Introduction to theta liftings (following D. Prasad)

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Highbrow motivation:

$$F := \text{number field}$$
$$H, G := \text{reductive } F \text{ groups}$$

Suppose given

$${}^L H \longrightarrow {}^L G.$$

Langlands functoriality:

There exists a corresponding **transfer** of automorphic forms.

Imply, e.g. relationship between L-functions.

Theta liftings give explicit examples.

The Weil representation:

$$k := \text{ field of characteristic } \neq 2$$
$$W := \text{ finite dimensional } k\text{-vector space}$$

Nondegenerate symplectic pairing

$$\langle \cdot, \cdot \rangle : W \times W \longrightarrow k.$$

Heisenberg group: k-group $H(W)$ such that for k-algebras R

$$H(W)(R) := \{(w,t) : w \in W \otimes_k R, t \in R\}$$

where

$$(w_1, t_1) \cdot (w_2, t_2) = (w_1 + w_2, t_1 + t_2 + \tfrac{1}{2}\langle w_1, w_2 \rangle.$$

Have nonsplit central extension

$$0 \longrightarrow R \longrightarrow H(W)(R) \longrightarrow W \otimes_k R \longrightarrow 0$$

Let ρ be a representation of $H(W)(R)$.

$\rho|_R$ yields $\psi : R^+ \to \mathbb{C}^\times$.

Theorem (Stone-von Neumann):

For any nontrivial character $\psi : F_v \to \mathbb{C}^\times$ there exists an irreducible representation

$$\rho_\psi : H(W)(F_v) \longrightarrow \mathrm{Aut}(V_\psi)$$

(unique up to equivalence) on which F_v acts via ψ.

Realization:

$k \in \{F, F_v\}$
$R \in \{F_v, \mathbb{A}_F^S\}$

$W = X \oplus Y$, X, Y isotropic.

Have smooth representation

$$\rho_\psi : H(W)(R) \longrightarrow V_\psi := \mathrm{Aut}(\mathcal{S}(X(R)))$$

where

$$\begin{aligned} \rho_\psi(w_1)f(x) &= f(x + w_1) \\ \rho_\psi(w_2)f(x) &= \psi(\langle x, w_2 \rangle)f(x) \\ \rho_\psi(t)f(x) &= \psi(t)f(x) \end{aligned}$$

for $x, w_1 \in X(R)$, $w_2 \in Y(R)$ and $t \in R$.

Weil representation:

Have action

$$\mathsf{Sp}(W)(R) \times H(W)(R) \longrightarrow H(W)(R)$$
$$(g,(w,t)) \longmapsto (gw,t).$$

By uniqueness of ρ_ψ, $\mathsf{Sp}(W)(R)$ acts on V_{ρ_ψ}: there exists $\omega_\psi(g)$ (unique up to scaling) s.t.

$$\rho_\psi(gw,t) = \omega_\psi(g)\rho_\psi(w,t)\omega_\psi(g)^{-1}$$

Obtain

$$\mathsf{Sp}(W)(R) \longrightarrow \mathsf{PGL}(V_{\rho_\psi})$$

and hence

$$\mathsf{Mp}(W) \longrightarrow \mathsf{GL}(V_{\rho_\psi}).$$

Note $\mathsf{Mp}(W)$ is **not** algebraic.

More details

Recall the equation

$$\rho_\psi(gw,t) = \omega_\psi(g)\rho_\psi(w,t)\omega_\psi(g)^{-1} \qquad (1)$$

Define

$\widehat{\mathsf{Sp}}_\psi(W) := \{(g,\omega_\psi(g)) \in \mathsf{Sp}(W)\times\mathbb{C}^\times : \text{(1) holds}\}$

Have exact sequence

$$1 \to \mathbb{C}^\times \to \widehat{\mathsf{Sp}}_\psi(W) \to \mathsf{Sp}(W) \to 1.$$

Theorem: Restriction to the commutator subgroup yields an exact sequence

$$0 \to \mathbb{Z}/2 \to [\widehat{\mathsf{Sp}}_\psi(W), \widehat{\mathsf{Sp}}_\psi(W)] \to \mathsf{Sp}(W) \to 1$$

and

$$[\widehat{\mathsf{Sp}}_\psi(W), \widehat{\mathsf{Sp}}_\psi(W)] \cong \mathsf{Mp}(W).$$

Realization:

$\mathsf{Sp}(W) \to \mathsf{PGL}(V_\psi)$ given by

$$\begin{pmatrix} A & 0 \\ 0 & A^{-t} \end{pmatrix} f(X) = |\det(A)|^{\frac{1}{2}} f(A^t X) \tag{2}$$

$$\begin{pmatrix} I & B \\ 0 & I \end{pmatrix} f(X) = \psi\left(\frac{X^t B X}{2}\right) f(X) \tag{3}$$

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} f(X) = \gamma \widehat{f}(X) \tag{4}$$

where γ is an 8th root of unity and $\widehat{\ }$ denotes Fourier transform.

Dual reductive pairs:

Let $G_1, G_2 \leq \mathsf{Sp}(W)$ be reductive subgroups.

Definition:

(G_1, G_2) is a **dual reductive pair** if

$$Z_{\mathsf{Sp}(W)}(G_1) = G_2 \text{ and } Z_{\mathsf{Sp}(W)}(G_2) = G_1.$$

Global theta liftings

Let $\psi : F\backslash\mathbb{A}_F \to \mathbb{C}$ be nontrivial. Have a Weil representation

$$\mathsf{Sp}(W)(\mathbb{A}_F) \longrightarrow \mathcal{S}(X(\mathbb{A}_F))$$

Theorem (Weil): The cover

$$\mathsf{Mp}(W)(\mathbb{A}_F) \to \mathsf{Sp}(W)(\mathbb{A}_F)$$

splits over $\mathsf{Sp}(W)(F)$.

Thus $\mathsf{Sp}(W)(F)$ operates on $S(X(\mathbb{A}_F))$.

Define a distribution

$$\begin{aligned}\theta : \mathcal{S}(X(\mathbb{A}_F)) &\longrightarrow \mathbb{C} \\ f &\longmapsto \sum_{x \in X(F)} f(x)\end{aligned}$$

and a function

$$\theta_f(x) := \theta(x \cdot f) : \mathsf{Sp}(W)(F)\backslash \mathsf{Mp}(W)(\mathbb{A}_F) \longrightarrow \mathbb{C}$$

This is the **theta function** attached to f.

$(G_1, G_2) \leq \mathsf{Sp}(W)$:=dual reductive pair.

$\widetilde{G}_i \leq \mathsf{Mp}(W)$:=the inverse images under $\mathsf{Mp}(W) \to \mathsf{Sp}(W)(F)$.

Suppose that $\mathsf{Mp}(W)$ splits over $G_1 \times \widetilde{G}_2$.

Have a map

$$\theta_f : \mathcal{A}(G_1(F)\backslash G_1(\mathbb{A}_F)) \longrightarrow \mathcal{A}(\widetilde{G}_2(F)\backslash \widetilde{G}_2(\mathbb{A}_F))$$

where

$$\theta_f(\phi)(g_2) = \int_{G_1(F)\backslash G_1(\mathbb{A}_F)} \theta_\phi(g_1, g_2) f(g_1) dg_1.$$

The function $\theta_f(\phi)$ is the **theta lift** of ϕ.

Of course, can switch roles of G_1 and G_2 to obtain inverse lifts.

Can restrict the domain to the π_1-isotypic subspace, where π_1 is an automorphic representation of G_1.

Basic question:

When is the space generated by $\theta_f(\phi)$ for $\phi \in \pi_1$ not identically zero?

Shintani's example

V/F :=orthogonal space W/F :=symplectic space

So $V \otimes W$ is a symplectic space

$$(O(V), \mathsf{Sp}(W)) \leq \mathsf{Sp}(W)$$

a dual reductive pair.

Assume $\dim(W) = 2$, isotropic basis e_1, e_2.

$$X = V \otimes e_1,\ Y = V \otimes e_2$$

For $f \in \mathcal{S}(V)$ obtain θ-function

$$\theta_f(g) = \sum_{x \in X(F)} f(xg)$$

Since $\mathsf{Sp}(W) \cong \mathsf{SL}_2$, obtain a theta lifting

$$\theta_f : \mathcal{A}(O(F)\backslash O(\mathbb{A}_F)) \longrightarrow \mathcal{A}(\mathsf{SL}_2(F)\backslash H(\mathbb{A}_F))$$

where $H \in \{\mathsf{SL}_2, \widetilde{\mathsf{SL}}_2\}$.

Concrete examples:

Let $F = \mathbb{Q}$, $q =$quadratic form on V.

(1) $\dim(V) = 1$, $q = x^2$, $O(V) = \{\pm 1\}$.

Obtain classical theta functions.

(2) $\dim(V) = 2$, $q =$norm form of $K/\mathbb{Q}$ ($[K : \mathbb{Q}] = 2$), $O(V) \cong \mathrm{Res}_{K/\mathbb{Q}}\mathrm{GL}_1/\mathrm{GL}_1$.

Lifting constructs automorphic induction of Hecke characters to "CM" or dihedral automorphic representations on SL_2.

(3) $\dim(V) = 3$, $q = x^2 - yz$, $SO(V) \cong \mathrm{PGL}_2$

Inverse of Shimura correspondence

π :=cuspidal representation of $\mathrm{PGL}_2(\mathbb{A}_\mathbb{Q})$
σ :=cuspidal representation of $\widetilde{\mathrm{SL}_2}(\mathbb{A}_\mathbb{Q})$.

Theorem (Waldspurger):

(1) $\theta(\pi) \neq 0$ iff $L(\frac{1}{2}, \pi) \neq 0$.
(2) $\theta(\sigma) \neq 0$ iff σ has a ψ-Whittaker model.