \qquad \textit{where}$$

$$a_\nu = \frac{1}{vol(\mathbb{R}/M)} \int_{\mathbb{R}^2/M} f(z) e(-tr(\nu z)) dx_1 dx_2.$$

In contrast to the one dimensional case, we have

Fourier expansion

If f is Hilbert modular form then it has Fourier expansion at the cusp ∞ as:
There is $M \subset F$, $\mathbb{Z}$-module of rank 2, such that

$$f(z+\mu) = f(z) \qquad \forall \mu \in M, \qquad \textit{and}$$

$$f = \sum_{\nu \in M^{\vee}} a_{\nu} e(tr(\nu z)) \qquad \textit{where}$$

$$a_{\nu} = \frac{1}{vol(\mathbb{R}/M)} \int_{\mathbb{R}^2/M} f(z) e(-tr(\nu z)) dx_1 dx_2.$$

In contrast to the one dimensional case, we have

Fourier expansion

If f is Hilbert modular form then it has Fourier expansion at the cusp ∞ as:
There is $M \subset F$, $\mathbb{Z}$-module of rank 2, such that

$$f(z+\mu) = f(z) \qquad \forall \mu \in M, \qquad and$$

$$f = \sum_{\nu \in M^\vee} a_\nu e(tr(\nu z)) \qquad where$$

$$a_\nu = \frac{1}{vol(\mathbb{R}/M)} \int_{\mathbb{R}^2/M} f(z) e(-tr(\nu z)) dx_1 dx_2.$$

In contrast to the one dimensional case, we have

Hilbert modular forms

Theorem (Koecher principle)

Let $f : \mathbb{H}^2 \longrightarrow \mathbb{C}$ *Hilbert modular form, then*

$$a_\nu \neq 0 \quad \textit{implies} \quad \nu = 0 \quad \textit{or} \quad \nu \gg 0.$$

We denote the space of all Hilbert modular forms of weight k by M_k. This has an interpretation as global section of line bundles over modular surface, and by using sheaf cohomology, M_k is **finite dimensional**.

Definition

A Hilbert modular form is called **cusp form** if it vanishes at all cusps of Γ. S_k is the space of all cusp forms of weight k.

Hilbert modular forms

Theorem (Koecher principle)

Let $f : \mathbb{H}^2 \longrightarrow \mathbb{C}$ *Hilbert modular form, then*

$$a_\nu \neq 0 \qquad \textit{implies} \qquad \nu = 0 \qquad \textit{or} \qquad \nu \gg 0.$$

We denote the space of all Hilbert modular forms of weight k by M_k. This has an interpretation as global section of line bundles over modular surface, and by using sheaf cohomology, M_k is **finite dimensional**.

Definition

A Hilbert modular form is called **cusp form** if it vanishes at all cusps of Γ. S_k is the space of all cusp forms of weight k.

Hilbert modular forms

Theorem (Koecher principle)

Let $f : \mathbb{H}^2 \longrightarrow \mathbb{C}$ *Hilbert modular form, then*

$$a_\nu \neq 0 \qquad \textit{implies} \qquad \nu = 0 \qquad \textit{or} \qquad \nu \gg 0.$$

We denote the space of all Hilbert modular forms of weight k by M_k. This has an interpretation as global section of line bundles over modular surface, and by using sheaf cohomology, M_k is **finite dimensional**.

Definition

A Hilbert modular form is called **cusp form** if it vanishes at all cusps of Γ. S_k is the space of all cusp forms of weight k.

Non-vanishing conditions

Theorem

Let f be Hilbert modular form of weight $k = (k_1, k_2)$ for Γ. The f vanishes identically unless k_1, k_2 are both positive or $k_1 = k_2 = 0$. In latter case f is constant.

Corollary

If $\pi : Z_{K_f} \longrightarrow X_{K_f}$ is the natural map, then any holomorphic 1-form on Z_{K_f} vanishes identically, i.e

$$H^1(Z_{K_f}, \mathcal{O}_{K_f}) = 0$$

Non-vanishing conditions

Theorem

Let f be Hilbert modular form of weight $k = (k_1, k_2)$ for Γ. The f vanishes identically unless k_1, k_2 are both positive or $k_1 = k_2 = 0$. In latter case f is constant.

Corollary

If $\pi : Z_{K_f} \longrightarrow X_{K_f}$ is the natural map, then any holomorphic 1-form on Z_{K_f} vanishes identically, i.e

$$H^1(Z_{K_f}, \mathcal{O}_{K_f}) = 0$$

Non-vanishing conditions

Theorem

Let f be Hilbert modular form of weight $k = (k_1, k_2)$ for Γ. The f vanishes identically unless k_1, k_2 are both positive or $k_1 = k_2 = 0$. In latter case f is constant.

Corollary

If $\pi : Z_{K_f} \longrightarrow X_{K_f}$ is the natural map, then any holomorphic 1-form on Z_{K_f} vanishes identically, i.e

$$H^1(Z_{K_f}, \mathcal{O}_{K_f}) = 0$$

Non-vanishing conditions

Proof.

Let ω be 1-form, and η be pullback on regular points of X_{k_f}. We have

$$\eta = f_1(z)dz_1 + f_2(z)dz_2$$

where f_1, f_2 are Hilbert modular form of weight $(2,0)$, and $(0,2)$. Using last theorem, we can say η vanishes. Therefore, ω is zero. □

Remark: Using Hodge theory we can show $H^1(Z_{k_f}, \mathbb{C})$ vanishes. This means that the interesting part of cohomology of Hilbert modular surfaces is in degree 2.

Non-vanishing conditions

Proof.

Let ω be 1-form, and η be pullback on regular points of X_{k_f}. We have

$$\eta = f_1(z)dz_1 + f_2(z)dz_2$$

where f_1, f_2 are Hilbert modular form of weight $(2,0)$, and $(0,2)$. Using last theorem, we can say η vanishes. Therefore, ω is zero. □

Remark: Using Hodge theory we can show $H^1(Z_{k_f}, \mathbb{C})$ vanishes. This means that the interesting part of cohomology of Hilbert modular surfaces is in degree 2.

Non-vanishing conditions

Proof.

Let ω be 1-form, and η be pullback on regular points of X_{k_f}. We have

$$\eta = f_1(z)dz_1 + f_2(z)dz_2$$

where f_1, f_2 are Hilbert modular form of weight $(2,0)$, and $(0,2)$. Using last theorem, we can say η vanishes. Therefore, ω is zero. □

Remark: Using Hodge theory we can show $H^1(Z_{k_f}, \mathbb{C})$ vanishes. This means that the interesting part of cohomology of Hilbert modular surfaces is in degree 2.

Non-vanishing conditions

Proof.

Let ω be 1-form, and η be pullback on regular points of X_{k_f}. We have

$$\eta = f_1(z)dz_1 + f_2(z)dz_2$$

where f_1, f_2 are Hilbert modular form of weight $(2,0)$, and $(0,2)$. Using last theorem, we can say η vanishes. Therefore, ω is zero. □

Remark: Using Hodge theory we can show $H^1(Z_{k_f}, \mathbb{C})$ vanishes. This means that the interesting part of cohomology of Hilbert modular surfaces is in degree 2.

$H^2(-,\mathbb{Q})$

We are looking at $H^2(Z_\Gamma,\mathbb{Q})$. Using Poincare duality, we have a non-degenerate pairing

$$H_2(Z_\Gamma) \times H_2(Z_\Gamma) \longrightarrow \mathbb{Q}.$$

Let E_σ be the subspace of $H_2(Z_\Gamma,\mathbb{Q})$ generated by the classes of the curves S_σ in the resolving of cusp σ.
We have the decomposition

$$H_2(Z_\Gamma,\mathbb{Q}) = \left(\bigoplus_\sigma E_\sigma\right) \oplus Im\left\{j_* : H_2(X'_\Gamma,\mathbb{Q}) \longrightarrow H_2(Z_\Gamma,\mathbb{Q})\right\}.$$

where $j : X'_\Gamma \longrightarrow Z_\Gamma$.

$H^2(-,\mathbb{Q})$

We are looking at $H^2(Z_\Gamma,\mathbb{Q})$.Using Poincare duality, we have a non-degenerate pairing

$$H_2(Z_\Gamma)\times H_2(Z_\Gamma)\longrightarrow \mathbb{Q}.$$

Let E_σ be the subspace of $H_2(Z_\Gamma,\mathbb{Q})$ generated by the classes of the curves S_σ in the resolving of cusp σ.
We have the decomposition

$$H_2(Z_\Gamma,\mathbb{Q})=\left(\bigoplus_\sigma E_\sigma\right)\oplus Im\left\{j_*:H_2(X'_\Gamma,\mathbb{Q})\longrightarrow H_2(Z_\Gamma,\mathbb{Q})\right\}.$$

where $j:X'_\Gamma\longrightarrow Z_\Gamma$.

$H^2(-, \mathbb{Q})$

We are looking at $H^2(Z_\Gamma, \mathbb{Q})$.Using Poincare duality, we have a non-degenerate pairing

$$H_2(Z_\Gamma) \times H_2(Z_\Gamma) \longrightarrow \mathbb{Q}.$$

Let E_σ be the subspace of $H_2(Z_\Gamma, \mathbb{Q})$ generated by the classes of the curves S_σ in the resolving of cusp σ.

We have the decomposition

$$H_2(Z_\Gamma, \mathbb{Q}) = \left(\bigoplus_\sigma E_\sigma \right) \oplus Im \left\{ j_* : H_2(X'_\Gamma, \mathbb{Q}) \longrightarrow H_2(Z_\Gamma, \mathbb{Q}) \right\}.$$

where $j : X'_\Gamma \longrightarrow Z_\Gamma$.

$H^2(-,\mathbb{Q})$

We are looking at $H^2(Z_\Gamma,\mathbb{Q})$.Using Poincare duality, we have a non-degenerate pairing

$$H_2(Z_\Gamma) \times H_2(Z_\Gamma) \longrightarrow \mathbb{Q}.$$

Let E_σ be the subspace of $H_2(Z_\Gamma,\mathbb{Q})$ generated by the classes of the curves S_σ in the resolving of cusp σ.
We have the decomposition

$$H_2(Z_\Gamma,\mathbb{Q}) = \left(\bigoplus_\sigma E_\sigma\right) \oplus Im\left\{j_* : H_2(X'_\Gamma,\mathbb{Q}) \longrightarrow H_2(Z_\Gamma,\mathbb{Q})\right\}.$$

where $j : X'_\Gamma \longrightarrow Z_\Gamma$.

$H^2(-,\mathbb{Q})$

We are looking at $H^2(Z_\Gamma,\mathbb{Q})$.Using Poincare duality, we have a non-degenerate pairing

$$H_2(Z_\Gamma) \times H_2(Z_\Gamma) \longrightarrow \mathbb{Q}.$$

Let E_σ be the subspace of $H_2(Z_\Gamma,\mathbb{Q})$ generated by the classes of the curves S_σ in the resolving of cusp σ.
We have the decomposition

$$H_2(Z_\Gamma,\mathbb{Q}) = \left(\bigoplus_\sigma E_\sigma\right) \oplus Im\left\{j_* : H_2(X'_\Gamma,\mathbb{Q}) \longrightarrow H_2(Z_\Gamma,\mathbb{Q})\right\}.$$

where $j : X'_\Gamma \longrightarrow Z_\Gamma$.

$H^2(-,\mathbb{Q})$

By duality for cohomology we have

$$H^2(Z_\Gamma,\mathbb{Q}) = \left(\bigoplus_\sigma E_\sigma^\vee\right) \oplus \pi^* H^2(X_\Gamma,\mathbb{Q}).$$

where $\pi : Z_\Gamma \longrightarrow X_\Gamma$.

There is an exact sequence

$$0 = H^1(X_\Gamma - X_\Gamma', \mathbb{Q}) \to H_c^2(X_\Gamma', \mathbb{Q}) \to H^2(X_\Gamma, \mathbb{Q}) \to H^2(X_\Gamma - X_\Gamma', \mathbb{Q}) = 0.$$

Proposition

$$H^2(X_\Gamma,\mathbb{Q}) \cong H_c^2(X_\Gamma',\mathbb{Q})$$

$H^2(-,\mathbb{Q})$

By duality for cohomology we have

$$H^2(Z_\Gamma,\mathbb{Q}) = \left(\bigoplus_\sigma E_\sigma^\vee\right) \oplus \pi^* H^2(X_\Gamma,\mathbb{Q}).$$

where $\pi : Z_\Gamma \longrightarrow X_\Gamma$.
There is an exact sequence

$$0 = H^1(X_\Gamma - X'_\Gamma,\mathbb{Q}) \to H_c^2(X'_\Gamma,\mathbb{Q}) \to H^2(X_\Gamma,\mathbb{Q}) \to H^2(X_\Gamma - X'_\Gamma,\mathbb{Q}) = 0.$$

Proposition

$$H^2(X_\Gamma,\mathbb{Q}) \cong H_c^2(X'_\Gamma,\mathbb{Q})$$

$H^2(-,\mathbb{Q})$

By duality for cohomology we have

$$H^2(Z_\Gamma,\mathbb{Q}) = \left(\bigoplus_\sigma E_\sigma^\vee\right) \oplus \pi^* H^2(X_\Gamma,\mathbb{Q}).$$

where $\pi : Z_\Gamma \longrightarrow X_\Gamma$.
There is an exact sequence

$$0 = H^1(X_\Gamma - X'_\Gamma,\mathbb{Q}) \to H^2_c(X'_\Gamma,\mathbb{Q}) \to H^2(X_\Gamma,\mathbb{Q}) \to H^2(X_\Gamma - X'_\Gamma,\mathbb{Q}) = 0.$$

Proposition

$$H^2(X_\Gamma,\mathbb{Q}) \cong H^2_c(X'_\Gamma,\mathbb{Q})$$

Mixed Hodge Structure

X'_Γ is quasi-projective algebraic variety, therefore by Deligne Hodge Theory, there is mixed Hodge structure as:

- There is decreasing filtration $\{F_p\}_{p\in\mathbb{Z}}$ on $H^i(X'_\Gamma, \mathbb{Q}) \otimes \mathbb{C}$,
- There is an increasing weight filtration $\{W_k\}_{k\in\mathbb{Z}}$ on $H^2(X'_\Gamma, \mathbb{Q})$ as
  - $W_k H^2(X'_\Gamma, \mathbb{Q}) = 0 \qquad k \leq 1$
  - $W_2 H^2(X'_\Gamma, \mathbb{Q}) = W_3 H^2(X'_\Gamma, \mathbb{Q}) = j^* H^2(Z_\Gamma, \mathbb{Q})$
  - $W_k H^2(X'_\Gamma, \mathbb{Q}) = H^2(X'_\Gamma, \mathbb{Q}) \qquad k \geq 4$
- **Remark**: There exists a pure hodge structure of weight k on $W_k H^2(X'_\Gamma, \mathbb{Q}) / W_{k-1} H^2(X'_\Gamma, \mathbb{Q})$.

Mixed Hodge Structure

X'_Γ is quasi-projective algebraic variety, therefore by Deligne Hodge Theory, there is mixed Hodge structure as:

- There is decreasing filtration $\{F_p\}_{p\in\mathbb{Z}}$ on $H^i(X'_\Gamma,\mathbb{Q})\otimes\mathbb{C}$,
- There is an increasing weight filtration $\{W_k\}_{k\in\mathbb{Z}}$ on $H^2(X'_\Gamma,\mathbb{Q})$ as
  - $W_kH^2(X'_\Gamma,\mathbb{Q})=0 \qquad k\leq 1$
  - $W_2H^2(X'_\Gamma,\mathbb{Q})=W_3H^2(X'_\Gamma,\mathbb{Q})=j^*H^2(Z_\Gamma,\mathbb{Q})$
  - $W_kH^2(X'_\Gamma,\mathbb{Q})=H^2(X'_\Gamma,\mathbb{Q}) \qquad k\geq 4$
- **Remark**: There exists a pure hodge structure of weight k on $W_kH^2(X'_\Gamma,\mathbb{Q})/W_{k-1}H^2(X'_\Gamma,\mathbb{Q})$.

Mixed Hodge Structure

X'_Γ is quasi-projective algebraic variety, therefore by Deligne Hodge Theory, there is mixed Hodge structure as:

- There is decreasing filtration $\{F_p\}_{p\in\mathbb{Z}}$ on $H^i(X'_\Gamma, \mathbb{Q}) \otimes \mathbb{C}$,
- There is an increasing weight filtration $\{W_k\}_{k\in\mathbb{Z}}$ on $H^2(X'_\Gamma, \mathbb{Q})$ as
  - $W_k H^2(X'_\Gamma, \mathbb{Q}) = 0 \qquad k \leq 1$
  - $W_2 H^2(X'_\Gamma, \mathbb{Q}) = W_3 H^2(X'_\Gamma, \mathbb{Q}) = j^* H^2(Z_\Gamma, \mathbb{Q})$
  - $W_k H^2(X'_\Gamma, \mathbb{Q}) = H^2(X'_\Gamma, \mathbb{Q}) \qquad k \geq 4$

- **Remark**: There exists a pure hodge structure of weight k on $W_k H^2(X'_\Gamma, \mathbb{Q})/W_{k-1} H^2(X'_\Gamma, \mathbb{Q})$.

Mixed Hodge Structure

X'_Γ is quasi-projective algebraic variety, therefore by Deligne Hodge Theory, there is mixed Hodge structure as:

- There is decreasing filtration $\{F_p\}_{p\in\mathbb{Z}}$ on $H^i(X'_\Gamma, \mathbb{Q}) \otimes \mathbb{C}$,
- There is an increasing weight filtration $\{W_k\}_{k\in\mathbb{Z}}$ on $H^2(X'_\Gamma, \mathbb{Q})$ as
  - $W_k H^2(X'_\Gamma, \mathbb{Q}) = 0 \qquad k \leq 1$
  - $W_2 H^2(X'_\Gamma, \mathbb{Q}) = W_3 H^2(X'_\Gamma, \mathbb{Q}) = j^* H^2(Z_\Gamma, \mathbb{Q})$
  - $W_k H^2(X'_\Gamma, \mathbb{Q}) = H^2(X'_\Gamma, \mathbb{Q}) \qquad k \geq 4$
- **Remark**: There exists a pure hodge structure of weight k on $W_k H^2(X'_\Gamma, \mathbb{Q})/W_{k-1} H^2(X'_\Gamma, \mathbb{Q})$.

Pure Hodge Structure

Recall

$$j^* H^2(Z_\Gamma, \mathbb{Q}) = j^* \left(\left(\bigoplus_\sigma E_\sigma^\vee \right) \oplus \pi^* H^2(X_\Gamma, \mathbb{Q}) \right).$$

Since $j^* E_\sigma^\vee = 0$ for all cusps,

Proposition

There is pure Hodge structure of weight 2 on

$$\mathbb{H}^2(X_\Gamma, \mathbb{C}) := j^* \pi^* H^2(X_\Gamma, \mathbb{C}).$$

- **Remark:** Notice that $\mathbb{H}^2(X_\Gamma, \mathbb{Q}) = Im\{H_c^2(X'_\Gamma, \mathbb{Q}) \to H^2(X'_\Gamma, \mathbb{Q})\}$

Pure Hodge Structure

Recall

$$j^* H^2(Z_\Gamma, \mathbb{Q}) = j^* \left(\left(\bigoplus_\sigma E_\sigma^\vee \right) \oplus \pi^* H^2(X_\Gamma, \mathbb{Q}) \right).$$

Since $j^* E_\sigma^\vee = 0$ for all cusps,

Proposition

There is pure Hodge structure of weight 2 on

$$\mathbb{H}^2(X_\Gamma, \mathbb{C}) := j^* \pi^* H^2(X_\Gamma, \mathbb{C}).$$

- **Remark:** Notice that $\mathbb{H}^2(X_\Gamma, \mathbb{Q}) = Im\{H_c^2(X_\Gamma', \mathbb{Q}) \to H^2(X_\Gamma', \mathbb{Q})\}$

Pure Hodge Structure

Recall

$$j^* H^2(Z_\Gamma, \mathbb{Q}) = j^* \left(\left(\bigoplus_\sigma E_\sigma^\vee \right) \oplus \pi^* H^2(X_\Gamma, \mathbb{Q}) \right).$$

Since $j^* E_\sigma^\vee = 0$ for all cusps,

Proposition

There is pure Hodge structure of weight 2 on

$$\mathbb{H}^2(X_\Gamma, \mathbb{C}) := j^* \pi^* H^2(X_\Gamma, \mathbb{C}).$$

- **Remark:** Notice that $\mathbb{H}^2(X_\Gamma, \mathbb{Q}) = Im\{H_c^2(X_\Gamma', \mathbb{Q}) \to H^2(X_\Gamma', \mathbb{Q})\}$

Pure Hodge Structure

Let $f = (..., f_j, ...)$ be Hilbert modular form of weight 2. This defines a 2-form ω_f by

$$\omega_f = (2\pi i)^2 f_j(z) dz_1 \wedge dz_2 \quad on \quad \Gamma_j \backslash \mathbb{H}^2.$$

Lemma

$$F^2 \mathbb{H}^2(X_\Gamma, \mathbb{C}) = \{\omega_f : f \in S_2\}$$

Let $\varepsilon_1 = (\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, 1)$, $\varepsilon_2 = (1, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})$ acts on $\mathbb{H}^\pm \times \mathbb{H}^\pm$ by

$$\varepsilon_1 : (z_1, z_2) \longmapsto (\bar{z}_1, z_2), \quad \varepsilon_1 : (z_1, z_2) \longmapsto (z_1, \bar{z}_2)$$

Pure Hodge Structure

Let $f = (..., f_j, ...)$ be Hilbert modular form of weight 2. This defines a 2-form ω_f by

$$\omega_f = (2\pi i)^2 f_j(z) dz_1 \wedge dz_2 \quad on \quad \Gamma_j \backslash \mathbb{H}^2.$$

Lemma

$$F^2 \mathbb{H}^2(X_\Gamma, \mathbb{C}) = \{\omega_f : f \in S_2\}$$

Let $\varepsilon_1 = (\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, 1)$, $\varepsilon_2 = (1, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})$ acts on $\mathbb{H}^{\pm} \times \mathbb{H}^{\pm}$ by

$$\varepsilon_1 : (z_1, z_2) \longmapsto (\bar{z}_1, z_2), \quad \varepsilon_1 : (z_1, z_2) \longmapsto (z_1, \bar{z}_2)$$

Pure Hodge Structure

Let $f = (..., f_j, ...)$ be Hilbert modular form of weight 2. This defines a 2-form ω_f by

$$\omega_f = (2\pi i)^2 f_j(z) dz_1 \wedge dz_2 \quad on \quad \Gamma_j \backslash \mathbb{H}^2.$$

Lemma

$$F^2\mathbb{H}^2(X_\Gamma, \mathbb{C}) = \{\omega_f : f \in S_2\}$$

Let $\varepsilon_1 = (\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, 1)$, $\varepsilon_2 = (1, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})$ acts on $\mathbb{H}^\pm \times \mathbb{H}^\pm$ by

$$\varepsilon_1 : (z_1, z_2) \longmapsto (\overline{z}_1, z_2), \quad \varepsilon_1 : (z_1, z_2) \longmapsto (z_1, \overline{z}_2)$$

Pure Hodge Structure

Let $f = (..., f_j, ...)$ be Hilbert modular form of weight 2. This defines a 2-form ω_f by

$$\omega_f = (2\pi i)^2 f_j(z) dz_1 \wedge dz_2 \quad on \quad \Gamma_j \backslash \mathbb{H}^2.$$

Lemma

$$F^2 \mathbb{H}^2(X_\Gamma, \mathbb{C}) = \{\omega_f : f \in S_2\}$$

Let $\varepsilon_1 = (\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, 1)$, $\varepsilon_2 = (1, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})$ acts on $\mathbb{H}^\pm \times \mathbb{H}^\pm$ by

$$\varepsilon_1 : (z_1, z_2) \longmapsto (\overline{z}_1, z_2), \quad \varepsilon_1 : (z_1, z_2) \longmapsto (z_1, \overline{z}_2)$$

Pure Hodge Structure

We define

$$\eta_f = \varepsilon_2^* \omega_f \qquad \eta_f' = \varepsilon_1^* \omega_f$$

Theorem

$\mathbb{H}^2(X_\Gamma, \mathbb{C})$ *is direct sum of*

1. *it is* $(2,0)$*- component* $\{\omega_f : f \in S_2\}$
2. *it is* $(0,2)$*- component* $\{\overline{\omega}_f : f \in S_2\}$
3. *it is* $(1,1)$*-component* $\{\eta_f + \eta_g' : f, g \in S_2\} \bigoplus W$ *, where* W *is the space generated by the forms* ω_1, ω_2 *on all components of* X_Γ*.*

Pure Hodge Structure

We define

$$\eta_f = \varepsilon_2^*\omega_f \qquad \eta_f' = \varepsilon_1^*\omega_f$$

Theorem

$\mathbb{H}^2(X_\Gamma, \mathbb{C})$ is direct sum of

1. *it is $(2,0)$- component $\{\omega_f : f \in S_2\}$*
2. *it is $(0,2)$- component $\{\overline{\omega}_f : f \in S_2\}$*
3. *it is $(1,1)$-component $\{\eta_f + \eta_g' : f, g \in S_2\} \oplus W$, where W is the space generated by the forms ω_1, ω_2 on all components of X_Γ.*

Pure Hodge Structure

We define

$$\eta_f = \varepsilon_2^* \omega_f \qquad \eta_f^{'} = \varepsilon_1^* \omega_f$$

Theorem

$\mathbb{H}^2(X_\Gamma, \mathbb{C})$ *is direct sum of*

1. *it is* $(2,0)$*- component* $\{\omega_f : f \in S_2\}$
2. *it is* $(0,2)$*- component* $\{\overline{\omega}_f : f \in S_2\}$
3. *it is* $(1,1)$*-component* $\{\eta_f + \eta_g^{'} : f, g \in S_2\} \oplus W$ *, where* W *is the space generated by the forms* ω_1*,* ω_2 *on all components of* X_Γ*.*

Cuspidal Cohomology

Definition

$\mathbb{H}^2_{cusp}(X_\Gamma, \mathbb{C})$ is the orthogonal complement of W in $\mathbb{H}^2(X_\Gamma, \mathbb{C})$.

Hecke ring

Let

$$B = G(\mathbb{A}_\mathbb{Q}) \cap \left(G(\mathbb{R})^0 \times \prod_{\nu \in S_f} GL_2(\mathcal{O}_{\mathcal{F}_\nu}) \right)$$

and

$$R = G(\mathbb{R})^0 \times K_0$$

Definition

Let H_K be the algebra over $\mathbb{Q}$ generated by

$$T(\mathfrak{m}) = \sum_b RbR \quad where \qquad det(b)\mathcal{O}_\mathcal{F} = \mathfrak{m}.$$

where $\mathfrak{m}$ is an integral ideal.

Hecke ring

Let

$$B = G(\mathbb{A}_{\mathbb{Q}}) \cap \left(G(\mathbb{R})^0 \times \prod_{\nu \in S_f} GL_2(\mathcal{O}_{\mathcal{F}_\nu}) \right)$$

and

$$R = G(\mathbb{R})^0 \times K_0$$

Definition

Let H_K be the algebra over $\mathbb{Q}$ generated by

$$T(\mathfrak{m}) = \sum_b RbR \quad where \qquad det(b)\mathcal{O}_{\mathcal{F}} = \mathfrak{m}.$$

where $\mathfrak{m}$ is an integral ideal.

Hecke ring

Let

$$B = G(\mathbb{A}_\mathbb{Q}) \cap \left(G(\mathbb{R})^0 \times \prod_{\nu \in S_f} GL_2(\mathcal{O}_{\mathcal{F}_\nu}) \right)$$

and

$$R = G(\mathbb{R})^0 \times K_0$$

Definition

Let H_K be the algebra over $\mathbb{Q}$ generated by

$$T(\mathfrak{m}) = \sum_b RbR \quad where \qquad det(b)\mathcal{O}_\mathcal{F} = \mathfrak{m}.$$

where $\mathfrak{m}$ is an integral ideal.

The action of Hecke operators on Modular Forms

- **Note:** There is a version of action of Hecke operators on Modular forms, we are going to use this fact that S_k, i. e the space of cusp forms has basis of of eigenforms for all Hecke operators and **Multiplicity one principle**, i. e two eigenforms with same eigenvalus are multiple of each other.

The action of Hecke operators on Cohomology

We have

$$\mathbb{H}^2(X_{K_0}, \mathbb{Q}) = \mathbb{H}^2_{cusp}(X_{K_0}, \mathbb{Q}) \oplus (\mathbb{Q}(-1))^{h^+}.$$

where $\mathbb{Q}(-1)$ is the rational hodge structure of type $(1,1)$ of $(2\pi i)\mathbb{Q}$.

Recall that $\mathbb{H}^2_{cusp}(X_{K_0}, \mathbb{Q})$ has Hodge decomposition where each term is isomorphic with a space of cusp forms.

Therefore, H_K acts on $\mathbb{H}^2_{cusp}(X_\Gamma, \mathbb{C})$ which is compatible with action on modular forms.

The action of Hecke operators on Cohomology

We have

$$\mathbb{H}^2(X_{K_0}, \mathbb{Q}) = \mathbb{H}^2_{cusp}(X_{K_0}, \mathbb{Q}) \oplus (\mathbb{Q}(-1))^{h^+}.$$

where $\mathbb{Q}(-1)$ is the rational hodge structure of type $(1,1)$ of $(2\pi i)\mathbb{Q}$.
Recall that $\mathbb{H}^2_{cusp}(X_{K_0}, \mathbb{Q})$ has Hodge decomposition where each term is isomorphic with a space of cusp forms.
Therefore, H_K acts on $\mathbb{H}^2_{cusp}(X_\Gamma, \mathbb{C})$ which is compatible with action on modular forms.

The action of Hecke operators on Cohomology

We have

$$\mathbb{H}^2(X_{K_0}, \mathbb{Q}) = \mathbb{H}^2_{cusp}(X_{K_0}, \mathbb{Q}) \oplus (\mathbb{Q}(-1))^{h^+}.$$

where $\mathbb{Q}(-1)$ is the rational hodge structure of type $(1, 1)$ of $(2\pi i)\mathbb{Q}$.
Recall that $\mathbb{H}^2_{cusp}(X_{K_0}, \mathbb{Q})$ has Hodge decomposition where each term is isomorphic with a space of cusp forms.
Therefore, H_K acts on $\mathbb{H}^2_{cusp}(X_\Gamma, \mathbb{C})$ which is compatible with action on modular forms.

The action of Hecke operators on $\mathbb{H}^2(X_\Gamma, \mathbb{C})$

Theorem

For any $T \in H_K$ we can attach T^*, an endomorphism of $\mathbb{H}^2(X_{K_0}, \mathbb{Q})$ such that preserves the Hodge decomposition, and

$$\int_{T_* c} \omega = \int_c T^* \omega \quad \forall c \in H_2(X'_{K_0}, \mathbb{Q}), \quad \omega \in H^2(X', \mathbb{Q})$$

where T_* is its dual endomorphism on $\mathbb{H}_2(X_{K_0}, \mathbb{Q})$. Also,

$$< \omega_1, T^*\omega_2 >=< T^*\omega_1, \omega_2 >$$

Decomposition of H_K

Proposition

H_K is a semi-simple finite dimensional algebra over $\mathbb{Q}$, and S_2 is an $H_K \otimes \mathbb{C}$-module of rank one. Moreover,

$$H_K = \oplus k_i$$

where k_i is finite field extension of $\mathbb{Q}$.

We can choose set of primitive idempotents $\{e_1, e_2, ..., e_n\}$ such that $k_i = e_i H_K$.

Decomposition of H_K

Proposition

H_K is a semi-simple finite dimensional algebra over $\mathbb{Q}$, and S_2 is an $H_K \otimes \mathbb{C}$-module of rank one. Moreover,

$$H_K = \oplus k_i$$

where k_i is finite field extension of $\mathbb{Q}$.

We can choose set of primitive idempotents $\{e_1, e_2, ..., e_n\}$ such that $k_i = e_i H_K$.

Embedding of k_i

If f is normalized eigenform f in $e_i S_2$, then there is embedding

$$\sigma_i : k_i \longrightarrow \mathbb{C}$$

$$t(f) = \sigma_i(t) f$$

therefore, $k_i \bigotimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\sigma} \mathbb{C}$.
Using above embedding

$$e_i S_2 = \bigoplus_{\sigma} \mathbb{C} f^{\sigma}$$

where the f^{σ} are the normalized eigenforms with $t(f^{\sigma}) = \sigma(t) f^{\sigma}$.

- **Remark:** f^{σ} is called **companion** of f.

Embedding of k_i

If f is normalized eigenform f in $e_i S_2$, then there is embedding

$$\sigma_i : k_i \longrightarrow \mathbb{C}$$

$$t(f) = \sigma_i(t) f$$

therefore, $k_i \bigotimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\sigma} \mathbb{C}$.
Using above embedding

$$e_i S_2 = \bigoplus_{\sigma} \mathbb{C} f^{\sigma}$$

where the f^{σ} are the normalized eigenforms with $t(f^{\sigma}) = \sigma(t) f^{\sigma}$.

- **Remark:** f^{σ} is called **companion** of f.

Embedding of k_i

If f is normalized eigenform f in $e_i S_2$, then there is embedding

$$\sigma_i : k_i \longrightarrow \mathbb{C}$$

$$t(f) = \sigma_i(t) f$$

therefore, $k_i \bigotimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\sigma} \mathbb{C}$.
Using above embedding

$$e_i S_2 = \bigoplus_{\sigma} \mathbb{C} f^{\sigma}$$

where the f^{σ} are the normalized eigenforms with $t(f^{\sigma}) = \sigma(t) f^{\sigma}$.

- **Remark:** f^{σ} is called **companion** of f.

Decomposition of $\mathbb{H}^2_{cusp}(X_{K_0}, \mathbb{Q})$

Let $F = \{f_1, ..., fn\}$ be a set of normalized eigenforms, one for each k_i (i.e $f_i \in e_i S_2$).

If $f \in e_i S_2$ the we shall write $e_i = e_f$ and $k_i = k_f$.

We let

$$H^2(M_f, \mathbb{Q}) := e_f \mathbb{H}^2_{cusp}(X, \mathbb{Q}).$$

Theorem

There is a decomposition of polarized Hodge structure on $\mathbb{H}^2_{cusp}(X, \mathbb{Q})$ as

$$\mathbb{H}^2_{cusp}(X, \mathbb{Q}) = \bigoplus_{f \in F} H^2(M_f, \mathbb{Q})$$

Decomposition of $\mathbb{H}^2_{cusp}(X_{K_0}, \mathbb{Q})$

Let $F = \{f_1, ..., fn\}$ be a set of normalized eigenforms, one for each k_i (i.e $f_i \in e_i S_2$).

If $f \in e_i S_2$ the we shall write $e_i = e_f$ and $k_i = k_f$.

We let

$$H^2(M_f, \mathbb{Q}) := e_f \mathbb{H}^2_{cusp}(X, \mathbb{Q}).$$

Theorem

There is a decomposition of polarized Hodge structure on $\mathbb{H}^2_{cusp}(X, \mathbb{Q})$ as

$$\mathbb{H}^2_{cusp}(X, \mathbb{Q}) = \bigoplus_{f \in F} H^2(M_f, \mathbb{Q})$$

Decomposition of $\mathbb{H}^2_{cusp}(X_{K_0}, \mathbb{Q})$

Let $F = \{f_1, ..., fn\}$ be a set of normalized eigenforms, one for each k_i (i.e $f_i \in e_i S_2$).
If $f \in e_i S_2$ the we shall write $e_i = e_f$ and $k_i = k_f$.
We let

$$H^2(M_f, \mathbb{Q}) := e_f \mathbb{H}^2_{cusp}(X, \mathbb{Q}).$$

Theorem

There is a decomposition of polarized Hodge structure on $\mathbb{H}^2_{cusp}(X, \mathbb{Q})$ as

$$\mathbb{H}^2_{cusp}(X, \mathbb{Q}) = \bigoplus_{f \in F} H^2(M_f, \mathbb{Q})$$

Decomposition of $\mathbb{H}^2_{cusp}(X_{K_0}, \mathbb{Q})$

Let $F = \{f_1, ..., fn\}$ be a set of normalized eigenforms, one for each k_i (i.e $f_i \in e_i S_2$).
If $f \in e_i S_2$ the we shall write $e_i = e_f$ and $k_i = k_f$.
We let

$$H^2(M_f, \mathbb{Q}) := e_f \mathbb{H}^2_{cusp}(X, \mathbb{Q}).$$

Theorem

There is a decomposition of polarized Hodge structure on $\mathbb{H}^2_{cusp}(X, \mathbb{Q})$ as

$$\mathbb{H}^2_{cusp}(X, \mathbb{Q}) = \bigoplus_{f \in F} H^2(M_f, \mathbb{Q})$$

Further Decomposition of $H^2(M_f, \mathbb{Q})$

Let consider ε_1, ε_2 as involutions on $\mathbb{H}^2(X, \mathbb{Q})$.

Because the actions of ε_1, ε_2 commutes with Hecke operators, therefore,

$$H^2(M_f, \mathbb{Q}) = \bigoplus_{s,s' \in \{+,-\}} H^2(M_f, \mathbb{Q})_{ss'}$$

where ε_1, ε_2 act on $H^2(M_f, \mathbb{Q})_{ss'}$ as $s.Id$ and $s'.Id$ respectively.

Proposition

For every normalized eigenform $f \in S_2$ we have

$$rank_{k_f} H^2(M_f, \mathbb{Q})_{ss'} = 1.$$

Further Decomposition of $H^2(M_f, \mathbb{Q})$

Let consider ε_1, ε_2 as involutions on $\mathbb{H}^2(X, \mathbb{Q})$.
Because the actions of ε_1, ε_2 commutes with Hecke operators, therefore,

$$H^2(M_f, \mathbb{Q}) = \bigoplus_{s,s' \in \{+,-\}} H^2(M_f, \mathbb{Q})_{ss'}$$

where ε_1, ε_2 act on $H^2(M_f, \mathbb{Q})_{ss'}$ as $s.Id$ and $s'.Id$ respectively.

Proposition

For every normalized eigenform $f \in S_2$ we have

$$rank_{k_f} H^2(M_f, \mathbb{Q})_{ss'} = 1.$$

Further Decomposition of $H^2(M_f, \mathbb{Q})$

Let consider ε_1, ε_2 as involutions on $\mathbb{H}^2(X, \mathbb{Q})$.
Because the actions of ε_1, ε_2 commutes with Hecke operators, therefore,

$$H^2(M_f, \mathbb{Q}) = \bigoplus_{s,s' \in \{+,-\}} H^2(M_f, \mathbb{Q})_{ss'}$$

where ε_1, ε_2 act on $H^2(M_f, \mathbb{Q})_{ss'}$ as $s.Id$ and $s'.Id$ respectively.

Proposition

For every normalized eigenform $f \in S_2$ we have

$$rank_{k_f} H^2(M_f, \mathbb{Q})_{ss'} = 1.$$

Thanks