Height pairings on Hilbert modular varieties: quartic CM points

A report on work of Howard and Yang

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Montreal-Toronto Workshop on Number Theory
Fields Institute, Toronto
April 10, 2011

The aim of this talk is to survey some results and some conjectures about the relations between:

height pairings of
special cycles on Hilbert modular varieties

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Fourier coefficients of modular forms.

Mostly, I will report[1] on the recent preprint:

- B. Howard and T. Yang,
  Intersections of Hirzebruch-Zagier divisors and CM cycles.

There they consider the pairing

$$\left\{ \text{Hirzerbuch-Zagier divisors} \right\} \times \left\{ \text{quartic CM points.} \right\}$$

[1] Disclaimer: Any errors, misunderstandings, and unfounded speculations in this report are the sole responsibility of SK.

§1. The moduli space $\mathcal{M}$.

Notation:

$$F = \mathbb{Q}(\sqrt{d_F}) = \text{a real quadratic field, with } F \subset \mathbb{R},$$

$$\partial = \partial_F = \text{the different}, \qquad \sigma = \text{the nontrivial Galois auto.}$$

$$\mathcal{M} = \text{moduli stack over } \mathrm{Spec}\,(\mathbb{Z}) \text{ of}$$
$$\text{principally polarized RM abelian surfaces}$$

So over a base S, and object is (A, ι, λ):

$A \longrightarrow S$	an abelian scheme
$\iota : O_F \longrightarrow \mathrm{End}(A)$	an action of O_F
$\lambda : A \longrightarrow A^\vee$	a principal polarization

where $\iota(a)^* = \iota(a)$ and the Kottwitz condition holds:

$$\mathrm{char}(T, \iota(a)|\mathrm{Lie}\,(A)) = (T - a)(T - \sigma(a)) \in \mathcal{O}_S[T].$$

Then

$$\mathcal{M} \longrightarrow \mathrm{Spec}\,(\mathbb{Z})$$

is flat and is smooth over $\mathrm{Spec}\,(\mathbb{Z}[d_F^{-1}])$.

It gives an integral model of the Hilbert modular variety.

A fundamental problem is to investigate its arithmetic Chow groups

$$\widehat{\mathrm{CH}}^{\bullet}(\mathcal{M})$$

in the sense of Gillet-Soulé.

More precisely, as we have seen, we should work on a compactified integral model $\mathcal{M}'$ and consider the Chow groups defined using Green functions (objects) with log-log growth, as defined by Burgos, Kuehn and Kramer:

$$\widehat{\mathrm{CH}}^{\bullet}_{BBK}(\mathcal{M}').$$

§2. Special cycles.

To define **special cycles**, e.g., Hirzebruch-Zagier cycles in this setting:

Definition: A **special endomorphism** of $\mathbf{A} = (A, \iota, \lambda)$:

$$j \in \mathrm{End}(A), \qquad j^* = j, \qquad j \circ \iota(a) = \iota(\sigma(a)) \circ j,$$

for all $a \in O_F$.

$$L(\mathbf{A}) = \mathbb{Z}\text{-module of special endomorphisms}$$

$$V(\mathbf{A}) = L(\mathbf{A}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$j^2 = Q(j) \cdot 1_A \qquad \mathbb{Z}\text{-valued quadratic form}$$

The quadratic lattice $(L(\mathbf{A}), Q)$ of special endomorphisms has rank at most 4 and is positive definite.

Special cycles are defined as follows:[2] [3]

For $m \in \mathbb{Z}_{>0}$, let $\mathcal{T}(m)$ be the stack over $\mathrm{Spec}\,(\mathbb{Z})$ where an object of $\mathcal{T}(m)(S)$ is a pair $(\mathbf{A}, j)$ for $\mathbf{A}$, as before, and $j \in L(\mathbf{A})$ is a special endomorphism with $Q(j) = m$.

Then

$$\mathcal{T}(m) \longrightarrow \mathcal{M}$$

is a special cycle with $\mathcal{T}(m)_{\mathbb{Q}}$ the m-th Hirzebruch-Zagier divisor.

It should be equipped with a suitable[4] Green function $G(m, v)$ to define a class, $v \in \mathbb{R}_{>0}$,

$$\widehat{\mathcal{T}}(m, v) = (\mathcal{T}(m), G(m, v)) \in \widehat{\mathrm{CH}}^1_{BBK}(\mathcal{M}').$$

[2] S. Kudla and M. Rapoport, *Arithmetic Hirzebruch-Zagier cycles*, Crelle **515** (1999), 155–244.

[3] and Howard-Yang, section

[4] This has not yet been done...

Here, if $m \in \mathbb{Z}$ and $m < 0$, then $G(m, v)$ should be a smooth function on $\mathcal{M}'(\mathbb{C})$ and

$$\widehat{\mathcal{T}}(m, v) = (0, G(m, v)) \in \widehat{\mathrm{CH}}^1_{BBK}(\mathcal{M}').$$

Speculation: For a suitable definition of the constant term $\widehat{\mathcal{T}}(0, v)$, the generating series

$$\widehat{\phi}_{\mathcal{M}}(\tau) = \sum_m \widehat{\mathcal{T}}(m, v)\, q^m$$

is a modular form of weight 2, level d_F and Nebentypus χ_F, valued in $\widehat{\mathrm{CH}}^1_{BBK}(\mathcal{M}')$. Here $q = e(\tau)$, $\tau = u + iv$.

The analogous statement forthe generating function for special divisors on Shimura curves is a main result of the K., Rapoport and Yang, Princeton book.

One way to test this speculation[5] would be to compute the image

$$\Lambda(\widehat{\phi}_{\mathcal{M}}(\tau))$$

of the generating function under various linear functionals

$$\Lambda : \widehat{\mathrm{CH}}^1_{BBK}(\mathcal{M}') \longrightarrow \mathbb{C}$$

and to show that these are classical scalar valued modular forms.

For example, the height pairing

$$\Lambda(\cdot) = \langle \cdot, \widehat{\mathcal{C}} \rangle,$$

with classes in $\widehat{\mathcal{C}} \in \widehat{\mathrm{CH}}^2_{BBK}(\mathcal{M}')$.

For the class $\widehat{\mathcal{C}} = \widehat{\omega}^2$, where $\widehat{\omega}$ is the metrized Hodge bundle, this was done by Bruinier, Burgos and Kühn, *Borcherds products and arithmetic intersection theory on Hilbert modular surfaces*, Duke Math. J. **139** (2007), 1–88.

[5] or, indeed, to prove it ...

Here is another possibility. Let

$$j : \mathcal{C} \longrightarrow \mathcal{M}$$

be a morphism of an arithmetic curve to $\mathcal{M}$.

Then the pullback

$$\widehat{\mathrm{CH}}^1(\mathcal{M}) \xrightarrow{j^*} \widehat{\mathrm{CH}}^1(\mathcal{C}) \xrightarrow{\widehat{\deg}} \mathbb{R}$$

composed with the arithmetic degree $\widehat{\deg}$ gives a functional:

$$\Lambda(\cdot) = \widehat{\deg}\, j^*(\cdot).$$

A big supply of such arithmetic curves is provided by the CM points on $\mathcal{M}$.

§3. CM points.

Let E be a quartic CM field with totally real subfield F.

There are 3 cases:

$$\begin{cases} E/\mathbb{Q} = \text{biquadratic} \\ E/\mathbb{Q} = \text{cyclic} \\ E/\mathbb{Q} = \text{non-Galois} \end{cases}$$

Let $\Sigma = \{\pi_1, \pi_2\}$ be a CM type for E, so that

$$\pi_i : E \hookrightarrow \mathbb{C}, \qquad \pi_1|_F = \pi_2 \circ \sigma|_F.$$

Let $\mathrm{tr}_\Sigma(\alpha) = \pi_1(\alpha) + \pi_2(\alpha)$, and let

$$E_\Sigma = \mathbb{Q}(\{\mathrm{tr}_\Sigma(\alpha) \mid \alpha \in E\}) \quad \supset O_\Sigma$$

be the reflex field and its ring of integers.

Let

$$\mathcal{CM}_E^{\Sigma} = \begin{pmatrix} \text{the moduli stack over } \mathrm{Spec}\,(O_{\Sigma}) \text{ of} \\ \text{CM abelian schemes} \\ \text{of type } (O_E, \Sigma) \end{pmatrix}$$

so a point of $\mathcal{CM}_E^{\Sigma}(S)$ is a triple (A, ι_E, λ) with

$A \longrightarrow S$	an abelian scheme
$\iota : O_E \longrightarrow \mathrm{End}(A)$	an action of O_E
$\lambda : A \longrightarrow A^{\vee}$	a principal polarization

where $\iota(a)^* = \iota(\bar{a})$ and the Σ-Kottwitz condition holds:

$$\mathrm{char}(T, \iota(a)|\mathrm{Lie}\,(A)) = (T - \pi_1(a))(T - \pi_2(a)) \in \mathcal{O}_S[T].$$

It is shown in HY that

$$\mathcal{CM}_E^{\Sigma} \longrightarrow \mathrm{Spec}\,(O_{\Sigma})$$

is étale and proper, and hence is a (usually not connected) arithmetic curve. Moreover, there is a natural forgetful morphism

$$j_E^{\Sigma} : \mathcal{CM}_E^{\Sigma} \longrightarrow \mathcal{M} \times_{\mathrm{Spec}\,(\mathbb{Z})} \mathrm{Spec}\,(O_{\Sigma}),$$

$$(A, \iota_E, \lambda) \mapsto (A, \iota_E|_{O_F}, \lambda)$$

Problem: Identify the pullback of the generating function

$$\widehat{\deg}\,(j_E^{\Sigma})^*(\widehat{\phi}_{\mathcal{M}}(\tau)) = \sum_m \widehat{\deg}\,(j_E^{\Sigma})^*(\widehat{\mathcal{T}}(m, v))\, q^m$$

§4. Special cycles on $\mathcal{CM}_E^\Sigma$.

The conjectural answer and the results of HY that support it begin with a nice bit of algebra.

For simplicity, from now on we restrict to the non-biquadratic case.

Let

$$R = E \otimes_{id,F,\sigma} E,$$

with automorphisms $\rho(a \otimes b) = \bar{b} \otimes a$ and $\tau(a \otimes b) = b \otimes a$.

Let

$$E^\sharp = R^{<\tau>} \supset F^\sharp = R^{<\tau,\rho^2>}.$$

Then $E^\sharp$ is a CM field with totally real subfield $F^\sharp$ and complex conjugation $\alpha \mapsto \alpha^\dagger = \rho^2(\alpha)$.

The CM types $\Sigma = \{\pi_1, \pi_2\}$ of E correspond to complex embeddings

$$\phi_\Sigma : E^\sharp \xrightarrow{\sim} E_\Sigma \subset \mathbb{C}, \qquad a \otimes b \mapsto \pi_1(a)\pi_2(b).$$

In order to compute the pullback, we have to consider the fiber product

$$\begin{array}{ccc} \mathcal{CM}_E^\Sigma \times_{\mathcal{M}} \mathcal{T}(m) & \longrightarrow & \mathcal{T}(m) \\ \downarrow & & \downarrow \\ \mathcal{CM}_E^\Sigma & \xrightarrow{j_E^\Sigma} & \mathcal{M} \end{array}$$

where an S-point of the top left corner is a collection

$$(\mathbf{A}, j) = (A, \iota_E, \lambda, j), \qquad j \in L(\mathbf{A}), \quad Q(j) = m.$$

Example: Since E is not biquadratic, if $\mathbf{A}$ is a point of $\mathcal{CM}_E^\Sigma(\mathbb{C})$, then $L(\mathbf{A}) = 0$, so the cycles $\mathcal{CM}_E^\Sigma$ and $\mathcal{T}(m)$ do not meet on the generic fiber.

In general, it turns out that, the O_E-action on A induces an additional structure on the module $L(\mathbf{A})$ of special endomorphisms. This allows us to decompose the fiber product.

More precisely:

Note that the reflex algebra $E^\sharp$ is spanned by elements

$$a \odot b = a \otimes b + b \otimes a.$$

Proposition. (HY) The reflex algebra $E^\sharp$ acts on $V(\mathbf{A}) = L(\mathbf{A}) \otimes_{\mathbb{Z}} \mathbb{Q}$ by the rule

$$(a \odot b) \bullet j = \iota_E(a) \circ j \circ \iota_E(\bar{b}) + \iota_E(b) \circ j \circ \iota_E(\bar{a}).$$

Moreover, for $\alpha \in E^\sharp$,

$$(\alpha \bullet x, y) = (x, \alpha^\dagger \bullet y),$$

where $(\ ,\ )$ is the bilinear form on $V(\mathbf{A})$ associated to Q.

Corollary. There is a unique $F^\sharp$ quadratic form $Q^\sharp_{\mathbf{A}}$ on $W(\mathbf{A}) = V(\mathbf{A})$ such that

$$Q(x) = \mathrm{tr}_{F^\sharp/\mathbb{Q}}(Q^\sharp_{\mathbf{A}}(x)).$$

The quadratic space $(W(\mathbf{A}), Q^\sharp_{\mathbf{A}})$ is totally positive definite.

As a result, there is a decomposition

$$\mathcal{CM}_E^\Sigma \times_{\mathcal{M}} \mathcal{T}(m) = \bigsqcup_{\substack{\alpha \in F^\sharp,\ \alpha \gg 0 \\ \mathrm{tr}_{F^\sharp/\mathbb{Q}}(\alpha) = m}} \mathcal{CM}_E^\Sigma(\alpha),$$

where

$$\mathcal{CM}_E^\Sigma(\alpha)(S) = \begin{pmatrix} \text{locus of collections } (\mathbf{A}, j) \\ \mathbf{A} \in \mathcal{CM}_E^\Sigma(S) \\ j \in L(\mathbf{A}),\ Q_{\mathbf{A}}^\sharp(j) = \alpha \end{pmatrix}.$$

It is shown in HY that $\mathcal{CM}_E^\Sigma(\alpha)$ is either empty or of dimension 0. There is a forgetful morphism

$$\mathcal{CM}_E^\Sigma(\alpha) \longrightarrow \mathcal{CM}_E^\Sigma.$$

Hence, we get **a special** 0**–cycle** on the arithmetic curve $\mathcal{CM}_E^\Sigma$.

§5. Another generating function

In fact, for each $\alpha \in (F^\sharp)^\times$, we have a class

$$\widehat{\mathcal{C}}(\alpha, \mathbf{v}) \in \widehat{\mathrm{CH}}^1(\mathcal{CM}_E^\Sigma), \qquad \mathbf{v} = (v_1, v_2) \in (\mathbb{R}_+^\times)^2,$$

the first arithmetic Chow group of $\mathcal{CM}_E^\Sigma$:

$$\widehat{\mathcal{C}}(\alpha, \mathbf{v}) = \begin{cases} [\mathcal{CM}_E^\Sigma(\alpha)] & \text{if } \alpha \gg 0, \\ \text{an archimedean class} & \text{if } \alpha \not\gg 0. \end{cases}$$

Applying $\widehat{\deg}$, we get a new generating series

$$\widehat{\phi}_E^\Sigma(\boldsymbol{\tau}) = \sum_{\alpha \in F^\sharp} \widehat{\deg}\, \widehat{\mathcal{C}}(\alpha, \mathbf{v})\, \mathbf{q}^\alpha,$$

where

$$\mathbf{q}^\alpha = e(\mathrm{tr}(\alpha\, \boldsymbol{\tau})), \qquad \boldsymbol{\tau} = (\tau_1, \tau_2) \in \mathfrak{H}^2.$$

Here a class for $\alpha = 0$ must be added.

Conjecture I: (vague version) The generating series

$$\widehat{\phi}_E^{\Sigma}(\boldsymbol{\tau}) = \sum_{\alpha \in F^\sharp} \widehat{\deg}\, \widehat{\mathcal{C}}(\alpha, \mathbf{v})\, \mathbf{q}^\alpha,$$

is a Hilbert modular form of weight 1 (and some level) for $\mathrm{SL}_2(O_{F^\sharp})$.

This is not stated in HY, but is nearly proved there.

To state their results, we need some incoherent Eisenstein series.

§6. Incoherent Eisenstein series.

For each object **A** in $\mathcal{CM}_E^{\Sigma}(\mathbb{C})$, let

$$\mathbf{M} = M(\mathbf{A}) = (M, \kappa_E, \lambda_M),$$

where

$$M = H_1(A, \mathbb{Z})$$

$$\kappa_E : O_E \longrightarrow \mathrm{End}(M), \qquad \text{the } O_E\text{-action}$$

$$\lambda_M : M \times M \longrightarrow \mathbb{Z}, \qquad \text{the Riemann form coming from } \lambda_A.$$

$$(L(\mathbf{M}), Q_{\mathbf{M}}) = \text{the lattice of special endomorphisms of } \mathbf{M}.$$

As before, there is an $E^\sharp$ action on $V(\mathbf{M}) = L(\mathbf{M}) \otimes_{\mathbb{Z}} \mathbb{Q}$, and an $F^\sharp$-quadratic space

$$(W(\mathbf{M}), Q^\sharp_{\mathbf{M}}).$$

Now

$$\mathrm{sig}(W(\mathbf{M}), Q^\sharp_{\mathbf{M}}) = ((0,2),(2,0)), \qquad \text{at} \quad (\phi_\Sigma, \phi_\Sigma \circ \sigma^\sharp).$$

Attached to this data, there is a Siegel-Weil Eisenstein series

$$E(\boldsymbol{\tau}, s, L(\mathbf{M}))$$

of weight $(-1, 1)$ for $F^\sharp$.

If we switch the weight in the first component, we obtain an **incoherent** Eisenstein series

$$E(\boldsymbol{\tau}, s, \widehat{L}(\mathbf{M}), \mathbf{1})$$

of weight $\mathbf{1} = (1, 1)$, where $\widehat{L}(\mathbf{M}) = L(\mathbf{M}) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$.

Then, the Siegel-Weil formula gives

$$\begin{aligned} E(\boldsymbol{\tau}, 0, L(\mathbf{M})) &= E(\boldsymbol{\tau}, 0, \widehat{L}(\mathbf{M}), (-1, 1)) \\ &= \text{binary theta series for } (L(\mathbf{M}), Q^\sharp_{\mathbf{M}}). \end{aligned}$$

and

$$E(\boldsymbol{\tau}, 0, \widehat{L}(\mathbf{M}), \mathbf{1}) = 0.$$

§7. Conjectures and results.

Let X_E^Σ be the set of isomorphism classes of objects **A** in $\mathcal{CM}_E^\Sigma(\mathbb{C})$.

Let

$$E(\boldsymbol{\tau}, s, \Sigma, \mathbf{1}) = \sum_{\mathbf{A} \in X_E^\Sigma} E(\boldsymbol{\tau}, s, \widehat{L}(\mathbf{M}), \mathbf{1}).$$

Let

$$E'(\boldsymbol{\tau}, 0, \Sigma, \mathbf{1}) = \sum_{\alpha \in F^\sharp} c_\Sigma(\alpha, \mathbf{v})\, \mathbf{q}^\alpha,$$

be the Fourier expansion of the central derivative.

Conjecture I:

$$E'(\boldsymbol{\tau}, 0, \Sigma, \mathbf{1}) = -w_E \cdot \widehat{\phi}_E^\Sigma(\boldsymbol{\tau}),$$

where w_E is the number of roots of unity in E.

Theorem: (HY)[6] Suppose that Hypothesis B holds for all primes p.
Then Conjecture A is true.

This amounts to a collection of Fourier coefficient identities:

$$-w_E \cdot \widehat{\deg}\, \widehat{\mathcal{C}}(\alpha, \mathbf{v})\, \mathbf{q}^\alpha = E'_\alpha(\boldsymbol{\tau}, 0, \Sigma, \mathbf{1}).$$

[6] Not stated in HY.

For a prime p, let

$$\mathbb{E}_p = \mathbb{Q}_p(\{\pi(x) \mid x \in E,\ \pi \in \mathrm{Hom}(E, \mathbb{Q}^{\mathrm{alg}})\}).$$

Hypothesis B:

1. $|\mathbb{E}_p : \mathbb{Q}_p| \le 4$,
2. $e(\mathbb{E}_p/\mathbb{Q}_p) \le p - 1$.

For example, Hypothesis B holds for all p if $E/\mathbb{Q}$ is cyclic and unramified for $p = 2,\ 3$.

For the constant term, we have 'fudged' and defined the class

$$\widehat{\mathcal{C}}(0, \mathbf{v}) := -\frac{1}{w_E} \cdot E_0'(\boldsymbol{\tau}, 0, \Sigma, \mathbf{1}).$$

This can no doubt be fixed...

Finally, we return to our original generating series for $\mathcal{M}$

$$\widehat{\phi}_{\mathcal{M}}(\tau) = \sum_m \widehat{\mathcal{T}}(m, v)\, q^m$$

valued in $\widehat{\mathrm{CH}}^2_{BBK}(\mathcal{M}')$ and its pullback

$$\widehat{\deg}\,(j_E^{\Sigma})^*(\widehat{\phi}_{\mathcal{M}}(\tau)) = \sum_m \widehat{\deg}\,(j_E^{\Sigma})^*(\widehat{\mathcal{T}}(m, v))\, q^m$$

under

$$j_E^{\Sigma} : \mathcal{CM}_E^{\Sigma} \longrightarrow \mathcal{M}$$

Conjecture II:

$$\widehat{\deg}\,(j_E^{\Sigma})^*(\widehat{\phi}_{\mathcal{M}}(\tau)) = \widehat{\phi}_E^{\Sigma}(\tau^{\Delta}).$$

This can be viewed as a collection of arithmetic intersection number identities

$$(\mathcal{T}(m), j_E^{\Sigma}(\mathcal{CM}_E^{\Sigma}))_{\mathcal{M},\text{finite}} = \sum_{\substack{\alpha \in F^{\sharp}, \alpha \gg o \\ \mathrm{tr}_{F^{\sharp}/\mathbb{Q}}(\alpha) = m}} \widehat{\deg}\, \mathcal{CM}_E^{\Sigma}(\alpha),$$

and an analogue for Green functions.

Under Hypothesis B for all primes *p*, these are proved in HY.

§8. Final remarks.

There are several remaining issues.

- The constant terms have to be handled. This will involve the Faltings heights of the CM abelian surfaces, cf.
  Tonghai Yang, *An arithmetic intersection on a Hilbert modular surface and the Faltings height*, preprint.
- The Green function used in HY is constructed as in the KRY book using the exponential integral Ei. The boundary behavior of this Green function is not well understood. In particular, it does not give classes in

$$\widehat{\mathrm{CH}}^1_{BKK}(\mathcal{M}').$$

So a modification of it is probably needed.

Finally, the conjectured identity can be viewed as a seesaw identity for arithmetic theta series:

$$\begin{array}{cccccc} \widehat{\phi}^{\Sigma}_{E}(\boldsymbol{\tau}) & \mathrm{SL}_2(F^\sharp) & & \mathcal{M} & \widehat{\phi}_{\mathcal{M}}(\tau) \\ & \uparrow & & \uparrow j^{\Sigma}_{E} & \\ & \mathrm{SL}_2(\mathbb{Q}) & & \mathcal{CM}^{\Sigma}_{E} & \end{array}$$

From the modularity of $\widehat{\phi}_{\mathcal{M}}(\tau)$, we get an arithmetic theta lift

$$S_2(\Gamma_0(D),\chi) \longrightarrow \widehat{\mathrm{CH}}^1(\mathcal{M}), \qquad f \mapsto \langle\, f, \widehat{\phi}_{\mathcal{M}}\,\rangle_{\mathrm{Pet.}} =: \widehat{\theta}_{\mathcal{M}}(f).$$

From Conjectures I and II, in particular the arithmetic seesaw identity

$$\begin{aligned}\widehat{\deg}\,(j_E^{\Sigma})^*(\widehat{\theta}_{\mathcal{M}}(f)) &\sim \frac{\partial}{\partial s}\langle\, f, \Delta^*(E(s,\Sigma))\,\rangle_{\mathrm{Pet}}\Big|_{s=0}\\ &\sim L'(1,f,E/F,\Sigma).\end{aligned}$$

Such a formula would give the values of the functionals $\widehat{\deg}\,(j_E^{\Sigma})^*$ on the classes

$$\widehat{\theta}_{\mathcal{M}}(f) \in \widehat{\mathrm{CH}}^1(\mathcal{M})$$

in terms of the central derivatives of the Rankin type integrals

$$L(s+1,f,E/F,\Sigma)) = \langle\, f(\tau), E(\tau^{\Delta},s,\Sigma,\mathbf{1})\,\rangle_{\mathrm{Pet.}}.$$