Toroidal compactifications of Hilbert modular varieties

Fritz Hörmann

Department of Mathematics and Statistics
McGill University

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affine Torus embeddings

k	a field
M	a lattice ($\cong \mathbb{Z}^n$)
$T = \mathrm{spec}(k[M^*])$	a split torus over k
$M = X_*(T)$	the cocharacter group of T

We want to look at a certain type of (partial) compactifications of T, called **torus embeddings.**

R	a discrete valuation ring (k-algebra)
K	quotient field
x	a point in $T(K)$ which does not extend to R

Goal: Look for open embeddings $T \hookrightarrow \overline{T}$ such that x extends to a section $x \in \widetilde{T}(R)$.

The valuation on K induces a linear morphism $\nu_x : M^* \to \mathbb{Z}$. Therefore the subring defining the open embedding above must not contain m^* with $\nu_x(m^*) < 0$.

affine Torus embeddings

- $\overline{T} := \text{spec}(k[\nu_x^{-1}\mathbb{Z}_{\geq 0}])$.
- more generally: Replace $\nu_x^{-1}\mathbb{Z}_{\geq 0}$ by any saturated (in order to get an open embedding) submonoid $\subsetneq M$ such that ν_x is non-negative.

All these $\overline{T}$ have the property that the action of T (by multiplication) extends to them

Proposition

The following are equivalent

1. affine open dense embeddings $T \hookrightarrow \overline{T}$ such that the action of T extends
2. finitely generated submonoids of M^*
3. polyhedral cones $\sigma = \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_n \subset M_{\mathbb{R}}$, which do not contain a line (here $v_i \in M$)

$\sigma \mapsto \quad \sigma^\vee = \{y \in M^* \mid \langle x, y \rangle \geq 0 \ \forall x \in \sigma\} \mapsto \quad T_\sigma := \text{spec}(k[\sigma^\vee])$
E.g.: $\mathbb{R}_{\geq 0}\nu_x \leftrightarrow \text{spec}(k[\nu_x^{-1}\mathbb{Z}_{\geq 0}])$

general Torus embeddings

Drop the assumption of affine — can we get something proper?

$$\begin{aligned} & \sigma, \tau \subset M_{\mathbb{R}} \\ & \sigma \subseteq \tau \\ \rightsquigarrow \quad & \text{natural map } T_\tau \to T_\sigma \end{aligned}$$

Given a collection $\Delta := \{\sigma_i\}$ with $\sigma_i \subset M_{\mathbb{R}}$, we can glue along all these natural maps (i.e. take the direct limit). Call this T_Δ.

T_Δ **separated**, if $\sigma \cap \tau$ for two r.p.c. in Δ is either empty or a common face of σ and τ which is included in T_τ. We call Δ a partial polyhedral cone decomposition in this case (and assume also that with $\sigma \in \Delta$, it contains all of the faces).

T_Δ **proper**, if for all possibe ν_x as above there is a cone σ with $\nu_x \in \sigma$, i.e. if $\bigcup_{\sigma\in\Delta} \sigma = M_{\mathbb{R}}$.

Theorem

The following define equivalent categories

1. open dense embeddings $T \hookrightarrow \overline{T}$ such that the action of T extends
2. partial polyhedral cone decompositions Δ of M^*.

Morphisms in the second case are **refinements**.

T_Δ **smooth**, if all r.p.c. in Δ are generated by part of a basis of M^*.

T_Δ **projective**, if there is a piecewise linear function $\mu : \bigcup \sigma \to \mathbb{R}$ with integral values on M, such that the $\sigma \in \Delta$ are the maximal sets on which μ is linear (together with the faces of those) which satisfied a certain convexity property.

Stratification

For each cone $\sigma \in \Delta$ there is an associated stratum $T_{[\sigma]}$, isomorphic to the quotient of T having cocharacter lattice $M/(<\sigma> \cap M)$. It embedds by the obvious map

$$\mathrm{spec}(k[\sigma^{\perp} \cap M^*]) \to \mathrm{spec}(k[\sigma^{\vee} \cap M^*])$$

Properties:

- $T_\Delta = \bigcup_{\sigma \in \Delta} T_{[\sigma]}$.
- $\sigma \subseteq \tau \Leftrightarrow \overline{T_{[\tau]}} \subseteq \overline{T_{[\sigma]}}$.
- $\kappa = \sigma \cap \tau \Leftrightarrow \overline{T_{[\sigma]}} \cap \overline{T_{[\tau]}} = \overline{T_{[\kappa]}}$.

The functor

Obviously:

$$T(S) = \{\pi : M_S \to (\mathcal{O}_S, \times) \text{ morphism of monoids}\}$$

We have:

$$T_\Delta(S) = \left\{ \begin{array}{ll} M' \subset M_S & \text{a subsheaf of monoids,} \\ \pi : M' \to (\mathcal{O}_S, \times) & \text{a (strict) morphism of} \\ & \text{sheaves of monoids such that} \\ & \forall s \in S\text{: } M'_s = \sigma^\vee \cap M \text{ for some } \sigma \in \Delta \end{array} \right\}$$

ϕ a strict morphism of monoids if $\phi(e) = e$ and $\phi(x)$ invertible $\Leftrightarrow x$ invertible.

Monodromy

Given a holomorphic map (with "bounded image")

$$\mu : B_1^*(0) \to T(\mathbb{C}) = M_{\mathbb{C}}/M$$

defines a monodromy element $x \in M$ (image of 1 under the monodromy representation $\mathbb{Z} = \pi_1(B_1^*(0)) \to M$).

Lemma

μ extends to $B_1(0) \to T(\mathbb{C})_\Delta$ if and only if $x \in \mathrm{supp}(\Delta)$.

Roughly: Via the embedding $B_1^*(0) \hookrightarrow \mathbb{G}_m(\mathbb{C})$, μ "looks like" the cocharacter x.

The symmetric space

Fix the following data:

F	totally real field of degree n
$\mathcal{L}$	fixed ideal of $\mathcal{O}_F$
V	projective $\mathcal{O}_F$-module of rank 2 with a $\mathcal{L}^{-1}$-valued symplectic form $\langle , \rangle_{\mathcal{O}_F}$ i.e. an isomorphism $\Lambda^2_{\mathcal{O}_F} V \cong \mathcal{L}$
$\langle v, w \rangle$	$\mathrm{tr}_{F\mid\mathbb{Q}} \langle v, w \rangle_{\mathcal{O}_F}$
G	$\{g \in \mathrm{Res}_{F\mid\mathbb{Q}} \mathrm{GL}_F(V_{\mathbb{Q}}) \mid \det(g) \in \mathbb{G}_{m,\mathbb{Q}}\}$ $= \mathrm{Res}_{F\mid\mathbb{Q}} \mathrm{GL}_F(V_{\mathbb{Q}}) \cap \mathrm{GSp}(V_{\mathbb{Q}})$
D	{polarized $\mathcal{O}_F$-Hodge structures on $V_{\mathbb{C}}$ } $= G(\mathbb{R})^+ / K \cdot Z$

$V_{\mathbb{R}} = \bigoplus_{\rho \in \mathrm{Hom}(F,\mathbb{R})} V^{\rho}$ with
V^{ρ} isomorphic to $\mathbb{R}^2$ with the F-representation induced by ρ.

The symmetric space

Definition of $\mathcal{O}_F$-Hodge structure

$F^0 := V^{-1,0}$ defines a $\mathcal{O}_F$-**Hodge structure** if one of the equivalent conditions hold

1. The representation $h : \mathbb{S} \to \mathrm{GL}(V_\mathbb{R})$ giving the Hodge structure factorizes via $G_\mathbb{R}$
2. The action of $\mathcal{O}_F$ on V induces endomorphims of Hodge structures
3. For each ρ, there is a $F^{0,\rho} \subset V_\mathbb{C}^\rho$ (1-dimensional) such that $F^0 = \bigoplus_\rho F^{0,\rho}$.
4. The complex torus $A := V_\mathbb{Z} \backslash V_\mathbb{C} / F^0$ is an $\mathcal{O}_F$-complex torus.

The symmetric space

Definition of polarization for $\mathcal{O}_F$-Hodge structures

Each $F^{0,\rho} \subset V_{\mathbb{C}}^{\rho}$ induces a sign $\operatorname{sgn} \frac{\langle v, \overline{v} \rangle}{2\pi i}$, where $v \in F^{0,\rho}$ is any non-zero element. We call the Hodge structure **polarized** if:

1. All signs above are positive

The Borel embedding

Relaxing the condition on $F^0 := V^{-1,0}$ of polarized Hodge structure — but not the condition of $\mathcal{O}_F$-compatibility — F^0 is just defined by a collection of 1-dimensional subspaces $F^{0,\rho} \subset V_{\mathbb{C}}^{\rho}$.
Therefore we get the open **Borel embedding**:

$$D \hookrightarrow D^{\vee} = (\mathrm{Res}_{F:\mathbb{Q}}\, \mathbb{P}_F(V_{\mathbb{Q}}))(\mathbb{C}) \cong \prod_{\rho} \mathbb{P}(V_{\mathbb{C}}^{\rho})$$

The closure $\overline{D}$ of the image decomposes into **boundary components**, which are products of boundary components of the $\mathbb{H} \subset \mathbb{P}(V_{\mathbb{C}}^{\rho})$ (either a real point or $\mathbb{H}$ itself). To each such boundary component one associates a real parabolic in $G_{\mathbb{R}}$. If $G_{\mathbb{R}}$ is simple only $G_{\mathbb{R}}$ and maximal parabolics of occur, otherwise products of those. For the compactification of the quotients $D/G(\mathbb{Z})^+$ only those boundary components whose parabolic is definied over $\mathbb{Q}$ matter. These are just the points

$$I \in (\mathrm{Res}_{F:\mathbb{Q}}\, \mathbb{P}_F(V_{\mathbb{Q}}))(\mathbb{Q}) = \mathbb{P}_F(V_{\mathbb{Q}}).$$

Siegel domain realization

The study of boundary components is intimately related to the realizations of D as a Siegel domain (of the first kind).

Consider the filtration given by I:

$$0 \subset I \subset V$$

of saturated $\mathcal{O}_F$-lattices. Since $\Lambda^2(V) \cong \mathcal{L}$, the lattice

$$U^I = I^{\otimes 2} \otimes_{\mathcal{O}_F} \mathcal{L}$$

acts as square zero elements shifting the filtration by 1. We let the algebraic group $\mathbb{G}_a(U^I_{\mathbb{Q}})$ act unipotently via exponentials of these. Define

P^I	$\{g \in G \mid gI \subseteq I\}$, the parabolic associated with I
	$= \mathbb{G}_m \cdot \mathrm{Res}_{F:\mathbb{Q}} \mathbb{G}_m \cdot \mathbb{G}_a(U_{\mathbb{Q}})$
G^I	$\mathbb{G}_a(U^I_{\mathbb{Q}}) \rtimes \mathbb{G}_m \subseteq P^I$
D^I	$\{\mathcal{O}_F\text{-mixed Hodge structures w.r.t. I}\} \cong U^I_{\mathbb{C}}$

Siegel domain realization / Boundary map

Ad hoc definition:
F^0 $\mathcal{O}_F$-mixed Hodge structures w.r.t. $I \Leftrightarrow F^{0,\rho} \neq I_{\mathbb{C}}^{\rho}\ \forall \rho$.
This condition may also be formulated as $h: \mathbb{S}_{\mathbb{C}} \to \mathrm{GL}_{\mathbb{C}}$ (appropriately defined) factorizing through $G^I(\mathbb{C})$.

Boundary map

We have an inclusion

$$D \to D^I$$

such that

$$D = \{x \in D^I \cong U_{\mathbb{C}}^I \mid \Im(x) \in C^I\}$$

where $C^I \subset U_{\mathbb{R}}^I$ is the cone of totally positive elements. This is a **Siegel domain of the first kind**.
Note: $\Im(x)$ well-defined.

The analytic boundary and the Baily-Borel compactification

On $D \subset D^I \cong U_{\mathbb{C}}^I$ one may define the "distance" to the boundary point I as follows

$$d^I(x) = \frac{1}{|N_{F:\mathbb{Q}} \Im(x)|},$$

where we chose any F-linear identification of $U_{\mathbb{Q}}$ with F.

This distance defines a topology on $\widetilde{D} = D \cap \bigcap_{I \in \mathbb{P}_F(V_{\mathbb{Q}})} I$ such that $\widetilde{D}/G(\mathbb{Z})^+$ becomes the structure of a normal projective (but singular) complex variety, **the Baily-Borel** compactification. Its boundary consists of finitely many *cusps* (class number of $\mathcal{O}_F$)

Summary

For each $I \in \mathbb{P}_F(V_\mathbb{Q})$, we have

$$G \hookleftarrow G^I$$

$$D \hookrightarrow D^I$$

$$D \subseteq D^I = (D^I)^\vee \subseteq D^\vee$$

$D/G(\mathbb{Z})^+$	Hilbert modular variety — want to compactify it.
$D^I/G^I(\mathbb{Z})^+$	is a torus — know how to compactify it!

Toroidal compactification: Glue the closure of $D/G^I(\mathbb{Z})^+$ in $(D^I/G^I(\mathbb{Z})^+)_{\Delta_I}$ for some r.p.c.d Δ_I to $D/G(\mathbb{Z})^+$ via the quotient map.

Analytic investigation of boundary

Given a holomorphic map

$$\mu : B_1^*(0) \to D/G(\mathbb{Z})^+$$

which does not extend to $B_1(0)$.

Lemma

(up to replacing μ by a finite cover)
A monodromy element $x \in G(\mathbb{Z})^+$ (unique up to conjugation) is unipotent.

By the very structure of $G(\mathbb{Z})$, x fixes an $\mathcal{O}_F$-line $I \subset V$, hence $x \in U^I(\mathbb{Z})$ and μ lifts along the map

$$D/G^I(\mathbb{Z})^+ \to D/G(\mathbb{Z})^+.$$

Analytic investigation of boundary

Lemma

x lies automatically in C^I

Sketch: If we consider x as a cocharacter of the torus $D^I/G^I(\mathbb{Z})^+$ then μ "looks like" x via the inclusion $B_1^*(0) \subseteq \mathbb{G}_m(\mathbb{C})$. $\Im(\widetilde{x(z)})) \in U_{\mathbb{R}}^I$ is well defined (independent of the lift to $U_{\mathbb{C}}^I$) and is just $\Im(\widetilde{x(z)})) = (-\frac{1}{2\pi} \log|z|) \cdot x \in C^I$.

We have seen that μ extends to a map $B_1(0) \to (D^I/U^I)_{\Delta^I}$ if and only if $x \in \mathrm{supp}(\Delta^I)$. Hence to compactify $D/G(\mathbb{Z})^+$, the support of Δ^I should cover precisely C^I. Such a Δ^I is called a **rational polyhedral cone decomposition** of C^I. In general it will be infinite.

Toroidal compactification over $\mathbb{C}$

For each I, choose a r.p.c.d. Δ^I of $C^I \subset U^I_{\mathbb{R}}$.
Define $(D/G^I(\mathbb{Z})^+)_{\Delta^I}$ as the closure of $D/G^I(\mathbb{Z})^+$ in $(D^I/G^I(\mathbb{Z})^+)_{\Delta^I}$.

Idea: Construct the quotient by an appropriate equivalence relation on

$$\coprod_I (D/G^I(\mathbb{Z})^+)_{\Delta^I}$$

For a $g \in G(\mathbb{Z})^+$ with $gI = J$, we get a map

$$\widetilde{g} : D^I/G^I(\mathbb{Z})^+ \to D^J/G^J(\mathbb{Z})^+$$

inducing g on

$$D/G^I(\mathbb{Z})^+ \to D/G^J(\mathbb{Z})^+$$

and hence projects to the **identity** on $D/G(\mathbb{Z})^+$.

Toroidal compactification over $\mathbb{C}$

Require that the maps $\widetilde{g}$ extend to maps

$$(D^I/G^I(\mathbb{Z})^+)_{\Delta^I} \to (D^J/G^J(\mathbb{Z})^+)_{\Delta^J}$$

which is equivalent to the conditions:

- Δ^I is invariant under $P^I(\mathbb{Z}) = \{g \in G(\mathbb{Z}) \mid gI = I\}$ (which boils down to invariance under $\mathrm{Res}_{F|\mathbb{Q}}(\mathbb{Z}) = \mathcal{O}_F^*$)
- $\{\Delta^I\}$ is determined by the choice of Δ_{I_k} for representatives $\{I_k\}$ of the ideal classes of F.

Toroidal compactification over $\mathbb{C}$

We define the following equivalence relation on $\coprod_I (D/G^I(\mathbb{Z})^+)_{\Delta^I}$:
$x_I \sim y_J$ if

- x_I and y_J are in the image of the same element $z \in D$ or
- $x_I = \widetilde{g} y_J$ for an elec.ment $g \in G(\mathbb{Z})^+$ with $gI = J$.

Theorem (Hirzebruch, Mumford)

If each Δ^I is smooth, the quotient $(D/G(\mathbb{Z})^+)_\Delta$ of this equivalence relation is a smooth compact analytic orbifold.

By introducing levels and requiring the Δ^I to be projective, one gets smooth projective complex varieties.
Remark: The map $(D/G^I(\mathbb{Z})^+)_{\Delta^I} \to (D/G(\mathbb{Z})^+)_\Delta$ factors through $(D/G^I(\mathbb{Z})^+)_{\Delta^I}/(P^I(\mathbb{Z})/G^I(\mathbb{Z})^+) \simeq (D/G^I(\mathbb{Z})^+)_{\Delta^I}/\mathcal{O}_F^*$.

Hilbert modular varieties

S a scheme over $spec(\mathbb{Q})$

$$X(S) = \left\{ \begin{array}{ll} A & \mathcal{O}_F\text{-abelian scheme over } S \\ \rho: & \mathrm{Hom}^{sym}_{\mathcal{O}_F}(A, {}^tA) \to \mathcal{L}_S \;\; \mathcal{O}_F\text{-iso. of etale sheaves} \\ & \text{mapping polarizations to totally positive elements} \end{array} \right\}$$

defines a Deligne-Mumford stack over $\mathbb{Q}$ with an isomorphism

$$X(\mathbb{C}) \to D/G(\mathbb{Z})^+$$

Recipe: Pullback the natural $\mathcal{O}_F$-Hodge structure on $H^1_{dR}(A)$ along

$$V_\mathbb{C} \xrightarrow{\beta_\mathbb{C}} H_1(A,\mathbb{Z}) \otimes \mathbb{C} \xrightarrow{period} H_1^{dR}(A),$$

where $\beta : V \to H_1(A,\mathbb{Z})$ is an isomorphism compatible with $\mathcal{O}_F$-action and polarization.
Note: β is precisely determined up to $G(\mathbb{Z})^+$.

A simple kind of one-motives

Definition

An one-motive M of dimension $(n, 0, n)$ is a morphism $\alpha : \underline{X} \to T$ from a locally constant etale sheaf of lattices $\underline{X}$ of dimension n to a torus T of dimension n.

Idea: Understand degenerating abelian varieties by representing them as quotients $T/\alpha(\underline{Y})$ where α is (infintesimally) close to the boundary of T (which we understand).
Define ${}^tM := (\alpha' : X^*(T) \to \underline{Y}^* \otimes \mathbb{G}_m)$.
Morphisms are commutative diagrams:

$$\begin{array}{ccc} \underline{Y} & \longrightarrow & \underline{Y}' \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} \\ T & \longrightarrow & T' \end{array}$$

A simple kind one-motives over $\mathbb{C}$

Define ($S = \mathrm{spec}(\mathbb{C})$):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_1(T,\mathbb{Z}) & \longrightarrow & H_1(M,\mathbb{Z}) & \xrightarrow{\pi} & \underline{Y} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow \alpha & & \\
0 & \longrightarrow & H_1(T,\mathbb{Z}) & \longrightarrow & \mathrm{Lie}(T) & \xrightarrow{\exp} & T(\mathbb{C}) & \longrightarrow & 0
\end{array}$$

Define (S arbitrary):

$$H_1^{dR}(M) := \mathrm{Lie}(T) \oplus \underline{Y} \otimes_{\mathbb{Z}} \mathcal{O}_S$$

Have an isomorphism ($S = \mathrm{spec}(\mathbb{C})$):

$$\begin{array}{rcl}
\mathit{period} : H_1(M,\mathbb{Z}) \otimes \mathbb{C} & \to & H_1^{dR}(M) \\
\gamma & \mapsto & (\omega \mapsto \int_\gamma \omega, \pi(\gamma)))
\end{array}$$

“mixed” Hilbert modular varieties

Choose $I \subset V$.

$$X^I(S) = \left\{ \begin{array}{ll} M & \mathcal{O}_F\text{-one-motive over } S \\ \rho: & \mathrm{Hom}^{sym}_{\mathcal{O}_F}(M, {}^tM) \to \mathcal{L} \;\; \mathcal{O}_F\text{-iso. of etale sheaves} \\ \iota: & I_S \to \underline{Y} \;\; \mathcal{O}_F\text{-iso. of etale sheaves} \end{array} \right\} / iso.$$

defines a **split torus** over $\mathbb{Q}$ (with cocharacter group U^I) with isomorphism:

$$X^I(\mathbb{C}) \to D^I / G^I(\mathbb{Z})^+$$

Recipe: Pullback the natural mixed $\mathcal{O}_F$-Hodge structure along:

$$V_{\mathbb{C}} \xrightarrow{\beta_{\mathbb{C}}} H^1(M, \mathbb{Z}) \otimes \mathbb{C} \xrightarrow{period} H^1_{dR}(M)$$

where $\beta : V \to H_1(M, \mathbb{Z})$ is an iso. compatible with $\mathcal{O}_F$-action and respecting the subspaces I pointwise.
Note: β is precisely determined up to $G^I(\mathbb{Z})^+$.

Comparison over $\mathbb{C}$

Over $\mathbb{C}$, given $(M = (\alpha : Y \to T), \rho)$, we can define

$$A := T(\mathbb{C})/\alpha(Y)$$

Then (A, ρ') is a polarized $\mathcal{O}_F$-abelian variety precisely if

(*) $(M, \rho) \in D^I/G^I(\mathbb{Z})^+$ lies actually in $D/G^I(\mathbb{Z})^+$

Under this condition, the map "forget the weight filtration":

$$D^I/G^I(\mathbb{Z})^+ \supset D/G^I(\mathbb{Z})^+ \to D/G(\mathbb{Z})^+$$

maps (M, ρ) to (A', ρ').
We saw: (*) is satisfied, if (A, ρ) is close enough to the boundary in $X^I(\mathbb{C})_{\Delta^I}$.

Algebraic comparison

We can apply the "torus embedding" functor to the *algebraic torus* X^I to get a torus embedding $X^I \to X^I_{\Delta^I}$ even defined over $\mathbb{Q}$.

R	a **complete** discrete valuation ring ($\mathbb{Q}$-algebra)
K	quotient field
$x = (M, \rho, \iota)$	a point in $X^I(K)$ which does not extend to R

(**) The corr. point extends to the partial compactification $(X^I)_{\Delta^I}$.

(this is the obvious algebraic analogue of (*))

Theorem (Mumford)

$$\left\{ \begin{array}{l} (A, \rho) \in X(K) \text{ extending to} \\ \text{an } \mathcal{O}_F\text{-semi-abelian scheme over } R \\ \text{with } A_{R/I} \cong \mathbb{G}_m \otimes I \end{array} \right\} \cong \left\{ \begin{array}{l} (M, \rho', \iota) \in X^I(K) \\ \text{s. t. (**) is satisfied} \end{array} \right\}$$

This construction is compatible with the complex map $D \subset D^I$, e.g. if $R = \mathrm{spec}(\mathbb{C}[[X]])$, $I = (X)$ and the map giving $x = (M, \rho, \iota)$ over R converges on $B_1^*(0)$.

Algebraic toroidal compactification

Theorem (Mumford, Rapoport)

The previous construction (for more general complete rings) can be used to glue an algebraic model X_Δ of $(D/G(\mathbb{Z})^+)_\Delta$ such that there are isomorphisms $\widehat{X^I}_{\Delta^I} \to \widehat{X}_\Delta$ of the formal completions along corresponding boundary strata.
Over $\mathbb{C}$ and in the interior the formal isomorphisms converge locally and give just the map $D/G^I(\mathbb{Z})^+ \to D/G(\mathbb{Z})^+$.