Montreal-Toronto Workshop

on

Hilbert Modular Varieties

Henri Darmon

Fields Institute, Toronto

April 2011

Algebraic Cycles on Hilbert modular varieties

Cycles on Hilbert modular varieties

Cycles on Hilbert modular surfaces

This is partly a survey of joint work with

Pierre Charollois (Paris),

Adam Logan (Ottawa),

Victor Rotger (Barcelona),

Yu Zhao (Montreal).

Algebraic Cycles on Hilbert modular varieties

Cycles on Hilbert modular varieties

Cycles on Hilbert modular surfaces

This is partly a survey of joint work with

Pierre Charollois (Paris),

Adam Logan (Ottawa),

Victor Rotger (Barcelona),

Yu Zhao (Montreal).

Algebraic Cycles on Hilbert modular varieties

Cycles on Hilbert modular varieties

Cycles on Hilbert modular surfaces

This is partly a survey of joint work with

Pierre Charollois (Paris),

Adam Logan (Ottawa),

Victor Rotger (Barcelona),

Yu Zhao (Montreal).

Algebraic Cycles on Hilbert modular varieties

Cycles on Hilbert modular varieties

Cycles on Hilbert modular surfaces

This is partly a survey of joint work with

Pierre Charollois (Paris),

Adam Logan (Ottawa),

Victor Rotger (Barcelona),

Yu Zhao (Montreal).

Special cycles on modular curves

Modular curves (and Shimura curves) are equipped with a rich supply of *arithmetically interesting* topological cycles.

Let $X_0(N)$= modular curve of level N,

$$X_0(N)(\mathbb{C}) = \Gamma_0(N)\backslash\mathcal{H}^*.$$

Special cycles on modular curves

Modular curves (and Shimura curves) are equipped with a rich supply of *arithmetically interesting* topological cycles.

Let $X_0(N)$= modular curve of level N,

$$X_0(N)(\mathbb{C}) = \Gamma_0(N)\backslash\mathcal{H}^*.$$

Quadratic embeddings

The cycles are naturally indexed by embeddings

$$\Psi : K \longrightarrow M_2(\mathbb{Q}),$$

where K is a commutative (quadratic) subring of $\mathbb{C}$.

$$\Sigma := \{\Psi : K \longrightarrow M_2(\mathbb{Q})\}/\Gamma_0(N).$$

$$\mathrm{Disc}(\Psi) = \mathrm{Disc}(\Psi(K) \cap M_0(N)).$$

Let D be a discriminant (not necessarily fundamental):

$$\Sigma_D := \{\Psi \in \Sigma : \mathrm{Disc}(\Psi) = D\}.$$

Quadratic embeddings

The cycles are naturally indexed by embeddings

$$\Psi : K \longrightarrow M_2(\mathbb{Q}),$$

where K is a commutative (quadratic) subring of $\mathbb{C}$.

$$\Sigma := \{\Psi : K \longrightarrow M_2(\mathbb{Q})\}/\Gamma_0(N).$$

$$\mathrm{Disc}(\Psi) = \mathrm{Disc}(\Psi(K) \cap M_0(N)).$$

Let D be a discriminant (not necessarily fundamental):

$$\Sigma_D := \{\Psi \in \Sigma : \mathrm{Disc}(\Psi) = D\}.$$

Quadratic embeddings

The cycles are naturally indexed by embeddings

$$\Psi : K \longrightarrow M_2(\mathbb{Q}),$$

where K is a commutative (quadratic) subring of $\mathbb{C}$.

$$\Sigma := \{\Psi : K \longrightarrow M_2(\mathbb{Q})\}/\Gamma_0(N).$$

$$\mathrm{Disc}(\Psi) = \mathrm{Disc}(\Psi(K) \cap M_0(N)).$$

Let D be a discriminant (not necessarily fundamental):

$$\Sigma_D := \{\Psi \in \Sigma : \mathrm{Disc}(\Psi) = D\}.$$

Quadratic embeddings

The cycles are naturally indexed by embeddings

$$\Psi : K \longrightarrow M_2(\mathbb{Q}),$$

where K is a commutative (quadratic) subring of $\mathbb{C}$.

$$\Sigma := \{\Psi : K \longrightarrow M_2(\mathbb{Q})\}/\Gamma_0(N).$$

$$\mathrm{Disc}(\Psi) = \mathrm{Disc}(\Psi(K) \cap M_0(N)).$$

Let D be a discriminant (not necessarily fundamental):

$$\Sigma_D := \{\Psi \in \Sigma : \mathrm{Disc}(\Psi) = D\}.$$

Some Key Facts

1. The (narrow) class group $G_D = \mathrm{cl}(D)$ acts naturally on Σ_D, without fixed points.
2. $\#\Sigma_D = \#G_D \cdot \#\{I \lhd \mathcal{O}_D : \mathcal{O}_D/I \simeq \mathbb{Z}/N\mathbb{Z}\}$.

Goal: Associate to each $\Psi \in \Sigma$ a (topological) cycle

$$\Delta_\Psi \subset X_0(N)(\mathbb{C}).$$

Some Key Facts

1. The (narrow) class group $G_D = \mathrm{cl}(D)$ acts naturally on Σ_D, without fixed points.
2. $\#\Sigma_D = \#G_D \cdot \#\{I \lhd \mathcal{O}_D : \mathcal{O}_D/I \simeq \mathbb{Z}/N\mathbb{Z}\}$.

Goal: Associate to each $\Psi \in \Sigma$ a (topological) cycle

$$\Delta_\Psi \subset X_0(N)(\mathbb{C}).$$

Some Key Facts

1. The (narrow) class group $G_D = \mathrm{cl}(D)$ acts naturally on Σ_D, without fixed points.
2. $\#\Sigma_D = \#G_D \cdot \#\{I \lhd \mathcal{O}_D : \mathcal{O}_D/I \simeq \mathbb{Z}/N\mathbb{Z}\}$.

Goal: Associate to each $\Psi \in \Sigma$ a (topological) cycle

$$\Delta_\Psi \subset X_0(N)(\mathbb{C}).$$

The cycle Δ_Ψ when $D < 0$: CM points.

The rational torus

$$\Psi(K^\times) \subset \mathbf{GL}_2(\mathbb{Q}) \circlearrowleft \mathcal{H}$$

has a unique fixed point $\tau_\Psi \in \mathcal{H}$. We set

$$\boxed{\Delta_\Psi := \{\tau_\Psi\}.}$$

The point τ_Ψ is a *CM point* on $X_0(N)$ (of discriminant D).

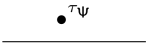
The cycle Δ_Ψ when $D < 0$: CM points.

The rational torus

$$\Psi(K^\times) \subset \mathbf{GL}_2(\mathbb{Q}) \circlearrowleft \mathcal{H}$$

has a unique fixed point $\tau_\Psi \in \mathcal{H}$. We set

$$\boxed{\Delta_\Psi := \{\tau_\Psi\}.}$$



The point τ_Ψ is a *CM point* on $X_0(N)$ (of discriminant D).

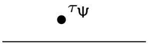
The cycle Δ_Ψ when $D < 0$: CM points.

The rational torus

$$\Psi(K^\times) \subset \mathbf{GL}_2(\mathbb{Q}) \circlearrowleft \mathcal{H}$$

has a unique fixed point $\tau_\Psi \in \mathcal{H}$. We set

$$\boxed{\Delta_\Psi := \{\tau_\Psi\}.}$$



The point τ_Ψ is a *CM point* on $X_0(N)$ (of discriminant D).

The cycle Δ_Ψ when $D = m^2$: modular symbols.

When $D = m^2$, the $\mathbb{Q}$-split torus $\Psi(K^\times)$ has two fixed points τ_Ψ and τ'_Ψ in $\mathbb{P}_1(\mathbb{Q}) \subset \mathcal{H}^*$.

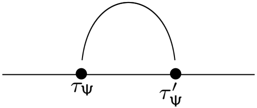
$\Delta_\Psi :=$ Geodesic joining τ_Ψ to τ'_Ψ.

The cycle $\Delta_\Psi \subset X_0(N)(\mathbb{C})$ of real dimension one is called a *modular symbol* (of conductor m).

The cycle Δ_Ψ when $D = m^2$: modular symbols.

When $D = m^2$, the $\mathbb{Q}$-split torus $\Psi(K^\times)$ has two fixed points τ_Ψ and τ'_Ψ in $\mathbb{P}_1(\mathbb{Q}) \subset \mathcal{H}^*$.

$$\boxed{\Delta_\Psi := \text{Geodesic joining } \tau_\Psi \text{ to } \tau'_\Psi.}$$

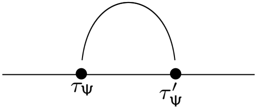


The cycle $\Delta_\Psi \subset X_0(N)(\mathbb{C})$ of real dimension one is called a *modular symbol* (of conductor m).

The cycle Δ_Ψ when $D = m^2$: modular symbols.

When $D = m^2$, the $\mathbb{Q}$-split torus $\Psi(K^\times)$ has two fixed points τ_Ψ and τ'_Ψ in $\mathbb{P}_1(\mathbb{Q}) \subset \mathcal{H}^*$.

$$\boxed{\Delta_\Psi := \text{Geodesic joining } \tau_\Psi \text{ to } \tau'_\Psi.}$$



The cycle $\Delta_\Psi \subset X_0(N)(\mathbb{C})$ of real dimension one is called a *modular symbol* (of conductor m).

The cycle Δ_Ψ when $D > 0$, $D \neq m^2$: Shintani cycles.

In the remaining cases where $D > 0$, the torus $\Psi(K^\times)$ has two fixed points τ_Ψ, τ'_Ψ in $\mathbb{P}_1(\mathbb{R}) - \mathbb{P}_1(\mathbb{Q})$.

$\Delta_\Psi :=$ Image of geodesic joining τ_Ψ to τ'_Ψ.

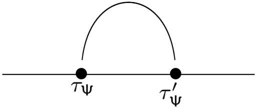
The cycle $\Delta_\Psi \subset X_0(N)(\mathbb{C})$ is closed, of real dimension one.

It is called a *Shintani cycle* (of discriminant D).

The cycle Δ_Ψ when $D > 0$, $D \neq m^2$: Shintani cycles.

In the remaining cases where $D > 0$, the torus $\Psi(K^\times)$ has two fixed points τ_Ψ, τ'_Ψ in $\mathbb{P}_1(\mathbb{R}) - \mathbb{P}_1(\mathbb{Q})$.

$$\boxed{\Delta_\Psi := \text{Image of geodesic joining } \tau_\Psi \text{ to } \tau'_\Psi.}$$



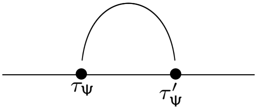
The cycle $\Delta_\Psi \subset X_0(N)(\mathbb{C})$ is closed, of real dimension one.

It is called a *Shintani cycle* (of discriminant D).

The cycle Δ_Ψ when $D > 0$, $D \neq m^2$: Shintani cycles.

In the remaining cases where $D > 0$, the torus $\Psi(K^\times)$ has two fixed points τ_Ψ, τ'_Ψ in $\mathbb{P}_1(\mathbb{R}) - \mathbb{P}_1(\mathbb{Q})$.

$$\boxed{\Delta_\Psi := \text{Image of geodesic joining } \tau_\Psi \text{ to } \tau'_\Psi.}$$



The cycle $\Delta_\Psi \subset X_0(N)(\mathbb{C})$ is closed, of real dimension one.

It is called a *Shintani cycle* (of discriminant D).

Some more definitions

Let $\chi : G_D \longrightarrow \mathbb{C}^\times$ be a (not necessarily quadratic!) character.

$$\Delta_{D,\chi} := \begin{cases} 0 & \text{if } \Sigma_D = \emptyset \\ \sum_{\sigma \in G_D} \chi(\sigma) \Delta_{\Psi^\sigma} & \text{with } \Psi \in \Sigma_D. \end{cases}$$

Important special case: χ is quadratic, i.e., a genus character. It cuts out a bi-quadratic extension $\mathbb{Q}(\sqrt{D_1}, \sqrt{D_2})$ where $D = D_1 D_2$.

$$\Delta_{D_1,D_2} := \Delta_{D,\chi}.$$

Some more definitions

Let $\chi : G_D \longrightarrow \mathbb{C}^\times$ be a (not necessarily quadratic!) character.

$$\Delta_{D,\chi} := \begin{cases} 0 & \text{if } \Sigma_D = \emptyset \\ \sum_{\sigma \in G_D} \chi(\sigma)\Delta_{\Psi^\sigma} & \text{with } \Psi \in \Sigma_D. \end{cases}$$

Important special case: χ is quadratic, i.e., a genus character. It cuts out a bi-quadratic extension $\mathbb{Q}(\sqrt{D_1}, \sqrt{D_2})$ where $D = D_1 D_2$.

$$\Delta_{D_1,D_2} := \Delta_{D,\chi}.$$

Periods attached to $\Delta_{D,\chi}$ when $D > 0$

Let $f \in S_2(\Gamma_0(N))$ be a newform of weight two.

$$\omega_f := 2\pi i f(z)dz = f(q)\frac{dq}{q} \in \Omega^1(X_0(N)/K_f).$$

We attach to f and the cycle $\Delta_{D,\chi}$ a *period*

$$\int_{\Delta_{D,\chi}} \omega_f \quad \in \quad \Lambda_{f,\chi}.$$

Let $L(f/K_D, \chi, s)$= Hasse-Weil L-series attached to f and $\chi \in G_D^\vee$.

Convention: if $D = m^2$ we set

$$L(f/K_D, \chi, s) = L(f, \chi, s)L(f, \bar{\chi}, s).$$

Periods attached to $\Delta_{D,\chi}$ when $D > 0$

Let $f \in S_2(\Gamma_0(N))$ be a newform of weight two.

$$\omega_f := 2\pi i f(z)dz = f(q)\frac{dq}{q} \in \Omega^1(X_0(N)/K_f).$$

We attach to f and the cycle $\Delta_{D,\chi}$ a *period*

$$\int_{\Delta_{D,\chi}} \omega_f \quad \in \quad \Lambda_{f,\chi}.$$

Let $L(f/K_D, \chi, s)$= Hasse-Weil L-series attached to f and $\chi \in G_D^\vee$.

Convention: if $D = m^2$ we set

$$L(f/K_D, \chi, s) = L(f, \chi, s)L(f, \bar{\chi}, s).$$

Periods attached to $\Delta_{D,\chi}$ when $D > 0$

Let $f \in S_2(\Gamma_0(N))$ be a newform of weight two.

$$\omega_f := 2\pi i f(z)dz = f(q)\frac{dq}{q} \in \Omega^1(X_0(N)/K_f).$$

We attach to f and the cycle $\Delta_{D,\chi}$ a *period*

$$\int_{\Delta_{D,\chi}} \omega_f \quad \in \quad \Lambda_{f,\chi}.$$

Let $L(f/K_D, \chi, s)$= Hasse-Weil L-series attached to f and $\chi \in G_D^\vee$.

Convention: if $D = m^2$ we set

$$L(f/K_D, \chi, s) = L(f, \chi, s)L(f, \bar{\chi}, s).$$

Periods attached to $\Delta_{D,\chi}$ when $D > 0$

Let $f \in S_2(\Gamma_0(N))$ be a newform of weight two.

$$\omega_f := 2\pi i f(z)dz = f(q)\frac{dq}{q} \in \Omega^1(X_0(N)/K_f).$$

We attach to f and the cycle $\Delta_{D,\chi}$ a *period*

$$\int_{\Delta_{D,\chi}} \omega_f \quad \in \quad \Lambda_{f,\chi}.$$

Let $L(f/K_D, \chi, s)$= Hasse-Weil L-series attached to f and $\chi \in G_D^\vee$.

Convention: if $D = m^2$ we set

$$L(f/K_D, \chi, s) = L(f, \chi, s)L(f, \bar{\chi}, s).$$

Relation with special values of L-series (the case $D > 0$).

Theorem

Let D be a positive discriminant.

1. *If $\Sigma_D \neq \emptyset$, then $L(f/K_D, \chi, s)$ vanishes to* even order *at $s = 1$ for all $\chi \in G_D^\vee$.*
2. *In that case,*

$$\left| \int_{\Delta_{D,\chi}} \omega_f \right|^2 = L(f/K_D, \chi, 1) \pmod{(K_f K_\chi)^\times}.$$

Heegner points attached to $\Delta_{D,\chi}$ when $D < 0$

The zero-dimensional cycles $\Delta_{D,\chi}$ are *homologically trivial* when $\chi \neq 1$.

$$J_{D,\chi} := \mathrm{AJ}(\Delta_{D,\chi}) = \int_{\partial^{-1}(\Delta_{D,\chi})} \omega_f \in \mathbb{C}/(\Lambda_f \otimes \mathbb{Z}(\chi)).$$

Assume for simplicity that $K_f = \mathbb{Q}$. Then f corresponds to a modular elliptic curve $E_f/\mathbb{Q}$ and $\mathbb{C}/\Lambda_f \sim E_f(\mathbb{C})$. We can view $J_{D,\chi}$ as a point, denoted $P_{D,\chi}$, in $E_f(\mathbb{C}) \otimes_Z \mathbb{Z}[\chi]$.

Theory of complex multiplication $\Rightarrow$ the point $P_{D,\chi}$ belongs to $E_f(H_D) \otimes \mathbb{Z}[\chi]$.

Heegner points attached to $\Delta_{D,\chi}$ when $D < 0$

The zero-dimensional cycles $\Delta_{D,\chi}$ are *homologically trivial* when $\chi \neq 1$.

$$J_{D,\chi} := \mathrm{AJ}(\Delta_{D,\chi}) = \int_{\partial^{-1}(\Delta_{D,\chi})} \omega_f \in \mathbb{C}/(\Lambda_f \otimes \mathbb{Z}(\chi)).$$

Assume for simplicity that $K_f = \mathbb{Q}$. Then f corresponds to a modular elliptic curve $E_f/\mathbb{Q}$ and $\mathbb{C}/\Lambda_f \sim E_f(\mathbb{C})$. We can view $J_{D,\chi}$ as a point, denoted $P_{D,\chi}$, in $E_f(\mathbb{C}) \otimes_Z \mathbb{Z}[\chi]$.

Theory of complex multiplication $\Rightarrow$ the point $P_{D,\chi}$ belongs to $E_f(H_D) \otimes \mathbb{Z}[\chi]$.

Heegner points attached to $\Delta_{D,\chi}$ when $D < 0$

The zero-dimensional cycles $\Delta_{D,\chi}$ are *homologically trivial* when $\chi \neq 1$.

$$J_{D,\chi} := \mathrm{AJ}(\Delta_{D,\chi}) = \int_{\partial^{-1}(\Delta_{D,\chi})} \omega_f \in \mathbb{C}/(\Lambda_f \otimes \mathbb{Z}(\chi)).$$

Assume for simplicity that $K_f = \mathbb{Q}$. Then f corresponds to a modular elliptic curve $E_f/\mathbb{Q}$ and $\mathbb{C}/\Lambda_f \sim E_f(\mathbb{C})$. We can view $J_{D,\chi}$ as a point, denoted $P_{D,\chi}$, in $E_f(\mathbb{C}) \otimes_Z \mathbb{Z}[\chi]$.

Theory of complex multiplication $\Rightarrow$ the point $P_{D,\chi}$ belongs to $E_f(H_D) \otimes \mathbb{Z}[\chi]$.

Heegner points attached to $\Delta_{D,\chi}$ when $D < 0$

The zero-dimensional cycles $\Delta_{D,\chi}$ are *homologically trivial* when $\chi \neq 1$.

$$J_{D,\chi} := \mathrm{AJ}(\Delta_{D,\chi}) = \int_{\partial^{-1}(\Delta_{D,\chi})} \omega_f \in \mathbb{C}/(\Lambda_f \otimes \mathbb{Z}(\chi)).$$

Assume for simplicity that $K_f = \mathbb{Q}$. Then f corresponds to a modular elliptic curve $E_f/\mathbb{Q}$ and $\mathbb{C}/\Lambda_f \sim E_f(\mathbb{C})$. We can view $J_{D,\chi}$ as a point, denoted $P_{D,\chi}$, in $E_f(\mathbb{C}) \otimes_Z \mathbb{Z}[\chi]$.

Theory of complex multiplication $\Rightarrow$ the point $P_{D,\chi}$ belongs to $E_f(H_D) \otimes \mathbb{Z}[\chi]$.

Heegner points attached to $\Delta_{D,\chi}$ when $D < 0$

The zero-dimensional cycles $\Delta_{D,\chi}$ are *homologically trivial* when $\chi \neq 1$.

$$J_{D,\chi} := \mathrm{AJ}(\Delta_{D,\chi}) = \int_{\partial^{-1}(\Delta_{D,\chi})} \omega_f \in \mathbb{C}/(\Lambda_f \otimes \mathbb{Z}(\chi)).$$

Assume for simplicity that $K_f = \mathbb{Q}$. Then f corresponds to a modular elliptic curve $E_f/\mathbb{Q}$ and $\mathbb{C}/\Lambda_f \sim E_f(\mathbb{C})$. We can view $J_{D,\chi}$ as a point, denoted $P_{D,\chi}$, in $E_f(\mathbb{C}) \otimes_Z \mathbb{Z}[\chi]$.

Theory of complex multiplication $\Rightarrow$ the point $P_{D,\chi}$ belongs to $E_f(H_D) \otimes \mathbb{Z}[\chi]$.

Relation with derivatives of L-series (the case $D < 0$).

Theorem (Gross-Zagier, Zhang)

Let D be a negative discriminant.

1. *If $\Sigma_D \neq \emptyset$, then $L(f/K_D, \chi, s)$ vanishes to* odd order *at $s = 1$ for all $\chi \in G_D^\vee$.*
2. *In that case,*

$$\langle P_{D,\chi}, P_{D,\bar{\chi}} \rangle = L'(f/K_D, \chi, 1) \pmod{(K_f K_\chi)^\times}.$$

Application to elliptic curves

Let E be a modular elliptic curve, attached to an eigenform $f \in S_2(\Gamma_0(N))$.

Theorem (Kolyvagin)

Assume that $D < 0$ and that $\Sigma_D \neq \emptyset$. If $P_{D,\chi} \neq 0$ in $E(H_D) \otimes \mathbb{Q}(\chi)$, then $(E(H_D) \otimes \mathbb{Q}(\chi))^\chi$ is spanned by $P_{D,\chi}$ and the corresponding (χ part of) the Shafarevich-Tate group is finite.

Conclusion: Heegner points give us a *tight control* on the arithmetic of elliptic curves over class fields of *imaginary quadratic fields.*

Application to elliptic curves

Let E be a modular elliptic curve, attached to an eigenform $f \in S_2(\Gamma_0(N))$.

Theorem (Kolyvagin)

Assume that $D < 0$ and that $\Sigma_D \neq \emptyset$. If $P_{D,\chi} \neq 0$ in $E(H_D) \otimes \mathbb{Q}(\chi)$, then $(E(H_D) \otimes \mathbb{Q}(\chi))^\chi$ is spanned by $P_{D,\chi}$ and the corresponding (χ part of) the Shafarevich-Tate group is finite.

Conclusion: Heegner points give us a *tight control* on the arithmetic of elliptic curves over class fields of *imaginary quadratic fields.*

Application to elliptic curves

Let E be a modular elliptic curve, attached to an eigenform $f \in S_2(\Gamma_0(N))$.

Theorem (Kolyvagin)

Assume that $D < 0$ and that $\Sigma_D \neq \emptyset$. If $P_{D,\chi} \neq 0$ in $E(H_D) \otimes \mathbb{Q}(\chi)$, then $(E(H_D) \otimes \mathbb{Q}(\chi))^{\chi}$ is spanned by $P_{D,\chi}$ and the corresponding (χ part of) the Shafarevich-Tate group is finite.

Conclusion: Heegner points give us a *tight control* on the arithmetic of elliptic curves over class fields of *imaginary quadratic fields.*

A tantalising open question when $D > 0$

Question

$$\int_{\Delta_{D,\chi}} \omega_f \neq 0 \overset{?}{\Longrightarrow} (E(H_D) \otimes \mathbb{Z}[\chi])^\chi, Ш(E/H_D)^\chi < \infty.$$

Possible strategy (ongoing work in progress with V. Rotger and I. Sols; cf. my AWS lectures) based on

1. Diagonal "Gross-Kudla-Schoen" cycles on triple products of modular curves;
2. p-adic deformations (à la Hida) of the images of these cycles under p-adic étale Abel-Jacobi maps.

To be discussed at next year's Toronto-Montreal meeting devoted to algebraic cycles!

A tantalising open question when $D > 0$

Question

$$\int_{\Delta_{D,\chi}} \omega_f \neq 0 \stackrel{?}{\Longrightarrow} (E(H_D)\otimes \mathbb{Z}[\chi])^{\chi}, Ш(E/H_D)^{\chi} < \infty.$$

Possible strategy (ongoing work in progress with V. Rotger and I. Sols; cf. my AWS lectures) based on

1. Diagonal "Gross-Kudla-Schoen" cycles on triple products of modular curves;
2. p-adic deformations (à la Hida) of the images of these cycles under p-adic étale Abel-Jacobi maps.

To be discussed at next year's Toronto-Montreal meeting devoted to algebraic cycles!

A tantalising open question when $D > 0$

Question

$$\int_{\Delta_{D,\chi}} \omega_f \neq 0 \stackrel{?}{\Longrightarrow} (E(H_D) \otimes \mathbb{Z}[\chi])^{\chi}, Ш(E/H_D)^{\chi} < \infty.$$

Possible strategy (ongoing work in progress with V. Rotger and I. Sols; cf. my AWS lectures) based on

1. Diagonal "Gross-Kudla-Schoen" cycles on triple products of modular curves;
2. p-adic deformations (à la Hida) of the images of these cycles under p-adic étale Abel-Jacobi maps.

To be discussed at next year's Toronto-Montreal meeting devoted to algebraic cycles!

A tantalising open question when $D > 0$

Question

$$\int_{\Delta_{D,\chi}} \omega_f \neq 0 \stackrel{?}{\Longrightarrow} (E(H_D) \otimes \mathbb{Z}[\chi])^{\chi}, Ш(E/H_D)^{\chi} < \infty.$$

Possible strategy (ongoing work in progress with V. Rotger and I. Sols; cf. my AWS lectures) based on

1. Diagonal "Gross-Kudla-Schoen" cycles on triple products of modular curves;
2. p-adic deformations (à la Hida) of the images of these cycles under p-adic étale Abel-Jacobi maps.

To be discussed at next year's Toronto-Montreal meeting devoted to algebraic cycles!

A tantalising open question when $D > 0$

Question

$$\int_{\Delta_{D,\chi}} \omega_f \neq 0 \stackrel{?}{\Longrightarrow} (E(H_D) \otimes \mathbb{Z}[\chi])^\chi, Ш(E/H_D)^\chi < \infty.$$

Possible strategy (ongoing work in progress with V. Rotger and I. Sols; cf. my AWS lectures) based on

1. Diagonal "Gross-Kudla-Schoen" cycles on triple products of modular curves;
2. p-adic deformations (à la Hida) of the images of these cycles under p-adic étale Abel-Jacobi maps.

To be discussed at next year's Toronto-Montreal meeting devoted to algebraic cycles!

Algebraic cycles on Hilbert modular surfaces

F= real quadratic field. $v_1, v_2 : F \longrightarrow \mathbb{R}$. Set $x_j := v_j(x)$.

X=associated Hilbert modular surface.

$$X(\mathbb{C}) = (\text{Compactification of}) \ \mathbf{SL}_2(\mathcal{O}_F)\backslash\mathcal{H} \times \mathcal{H}.$$

The surface X contains an interesting supply of *algebraic cycles*.

1. Codimension 2: CM points.
2. Codimension 1: Hirzebruch-Zagier divisors.

We will probably hear more about these in the lectures by Kumar and Steve tomorrow.

Algebraic cycles on Hilbert modular surfaces

F= real quadratic field. $v_1, v_2 : F \longrightarrow \mathbb{R}$. Set $x_j := v_j(x)$.

X=associated Hilbert modular surface.

$$X(\mathbb{C}) = (\text{Compactification of}) \ \mathbf{SL}_2(\mathcal{O}_F)\backslash\mathcal{H} \times \mathcal{H}.$$

The surface X contains an interesting supply of *algebraic cycles*.

1. Codimension 2: CM points.
2. Codimension 1: Hirzebruch-Zagier divisors.

We will probably hear more about these in the lectures by Kumar and Steve tomorrow.

Algebraic cycles on Hilbert modular surfaces

F= real quadratic field. $v_1, v_2 : F \longrightarrow \mathbb{R}$. Set $x_j := v_j(x)$.

X=associated Hilbert modular surface.

$$X(\mathbb{C}) = \text{(Compactification of) } \mathbf{SL}_2(\mathcal{O}_F)\backslash\mathcal{H} \times \mathcal{H}.$$

The surface X contains an interesting supply of *algebraic cycles*.

1. Codimension 2: CM points.
2. Codimension 1: Hirzebruch-Zagier divisors.

We will probably hear more about these in the lectures by Kumar and Steve tomorrow.

Algebraic cycles on Hilbert modular surfaces

F= real quadratic field. $v_1, v_2 : F \longrightarrow \mathbb{R}$. Set $x_j := v_j(x)$.

X=associated Hilbert modular surface.

$$X(\mathbb{C}) = \text{(Compactification of) } \mathbf{SL}_2(\mathcal{O}_F)\backslash\mathcal{H} \times \mathcal{H}.$$

The surface X contains an interesting supply of *algebraic cycles*.

1. Codimension 2: CM points.
2. Codimension 1: Hirzebruch-Zagier divisors.

We will probably hear more about these in the lectures by Kumar and Steve tomorrow.

Algebraic cycles on Hilbert modular surfaces

F= real quadratic field. $v_1, v_2 : F \longrightarrow \mathbb{R}$. Set $x_j := v_j(x)$.

X=associated Hilbert modular surface.

$$X(\mathbb{C}) = (\text{Compactification of}) \ \mathbf{SL}_2(\mathcal{O}_F)\backslash\mathcal{H} \times \mathcal{H}.$$

The surface X contains an interesting supply of *algebraic cycles*.

1. Codimension 2: CM points.
2. Codimension 1: Hirzebruch-Zagier divisors.

We will probably hear more about these in the lectures by Kumar and Steve tomorrow.

Cycles on Hilbert modular surfaces

I will focus on cycles that are very analogous to Shintani cycles, in the four-manifold $X(\mathbb{C})$. They are indexed by F-algebra embeddings

$$\Psi : K \longrightarrow M_2(F),$$

where $K = F(\sqrt{D})$ is a quadratic extension of F.

There are now *three cases* to consider.

1. $D_1, D_2 > 0$: the totally real case.
2. $D_1, D_2 < 0$: the complex multiplication (CM) case.
3. $D_1 < 0, D_2 > 0$: the "almost totally real" (ATR) case.

Cycles on Hilbert modular surfaces

I will focus on cycles that are very analogous to Shintani cycles, in the four-manifold $X(\mathbb{C})$. They are indexed by F-algebra embeddings

$$\Psi : K \longrightarrow M_2(F),$$

where $K = F(\sqrt{D})$ is a quadratic extension of F.

There are now *three cases* to consider.

1. $D_1, D_2 > 0$: the totally real case.
2. $D_1, D_2 < 0$: the complex multiplication (CM) case.
3. $D_1 < 0, D_2 > 0$: the "almost totally real" (ATR) case.

Cycles on Hilbert modular surfaces

I will focus on cycles that are very analogous to Shintani cycles, in the four-manifold $X(\mathbb{C})$. They are indexed by F-algebra embeddings

$$\Psi : K \longrightarrow M_2(F),$$

where $K = F(\sqrt{D})$ is a quadratic extension of F.

There are now *three cases* to consider.

1. $D_1, D_2 > 0$: the totally real case.
2. $D_1, D_2 < 0$: the complex multiplication (CM) case.
3. $D_1 < 0, D_2 > 0$: the "almost totally real" (ATR) case.

Cycles on Hilbert modular surfaces

I will focus on cycles that are very analogous to Shintani cycles, in the four-manifold $X(\mathbb{C})$. They are indexed by F-algebra embeddings

$$\Psi : K \longrightarrow M_2(F),$$

where $K = F(\sqrt{D})$ is a quadratic extension of F.

There are now *three cases* to consider.

1. $D_1, D_2 > 0$: the totally real case.

2. $D_1, D_2 < 0$: the complex multiplication (CM) case.

3. $D_1 < 0, D_2 > 0$: the "almost totally real" (ATR) case.

Cycles on Hilbert modular surfaces

I will focus on cycles that are very analogous to Shintani cycles, in the four-manifold $X(\mathbb{C})$. They are indexed by F-algebra embeddings

$$\Psi : K \longrightarrow M_2(F),$$

where $K = F(\sqrt{D})$ is a quadratic extension of F.

There are now *three cases* to consider.

1. $D_1, D_2 > 0$: the totally real case.

2. $D_1, D_2 < 0$: the complex multiplication (CM) case.

3. $D_1 < 0, D_2 > 0$: the "almost totally real" (ATR) case.

The totally real case

For $j = 1, 2$,

$$\Psi(K \otimes_{v_j} \mathbb{R})^\times \text{ has } \textit{two fixed points } \tau_j, \tau_j' \in \mathbb{R}.$$

Let $\Upsilon_j :=$ geodesic from τ_j to τ_j'.

$$\boxed{\Delta_\Psi := \text{Image of } \Upsilon_1 \times \Upsilon_2 \text{ in } X(\mathbb{C}).}$$

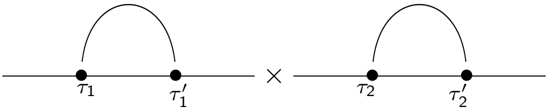
The totally real case

For $j = 1, 2$,

$$\Psi(K \otimes_{v_j} \mathbb{R})^\times \text{ has } \textit{two fixed points } \tau_j, \tau_j' \in \mathbb{R}.$$

Let $\Upsilon_j :=$ geodesic from τ_j to τ_j'.

$$\boxed{\Delta_\Psi := \text{Image of } \Upsilon_1 \times \Upsilon_2 \text{ in } X(\mathbb{C}).}$$



The CM case

For $j = 1, 2$,

$$(\Psi(K) \otimes_{v_j} \mathbb{R})^\times \text{ has } \textit{a single fixed point } \tau_j \in \mathcal{H}.$$

$$\boxed{\Delta_\Psi := \{(\tau_1, \tau_2)\}}$$

The CM case

For $j = 1, 2$,

$$(\Psi(K) \otimes_{v_j} \mathbb{R})^\times \text{ has } \textit{a single fixed point } \tau_j \in \mathcal{H}.$$

$$\boxed{\Delta_\Psi := \{(\tau_1, \tau_2)\}}$$

The ATR case

$\tau_1 :=$ fixed point of $\Psi(K^\times) \circlearrowleft \mathcal{H}_1$;

$\tau_2, \tau_2' :=$ fixed points of $\Psi(K^\times) \circlearrowleft (\mathcal{H}_2 \cup \mathbb{R})$;

$$\boxed{\Delta_\Psi = \{\tau_1\} \times \text{geodesic}(\tau_2 \to \tau_2').}$$

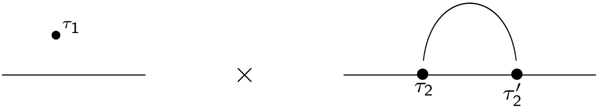
Key fact: The cycles $\Delta_\Psi \subset X(\mathbb{C})$ are *null-homologous*.

The ATR case

$\tau_1 :=$ fixed point of $\Psi(K^\times) \circlearrowleft \mathcal{H}_1$;

$\tau_2, \tau_2' :=$ fixed points of $\Psi(K^\times) \circlearrowleft (\mathcal{H}_2 \cup \mathbb{R})$;

$$\boxed{\Delta_\Psi = \{\tau_1\} \times \text{geodesic}(\tau_2 \to \tau_2').}$$



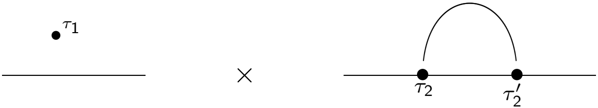
Key fact: The cycles $\Delta_\Psi \subset X(\mathbb{C})$ are *null-homologous*.

The ATR case

$\tau_1 :=$ fixed point of $\Psi(K^\times) \circlearrowleft \mathcal{H}_1$;

$\tau_2, \tau_2' :=$ fixed points of $\Psi(K^\times) \circlearrowleft (\mathcal{H}_2 \cup \mathbb{R})$;

$$\boxed{\Delta_\Psi = \{\tau_1\} \times \text{geodesic}(\tau_2 \to \tau_2').}$$



Key fact: The cycles $\Delta_\Psi \subset X(\mathbb{C})$ are *null-homologous*.

Elliptic curves

Let E be an elliptic curve over F, of *conductor* 1.

Simplifying Assumptions: $h^+(F) = 1$, $N = 1$.

Counting points mod $\mathfrak{p}$ yields $\mathfrak{n} \mapsto a(\mathfrak{n}) \in \mathbb{Z}$, on the integral ideals of $\mathcal{O}_F$.

Generating series

$$G(z_1, z_2) := \sum_{n>>0} a((n)) e^{2\pi i \left(\frac{n_1}{d_1} z_1 + \frac{n_2}{d_2} z_2\right)},$$

where $d :=$ totally positive generator of the different of F.

Elliptic curves

Let E be an elliptic curve over F, of *conductor* 1.

Simplifying Assumptions: $h^+(F) = 1$, $N = 1$.

Counting points mod $\mathfrak{p}$ yields $\mathfrak{n} \mapsto a(\mathfrak{n}) \in \mathbb{Z}$, on the integral ideals of $\mathcal{O}_F$.

Generating series

$$G(z_1, z_2) := \sum_{n>>0} a((n)) e^{2\pi i \left(\frac{n_1}{d_1} z_1 + \frac{n_2}{d_2} z_2 \right)},$$

where $d :=$ totally positive generator of the different of F.

Elliptic curves

Let E be an elliptic curve over F, of *conductor* 1.

Simplifying Assumptions: $h^+(F) = 1$, $N = 1$.

Counting points mod $\mathfrak{p}$ yields $\mathfrak{n} \mapsto a(\mathfrak{n}) \in \mathbb{Z}$, on the integral ideals of $\mathcal{O}_F$.

Generating series

$$G(z_1, z_2) := \sum_{n>>0} a((n))e^{2\pi i\left(\frac{n_1}{d_1}z_1+\frac{n_2}{d_2}z_2\right)},$$

where $d :=$ totally positive generator of the different of F.

Elliptic curves

Let E be an elliptic curve over F, of *conductor* 1.

Simplifying Assumptions: $h^+(F) = 1$, $N = 1$.

Counting points mod $\mathfrak{p}$ yields $\mathfrak{n} \mapsto a(\mathfrak{n}) \in \mathbb{Z}$, on the integral ideals of $\mathcal{O}_F$.

Generating series

$$G(z_1, z_2) := \sum_{n>>0} a((n)) e^{2\pi i \left(\frac{n_1}{d_1} z_1 + \frac{n_2}{d_2} z_2 \right)},$$

where $d :=$ totally positive generator of the different of F.

Modularity

Definition

The elliptic curve E is said to be *modular* if G is a Hilbert modular form of weight $(2, 2)$:

$$G(\gamma_1 z_1, \gamma_2 z_2) = (c_1 z_1 + d_2)^2 (c_2 z_2 + d_2)^2 G(z_1, z_2),$$

for all

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathcal{O}_F).$$

We will assume that E is modular in this sense. (This is known to be true, in many cases.)

Modularity

Definition

The elliptic curve E is said to be *modular* if G is a Hilbert modular form of weight $(2,2)$:

$$G(\gamma_1 z_1, \gamma_2 z_2) = (c_1 z_1 + d_2)^2 (c_2 z_2 + d_2)^2 G(z_1, z_2),$$

for all

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathcal{O}_F).$$

We will assume that E is modular in this sense. (This is known to be true, in many cases.)

Geometric formulation of modularity

The differential form

$$\alpha_G := G(z_1, z_2) dz_1 dz_2$$

is a *holomorphic* (hence closed) 2-form on

$$X(\mathbb{C}) := \mathbf{SL}_2(\mathcal{O}_F) \backslash (\mathcal{H} \times \mathcal{H}).$$

We will also work with the harmonic form

$$\omega_G := G(z_1, z_2) dz_1 dz_2 + G(\epsilon_1 z_1, \epsilon_2 \bar{z}_2) dz_1 d\bar{z}_2,$$

where $\epsilon \in \mathcal{O}_F^\times$ satisfies $\epsilon_1 > 0$, $\epsilon_2 < 0$.

Claim: The periods of ω_G against the cycles Δ_Ψ encode information about the arithmetic of E.

Geometric formulation of modularity

The differential form

$$\alpha_G := G(z_1, z_2) dz_1 dz_2$$

is a *holomorphic* (hence closed) 2-form on

$$X(\mathbb{C}) := \mathbf{SL}_2(\mathcal{O}_F)\backslash(\mathcal{H} \times \mathcal{H}).$$

We will also work with the harmonic form

$$\omega_G := G(z_1, z_2) dz_1 dz_2 + G(\epsilon_1 z_1, \epsilon_2 \bar{z}_2) dz_1 d\bar{z}_2,$$

where $\epsilon \in \mathcal{O}_F^\times$ satisfies $\epsilon_1 > 0$, $\epsilon_2 < 0$.

Claim: The periods of ω_G against the cycles Δ_Ψ encode information about the arithmetic of E.

Geometric formulation of modularity

The differential form

$$\alpha_G := G(z_1, z_2) dz_1 dz_2$$

is a *holomorphic* (hence closed) 2-form on

$$X(\mathbb{C}) := \mathbf{SL}_2(\mathcal{O}_F)\backslash(\mathcal{H} \times \mathcal{H}).$$

We will also work with the harmonic form

$$\omega_G := G(z_1, z_2) dz_1 dz_2 + G(\epsilon_1 z_1, \epsilon_2 \bar{z}_2) dz_1 d\bar{z}_2,$$

where $\epsilon \in \mathcal{O}_F^\times$ satisfies $\epsilon_1 > 0$, $\epsilon_2 < 0$.

Claim: The periods of ω_G against the cycles Δ_Ψ encode information about the arithmetic of E.

Periods of ω_G: the totally real case.

Theorem

$$\left|\int_{\Delta_{D.\chi}} \omega_G\right|^2 = L(E/K, \chi, 1) \pmod{K(\chi)^\times}.$$

Shimura-Oda period relations: It is conjectured that

$$\Lambda_G := \left\langle \int_{\Delta_\Psi} \omega_G, \quad \Psi \in \Sigma_D \text{ with } D \gg 0 \right\rangle \subset \mathbb{C}$$

is a lattice in $\mathbb{C}$ which is commensurable with the Weierstrass lattice of E.

Periods of ω_G: the totally real case.

Theorem

$$\left|\int_{\Delta_{D.\chi}} \omega_G\right|^2 = L(E/K, \chi, 1) \pmod{K(\chi)^\times}.$$

Shimura-Oda period relations: It is conjectured that

$$\Lambda_G := \left\langle \int_{\Delta_\Psi} \omega_G, \quad \Psi \in \Sigma_D \text{ with } D \gg 0 \right\rangle \subset \mathbb{C}$$

is a lattice in $\mathbb{C}$ which is commensurable with the Weierstrass lattice of E.

Periods of ω_G: the CM case

Since $\Delta_{D,\chi}$ is 0-dimensional, expressions like

$$\int_{\Delta_{D,\chi}} \omega_G, \qquad \int_{\partial^{-1}\Delta_{D,\chi}} \omega_G$$

do not make sense!

Question: Can CM cycles on X be used to construct points on E?

Related Question (Eyal Goren's thesis). Can CM cycles on X be used to construct canonical units in abelian extensions of CM fields, generalising elliptic units?

Answer: For elliptic curves, probably not...

Periods of ω_G: the CM case

Since $\Delta_{D,\chi}$ is 0-dimensional, expressions like

$$\int_{\Delta_{D,\chi}} \omega_G, \qquad \int_{\partial^{-1}\Delta_{D,\chi}} \omega_G$$

do not make sense!

Question: Can CM cycles on X be used to construct points on E?

Related Question (Eyal Goren's thesis). Can CM cycles on X be used to construct canonical units in abelian extensions of CM fields, generalising elliptic units?

Answer: For elliptic curves, probably not...

Periods of ω_G: the CM case

Since $\Delta_{D,\chi}$ is 0-dimensional, expressions like

$$\int_{\Delta_{D,\chi}} \omega_G, \qquad \int_{\partial^{-1}\Delta_{D,\chi}} \omega_G$$

do not make sense!

Question: Can CM cycles on X be used to construct points on E?

Related Question (Eyal Goren's thesis). Can CM cycles on X be used to construct canonical units in abelian extensions of CM fields, generalising elliptic units?

Answer: For elliptic curves, probably not...

Periods of ω_G: the CM case

Since $\Delta_{D,\chi}$ is 0-dimensional, expressions like

$$\int_{\Delta_{D,\chi}} \omega_G, \qquad \int_{\partial^{-1}\Delta_{D,\chi}} \omega_G$$

do not make sense!

Question: Can CM cycles on X be used to construct points on E?

Related Question (Eyal Goren's thesis). Can CM cycles on X be used to construct canonical units in abelian extensions of CM fields, generalising elliptic units?

Answer: For elliptic curves, probably not...

Elliptic curves of conductor 1 and the BSD conjecture

Consider the twist E_K of E by a quadratic extension K/F.

Proposition

1. *If K is totally real or CM, then E_K has even analytic rank.*
2. *If K is an ATR (*Almost Totally Real*) extension, then E_K has odd analytic rank.*

In particular, we do not expect points in $E(K)$ when K is CM...

Suggestion: *ATR cycles on Hilbert modular surfaces are a more appropriate generalisation of CM cycles on modular curves.*

Elliptic curves of conductor 1 and the BSD conjecture

Consider the twist E_K of E by a quadratic extension K/F.

Proposition

1. *If K is totally real or CM, then E_K has even analytic rank.*
2. *If K is an ATR (Almost Totally Real) extension, then E_K has odd analytic rank.*

In particular, we do not expect points in $E(K)$ when K is CM...

Suggestion: *ATR cycles on Hilbert modular surfaces are a more appropriate generalisation of CM cycles on modular curves.*

Periods of ω_G: the ATR case

Recall: The cycles Δ_Ψ are *homologically trivial* (after eventually tensoring with $\mathbb{Q}$), because $H_1(X(\mathbb{C}), \mathbb{Q}) = 0$.

$$P_\Psi := \int_{\partial^{-1}\Delta_\Psi} \omega_G \in \mathbb{C}/\Lambda_G \simeq E(\mathbb{C}).$$

Conjecture (Adam Logan,D)

If $\Psi \in \Sigma_D$, then the point P_Ψ belongs to $E(H_D) \otimes \mathbb{Q}$, where H_D is the Hilbert class field of the ATR extension $K = F(\sqrt{D})$.

Periods of ω_G: the ATR case

Recall: The cycles Δ_Ψ are *homologically trivial* (after eventually tensoring with $\mathbb{Q}$), because $H_1(X(\mathbb{C}), \mathbb{Q}) = 0$.

$$P_\Psi := \int_{\partial^{-1}\Delta_\Psi} \omega_G \in \mathbb{C}/\Lambda_G \simeq E(\mathbb{C}).$$

Conjecture (Adam Logan,D)

If $\Psi \in \Sigma_D$, then the point P_Ψ belongs to $E(H_D) \otimes \mathbb{Q}$, where H_D is the Hilbert class field of the ATR extension $K = F(\sqrt{D})$.

Elliptic units for totally real fields

Elliptic units: $\alpha \in \mathcal{O}_{Y_0(N)}^\times$, Δ a CM divisor $\Rightarrow \alpha(\Delta) \in \mathcal{O}_H^\times$.

$$\log \alpha(\Delta) = \int_{\partial^{-1}\Delta} \frac{d\alpha}{\alpha} = \int_{\partial^{-1}\Delta} \omega_{E_\alpha},$$

where E_α = an Eisenstein series of weight two.

ATR units: Let E=Eisenstein series of weight two on $\mathbf{SL}_2(\mathcal{O}_F)$,

$$z_\Psi = \int_{\partial^{-1}\Delta_\Psi} \omega_E \in \mathbb{C}/(2\pi i\mathbb{Z})$$

Conjecture (Pierre Charollois,D)

If $\Psi \in \Sigma_D$, then $u_\Psi = exp(z_\Psi)$ belongs to $\mathcal{O}_{H_D}^\times$, where H_D is the Hilbert class field of the ATR extension $K = F(\sqrt{D})$.

Elliptic units for totally real fields

Elliptic units: $\alpha \in \mathcal{O}_{Y_0(N)}^{\times}$, Δ a CM divisor $\Rightarrow \alpha(\Delta) \in \mathcal{O}_H^{\times}$.

$$\log \alpha(\Delta) = \int_{\partial^{-1}\Delta} \frac{d\alpha}{\alpha} = \int_{\partial^{-1}\Delta} \omega_{E_\alpha},$$

where E_α = an Eisenstein series of weight two.

ATR units: Let E=Eisenstein series of weight two on $\mathbf{SL}_2(\mathcal{O}_F)$,

$$z_\Psi = \int_{\partial^{-1}\Delta_\Psi} \omega_E \in \mathbb{C}/(2\pi i\mathbb{Z})$$

Conjecture (Pierre Charollois,D)

If $\Psi \in \Sigma_D$, then $u_\Psi = exp(z_\Psi)$ belongs to $\mathcal{O}_{H_D}^{\times}$. where H_D is the Hilbert class field of the ATR extension $K = F(\sqrt{D})$.

Elliptic units for totally real fields

Elliptic units: $\alpha \in \mathcal{O}_{Y_0(N)}^\times$, Δ a CM divisor $\Rightarrow \alpha(\Delta) \in \mathcal{O}_H^\times$.

$$\log \alpha(\Delta) = \int_{\partial^{-1}\Delta} \frac{d\alpha}{\alpha} = \int_{\partial^{-1}\Delta} \omega_{E_\alpha},$$

where E_α = an Eisenstein series of weight two.

ATR units: Let E=Eisenstein series of weight two on $\mathbf{SL}_2(\mathcal{O}_F)$,

$$z_\Psi = \int_{\partial^{-1}\Delta_\Psi} \omega_E \in \mathbb{C}/(2\pi i \mathbb{Z})$$

Conjecture (Pierre Charollois,D)

If $\Psi \in \Sigma_D$, then $u_\Psi = \exp(z_\Psi)$ belongs to $\mathcal{O}_{H_D}^\times$, where H_D is the Hilbert class field of the ATR extension $K = F(\sqrt{D})$.

Elliptic units for totally real fields

Elliptic units: $\alpha \in \mathcal{O}^\times_{Y_0(N)}$, Δ a CM divisor $\Rightarrow \alpha(\Delta) \in \mathcal{O}^\times_H$.

$$\log \alpha(\Delta) = \int_{\partial^{-1}\Delta} \frac{d\alpha}{\alpha} = \int_{\partial^{-1}\Delta} \omega_{E_\alpha},$$

where E_α = an Eisenstein series of weight two.

ATR units: Let E=Eisenstein series of weight two on $\mathbf{SL}_2(\mathcal{O}_F)$,

$$z_\Psi = \int_{\partial^{-1}\Delta_\Psi} \omega_E \in \mathbb{C}/(2\pi i \mathbb{Z})$$

Conjecture (Pierre Charollois,D)

If $\Psi \in \Sigma_D$, then $u_\Psi = exp(z_\Psi)$ belongs to $\mathcal{O}^\times_{H_D}$. where H_D is the Hilbert class field of the ATR extension $K = F(\sqrt{D})$.

The general question

Understand the process whereby ATR cycles on $X(\mathbb{C})$ lead to the construction of global invariants such as algebraic points on elliptic curves and Stark units.

Eventual applications:

a) Construction of *Euler systems* attached to elliptic curves.

b) “Explicit” construction of class fields.

c) Stark’s conjectures for abelian extensions of ATR fields.

The general question

Understand the process whereby ATR cycles on $X(\mathbb{C})$ lead to the construction of global invariants such as algebraic points on elliptic curves and Stark units.

Eventual applications:

a) Construction of *Euler systems* attached to elliptic curves.

b) "Explicit" construction of class fields.

c) Stark's conjectures for abelian extensions of ATR fields.

The general question

Understand the process whereby ATR cycles on $X(\mathbb{C})$ lead to the construction of global invariants such as algebraic points on elliptic curves and Stark units.

Eventual applications:

a) Construction of *Euler systems* attached to elliptic curves.

b) "Explicit" construction of class fields.

c) Stark's conjectures for abelian extensions of ATR fields.

The general question

Understand the process whereby ATR cycles on $X(\mathbb{C})$ lead to the construction of global invariants such as algebraic points on elliptic curves and Stark units.

Eventual applications:

a) Construction of *Euler systems* attached to elliptic curves.

b) "Explicit" construction of class fields.

c) Stark's conjectures for abelian extensions of ATR fields.

The BSD conjecture for curves of conductor 1

Conjecture (on ATR twists)

Let K be an ATR extension of F and let E_K be the associated twist of E. If $L'(E_K/F, 1) \neq 0$, then $E_K(F)$ has rank one and $Ш(E_K/F) < \infty$.

The BSD conjecture over totally real fields is very well understood in analytic rank ≤ 1, thanks mostly to the work of Zhang and his school.

Yet the conjecture on ATR twists continues to present a genuine mystery.

Modest proposal: Exhibit settings where the mysterious ATR construction can be directly compared with a classical Heegner point construction.

The BSD conjecture for curves of conductor 1

Conjecture (on ATR twists)

Let K be an ATR extension of F and let E_K be the associated twist of E. If $L'(E_K/F, 1) \neq 0$, then $E_K(F)$ has rank one and $Ш(E_K/F) < \infty$.

The BSD conjecture over totally real fields is very well understood in analytic rank ≤ 1, thanks mostly to the work of Zhang and his school.

Yet the conjecture on ATR twists continues to present a genuine mystery.

Modest proposal: Exhibit settings where the mysterious ATR construction can be directly compared with a classical Heegner point construction.

The BSD conjecture for curves of conductor 1

Conjecture (on ATR twists)

Let K be an ATR extension of F and let E_K be the associated twist of E. If $L'(E_K/F, 1) \neq 0$, then $E_K(F)$ has rank one and $Ш(E_K/F) < \infty$.

The BSD conjecture over totally real fields is very well understood in analytic rank ≤ 1, thanks mostly to the work of Zhang and his school.

Yet the conjecture on ATR twists continues to present a genuine mystery.

Modest proposal: Exhibit settings where the mysterious ATR construction can be directly compared with a classical Heegner point construction.

The BSD conjecture for curves of conductor 1

Conjecture (on ATR twists)

Let K be an ATR extension of F and let E_K be the associated twist of E. If $L'(E_K/F,1) \neq 0$, then $E_K(F)$ has rank one and $Ш(E_K/F) < \infty$.

The BSD conjecture over totally real fields is very well understood in analytic rank ≤ 1, thanks mostly to the work of Zhang and his school.

Yet the conjecture on ATR twists continues to present a genuine mystery.

Modest proposal: Exhibit settings where the mysterious ATR construction can be directly compared with a classical Heegner point construction.

$\mathbb{Q}$-curves

Definition

A $\mathbb{Q}$-curve over F is an elliptic curve E/F which is F-isogenous to its Galois conjugate.

Pinch, Cremona: For $N = \mathrm{disc}(F)$ prime and ≤ 1000, there are exactly 17 isogeny classes of elliptic curves of conductor 1 over $\mathbb{Q}(\sqrt{N})$,

$$N = 29, 37, 41, 109, 157, 229, 257, 337, 349,$$

$$397, 461, 509, 509, 877, 733, 881, 997.$$

All but two ($N = 509, 877$) are $\mathbb{Q}$-curves.

$\mathbb{Q}$-curves

Definition

A $\mathbb{Q}$-curve over F is an elliptic curve E/F which is F-isogenous to its Galois conjugate.

Pinch, Cremona: For $N = \mathrm{disc}(F)$ prime and ≤ 1000, there are exactly 17 isogeny classes of elliptic curves of conductor 1 over $\mathbb{Q}(\sqrt{N})$,

$$N = 29, 37, 41, 109, 157, 229, 257, 337, 349,$$

$$397, 461, 509, 509, 877, 733, 881, 997.$$

All but two ($N = 509, 877$) are $\mathbb{Q}$-curves.

$\mathbb{Q}$-curves and elliptic modular forms

Theorem (Ribet)

Let E be a $\mathbb{Q}$-curve of conductor 1 over $F = \mathbb{Q}(\sqrt{N})$. Then there is an elliptic modular form $f \in S_2(\Gamma_1(N), \varepsilon_F)$ with fourier coefficients in a quadratic (imaginary) field such that

$$L(E/F, s) = L(f, s)L(f^\sigma, s).$$

The Hilbert modular form G on $GL_2(\mathbb{A}_F)$ is the Doi-Naganuma lift of f. Modular parametrisation defined over F:

$$J_1(N) \longrightarrow E.$$

$\mathbb{Q}$-curves and elliptic modular forms

Theorem (Ribet)

Let E be a $\mathbb{Q}$-curve of conductor 1 over $F = \mathbb{Q}(\sqrt{N})$. Then there is an elliptic modular form $f \in S_2(\Gamma_1(N), \varepsilon_F)$ with fourier coefficients in a quadratic (imaginary) field such that

$$L(E/F, s) = L(f, s)L(f^\sigma, s).$$

The Hilbert modular form G on $GL_2(\mathbb{A}_F)$ is the Doi-Naganuma lift of f. Modular parametrisation defined over F:

$$J_1(N) \longrightarrow E.$$

Birch and Swinnerton-Dyer for $\mathbb{Q}$-curves

Theorem (Victor Rotger, Yu Zhao, D)

Let E be a $\mathbb{Q}$-curve of conductor 1 over a real quadratic field F, and let M/F be an ATR extension of F. If $L'(E_M/F,1) \neq 0$, then $E_M(F)$ has rank one and $Ш(E_M/F)$ is finite.

Caveat: Note the change in notation: the ATR extension is now denoted as M rather than K.

Birch and Swinnerton-Dyer for $\mathbb{Q}$-curves

Theorem (Victor Rotger, Yu Zhao, D)

Let E be a $\mathbb{Q}$-curve of conductor 1 over a real quadratic field F, and let M/F be an ATR extension of F. If $L'(E_M/F, 1) \neq 0$, then $E_M(F)$ has rank one and $Ш(E_M/F)$ is finite.

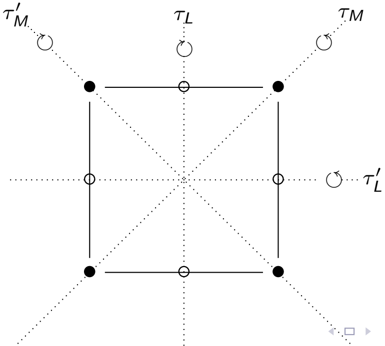
Caveat: Note the change in notation: the ATR extension is now denoted as M rather than K.

Some Galois theory

Let $\mathcal{M}$= Galois closure of M over $\mathbb{Q}$. Then $\mathrm{Gal}(\mathcal{M}/\mathbb{Q}) = D_8$.

This group contains two copies of the Klein 4-group:

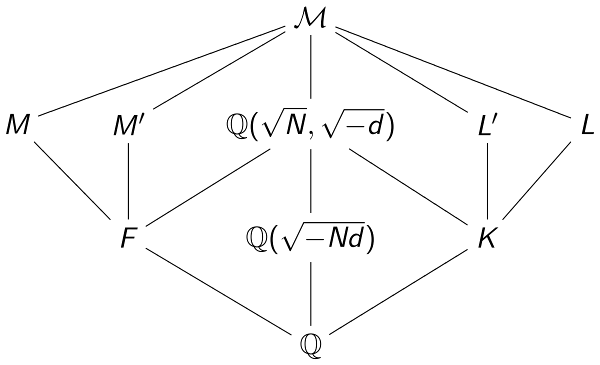
$$V_F = \langle \tau_M, \tau_M' \rangle, \qquad V_K = \langle \tau_L, \tau_L' \rangle.$$



Some Galois theory

Suppose that $\quad F = \mathcal{M}^{V_F} \quad M = \mathcal{M}^{\tau_M} \quad M' = \mathcal{M}^{\tau'_M},$

and set $\quad K = \mathcal{M}^{V_K} \quad L = \mathcal{M}^{\tau_L} \quad L' = \mathcal{M}^{\tau'_L}.$



Key facts about K and L

Let $\begin{cases} \chi_M : \mathbb{A}_F^\times \longrightarrow \pm 1 \text{ be the quadratic character attached to } M/F; \\ \chi_L : \mathbb{A}_K^\times \longrightarrow \pm 1 \text{ be the quadratic character attached to } L/K. \end{cases}$

1. $K = \mathbb{Q}(\sqrt{-d})$ is an imaginary quadratic field, and satisfies a suitable "Heegner hypothesis";
2. The central character $\chi_L|_{\mathbb{A}_\mathbb{Q}^\times}$ is equal to ε_F.
3. $\mathrm{Ind}_F^\mathbb{Q} \chi_M = \mathrm{Ind}_K^\mathbb{Q} \chi_L$;

Key facts about K and L

Let $\begin{cases} \chi_M : \mathbb{A}_F^\times \longrightarrow \pm 1 \text{ be the quadratic character attached to } M/F; \\ \chi_L : \mathbb{A}_K^\times \longrightarrow \pm 1 \text{ be the quadratic character attached to } L/K. \end{cases}$

1. $K = \mathbb{Q}(\sqrt{-d})$ is an imaginary quadratic field, and satisfies a suitable "Heegner hypothesis";
2. The central character $\chi_L|_{\mathbb{A}_\mathbb{Q}^\times}$ is equal to ε_F.
3. $\mathrm{Ind}_F^\mathbb{Q} \chi_M = \mathrm{Ind}_K^\mathbb{Q} \chi_L$;

Key facts about K and L

Let $\begin{cases} \chi_M : \mathbb{A}_F^\times \longrightarrow \pm 1 \text{ be the quadratic character attached to } M/F; \\ \chi_L : \mathbb{A}_K^\times \longrightarrow \pm 1 \text{ be the quadratic character attached to } L/K. \end{cases}$

1. $K = \mathbb{Q}(\sqrt{-d})$ is an imaginary quadratic field, and satisfies a suitable "Heegner hypothesis";
2. The central character $\chi_L|_{\mathbb{A}_\mathbb{Q}^\times}$ is equal to ε_F.
3. $\mathrm{Ind}_F^\mathbb{Q} \chi_M = \mathrm{Ind}_K^\mathbb{Q} \chi_L$;

Key facts about K and L

Let $\begin{cases} \chi_M : \mathbb{A}_F^\times \longrightarrow \pm 1 \text{ be the quadratic character attached to } M/F; \\ \chi_L : \mathbb{A}_K^\times \longrightarrow \pm 1 \text{ be the quadratic character attached to } L/K. \end{cases}$

1. $K = \mathbb{Q}(\sqrt{-d})$ is an imaginary quadratic field, and satisfies a suitable "Heegner hypothesis";
2. The central character $\chi_L|_{\mathbb{A}_\mathbb{Q}^\times}$ is equal to ε_F.
3. $\mathrm{Ind}_F^\mathbb{Q} \chi_M = \mathrm{Ind}_K^\mathbb{Q} \chi_L$;

The Artin formalism

Let $f \in S_2(\Gamma_0(N), \varepsilon_F)$ and let E/F be associated elliptic curve.

$$\begin{aligned} L(E_M/F, s) &= L(E/F, \chi_M, s) \\ &= L(f/F, \chi_M, s) \\ &= L(f \otimes \mathrm{Ind}_F^{\mathbb{Q}} \chi_M, s) \\ &= L(f \otimes \mathrm{Ind}_K^{\mathbb{Q}} \chi_L, s) \\ &= L(f/K, \chi_L, s) \end{aligned}$$

In particular, $L'(E_M/F, 1) \neq 0$ implies that $L'(f/K, \chi_L, 1) \neq 0$.

The Artin formalism

Let $f \in S_2(\Gamma_0(N), \varepsilon_F)$ and let E/F be associated elliptic curve.

$$\begin{aligned} L(E_M/F, s) &= L(E/F, \chi_M, s) \\ &= L(f/F, \chi_M, s) \\ &= L(f \otimes \mathrm{Ind}_F^{\mathbb{Q}} \chi_M, s) \\ &= L(f \otimes \mathrm{Ind}_K^{\mathbb{Q}} \chi_L, s) \\ &= L(f/K, \chi_L, s) \end{aligned}$$

In particular, $L'(E_M/F, 1) \neq 0$ implies that $L'(f/K, \chi_L, 1) \neq 0$.

The Artin formalism

Let $f \in S_2(\Gamma_0(N), \varepsilon_F)$ and let E/F be associated elliptic curve.

$$\begin{aligned} L(E_M/F, s) &= L(E/F, \chi_M, s) \\ &= L(f/F, \chi_M, s) \\ &= L(f \otimes \mathrm{Ind}_F^{\mathbb{Q}} \chi_M, s) \\ &= L(f \otimes \mathrm{Ind}_K^{\mathbb{Q}} \chi_L, s) \\ &= L(f/K, \chi_L, s) \end{aligned}$$

In particular, $L'(E_M/F, 1) \neq 0$ implies that $L'(f/K, \chi_L, 1) \neq 0$.

The work of Tian, Yuan, Zhang and Zhang

The following strikingly general theorem applies to forms on $\Gamma_1(N)$ with non-trivial nebentype character.

Theorem (Ye Tian, Xinyi Yuan, Shou-Wu Zhang, Wei Zhang)

If $L'(f/K, \chi_L, 1) \neq 0$, then $A_f(L)^- \otimes \mathbb{Q}$ has dimension one over T_f, and therefore

$$\mathrm{rank}(A_f(L)^-) = 2.$$

Furthermore $Ш(A_f/L)^-$ is finite.

$$\mathrm{rank}(A_f(L)^-) = \mathrm{rank}(A_f(M)^-), \qquad A_f(M)^- = E(M)^- \oplus E(M)^-.$$

Corollary

If $L'(E_M/F, 1) \neq 0$, then $\mathrm{rank}(E_M(F)) = 1$ and $Ш(E_M/F) < \infty$.

The work of Tian, Yuan, Zhang and Zhang

The following strikingly general theorem applies to forms on $\Gamma_1(N)$ with non-trivial nebentype character.

Theorem (Ye Tian, Xinyi Yuan, Shou-Wu Zhang, Wei Zhang)

If $L'(f/K, \chi_L, 1) \neq 0$, then $A_f(L)^- \otimes \mathbb{Q}$ has dimension one over T_f, and therefore

$$\mathrm{rank}(A_f(L)^-) = 2.$$

Furthermore $Ш(A_f/L)^-$ is finite.

$$\mathrm{rank}(A_f(L)^-) = \mathrm{rank}(A_f(M)^-), \qquad A_f(M)^- = E(M)^- \oplus E(M)^-.$$

Corollary

If $L'(E_M/F, 1) \neq 0$, then $\mathrm{rank}(E_M(F)) = 1$ and $Ш(E_M/F) < \infty$.

The work of Tian, Yuan, Zhang and Zhang

The following strikingly general theorem applies to forms on $\Gamma_1(N)$ with non-trivial nebentype character.

Theorem (Ye Tian, Xinyi Yuan, Shou-Wu Zhang, Wei Zhang)

If $L'(f/K, \chi_L, 1) \neq 0$, then $A_f(L)^- \otimes \mathbb{Q}$ has dimension one over T_f, and therefore

$$\mathrm{rank}(A_f(L)^-) = 2.$$

Furthermore $Ш(A_f/L)^-$ is finite.

$$\mathrm{rank}(A_f(L)^-) = \mathrm{rank}(A_f(M)^-), \qquad A_f(M)^- = E(M)^- \oplus E(M)^-.$$

Corollary

If $L'(E_M/F, 1) \neq 0$, then $\mathrm{rank}(E_M(F)) = 1$ and $Ш(E_M/F) < \infty$.

The work of Tian, Yuan, Zhang and Zhang

The following strikingly general theorem applies to forms on $\Gamma_1(N)$ with non-trivial nebentype character.

Theorem (Ye Tian, Xinyi Yuan, Shou-Wu Zhang, Wei Zhang)

If $L'(f/K, \chi_L, 1) \neq 0$, then $A_f(L)^- \otimes \mathbb{Q}$ has dimension one over T_f, and therefore

$$\operatorname{rank}(A_f(L)^-) = 2.$$

Furthermore $Ш(A_f/L)^-$ is finite.

$$\operatorname{rank}(A_f(L)^-) = \operatorname{rank}(A_f(M)^-), \qquad A_f(M)^- = E(M)^- \oplus E(M)^-.$$

Corollary

If $L'(E_M/F, 1) \neq 0$, then $\operatorname{rank}(E_M(F)) = 1$ and $Ш(E_M/F) < \infty$.

A final question

In the setting of $\mathbb{Q}$-curves, we have two constructions of a point in $E_M(F)$, with $M = F(\sqrt{D})$ ATR:

1. A "classical" Heegner point $P_M(f)$ attached to the elliptic cusp form $f \in S_2(\Gamma_1(N), \varepsilon_N)$.
2. A *conjectural* ATR point $P_M^?(G) = P_{D,1}(\omega_G)$ attached to the Hilbert modular form $G = DN(f)$.

Conjecture (Rotger, Zhao, D)

There exists a constant $\ell \in \mathbb{Q}^\times$, not depending on M, such that

$$P_M(f) = P_M^?(G).$$

This statement relates Abel-Jacobi type invariants attached to an elliptic cusp form, with those of its Doi-Naganuma lift.

A final question

In the setting of $\mathbb{Q}$-curves, we have two constructions of a point in $E_M(F)$, with $M = F(\sqrt{D})$ ATR:

1. A "classical" Heegner point $P_M(f)$ attached to the elliptic cusp form $f \in S_2(\Gamma_1(N), \varepsilon_N)$.
2. A *conjectural* ATR point $P_M^?(G) = P_{D,1}(\omega_G)$ attached to the Hilbert modular form $G = DN(f)$.

Conjecture (Rotger, Zhao, D)

There exists a constant $\ell \in \mathbb{Q}^\times$, not depending on M, such that

$$P_M(f) = P_M^?(G).$$

This statement relates Abel-Jacobi type invariants attached to an elliptic cusp form, with those of its Doi-Naganuma lift.

A final question

In the setting of $\mathbb{Q}$-curves, we have two constructions of a point in $E_M(F)$, with $M = F(\sqrt{D})$ ATR:

1. A "classical" Heegner point $P_M(f)$ attached to the elliptic cusp form $f \in S_2(\Gamma_1(N), \varepsilon_N)$.
2. A *conjectural* ATR point $P^?_M(G) = P_{D,1}(\omega_G)$ attached to the Hilbert modular form $G = DN(f)$.

Conjecture (Rotger, Zhao, D)

There exists a constant $\ell \in \mathbb{Q}^\times$, not depending on M, such that

$$P_M(f) = P^?_M(G).$$

This statement relates Abel-Jacobi type invariants attached to an elliptic cusp form, with those of its Doi-Naganuma lift.

A final question

In the setting of $\mathbb{Q}$-curves, we have two constructions of a point in $E_M(F)$, with $M = F(\sqrt{D})$ ATR:

1. A "classical" Heegner point $P_M(f)$ attached to the elliptic cusp form $f \in S_2(\Gamma_1(N), \varepsilon_N)$.
2. A *conjectural* ATR point $P_M^?(G) = P_{D,1}(\omega_G)$ attached to the Hilbert modular form $G = DN(f)$.

Conjecture (Rotger, Zhao, D)

There exists a constant $\ell \in \mathbb{Q}^\times$, not depending on M, such that

$$P_M(f) = P_M^?(G).$$

This statement relates Abel-Jacobi type invariants attached to an elliptic cusp form, with those of its Doi-Naganuma lift.

A final question

In the setting of $\mathbb{Q}$-curves, we have two constructions of a point in $E_M(F)$, with $M = F(\sqrt{D})$ ATR:

1. A "classical" Heegner point $P_M(f)$ attached to the elliptic cusp form $f \in S_2(\Gamma_1(N), \varepsilon_N)$.
2. A *conjectural* ATR point $P_M^?(G) = P_{D,1}(\omega_G)$ attached to the Hilbert modular form $G = DN(f)$.

Conjecture (Rotger, Zhao, D)

There exists a constant $\ell \in \mathbb{Q}^\times$, not depending on M, such that

$$P_M(f) = P_M^?(G).$$

This statement relates Abel-Jacobi type invariants attached to an elliptic cusp form, with those of its Doi-Naganuma lift.

A Big Thank You to

Eyal,

Steve,

for organising the Toronto-Montreal workshop!