Symmetric Spaces

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What are Hermitian Symmetric Spaces?

Definition

A Riemannian manifold M is called a *Riemannian symmetric space* if for each point $x \in M$ there exists an involution s_x which is an isometry of M and a neighbourhood N_x of x where x is the unique fixed point of s_x in N_x.

Definition

A Riemannian symmetric space M is said to be *Hermitian* if M has a complex structure making the Riemannian structure a Hermitian structure.

What are they concretely?

Theorem

Let M be a Riemannian symmetric space and $x \in M$ be any point. Furthermore, let $G = \mathrm{Isom}(M)$ and $K = \mathrm{Stab}_G(x)$. Then G is a real Lie group, K is a compact subgroup and $G/K \simeq M$. Moreover, we have that the involution s_x extends to an involution of G with $(G^{s_x})^0 \subset K \subset G^{s_x}$.

Theorem

If in particular M is a Hermitian symmetric space, then $\mathrm{SO}_2(\mathbb{R}) \subset Z(K)$. If moreover M is irreducible and $Z(G) = \{e\}$ then $Z(K) = \mathrm{SO}_2(\mathbb{R})$.

We remark that because $\mathrm{Isom}(M)$ acts transitively, it suffices to specifty s_x for a single point x.

Example - The Upper half plane

The upper half plane

$$\mathbb{H} = \{x + iy \in \mathbb{C} | y > 0\}, \text{ with metric } \frac{1}{y^2}(dx^2 + dy^2)$$

is a Hermitian Symmetric space. The isometry group is

$$G = \mathrm{Isom}(\mathbb{H}) \simeq \mathrm{PSL}_2(\mathbb{R}) \simeq \mathrm{PSO}(2,1)(\mathbb{R}).$$

The action on $\mathbb{H}$ is through fractional linear transformation

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \circ \tau = \frac{a\tau+b}{c\tau+d}.$$

Fixing $i \in \mathbb{H}$ as the base point, the compact subgroup is

$$K = \mathrm{Stab}_G(i) = \mathrm{PSO}_2(\mathbb{R}) \simeq \mathrm{SO}_2(\mathbb{R}).$$

At the point $i \in \mathbb{H}$ the involution is $\tau \mapsto \frac{-1}{\tau}$. The extension of this involution to G is

$$s_i : g \mapsto (g^T)^{-1}.$$

The Lie Algebra Structures

Given that $M = G/K$ we are naturally drawn to look at the Lie algebra structure of $\mathfrak{g} = \text{Lie}(G)$. The *Killing form* on $\mathfrak{g}$ is $B(X,Y) = \text{Tr}(\text{Ad}(X) \circ \text{Ad}(Y))$. We make several observations:

1. The Lie algebra decomposes as $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{k}$ is the Lie algebra for K and $\mathfrak{p} = \mathfrak{k}^{\perp}$ relative to B.
2. The involution s_x on M induces an involution on $\mathfrak{g}$ such that:

$$s_x : \mathfrak{k} + \mathfrak{p} \mapsto \mathfrak{k} - \mathfrak{p}.$$

3. Since K is compact it follows that $B|_{\mathfrak{k}}$ is negative-definite.

Definition

A *Cartan involution* $\theta : \mathfrak{g} \to \mathfrak{g}$ is an $\mathbb{R}$-linear map such that $B(X, \theta(Y))$ is negative-definite.
A decomposition of $\mathfrak{g}$ into the $+1, -1$ eigenspaces for a Cartan involution is called a *Cartan decomposition*.

Decomposition of Symmetric Spaces

Definition

A symmetric space M is said to be:

- *Compact Type* if $B|_{\mathfrak{p}}$ negative-definite (if and only if $\mathfrak{g}$ is compact).
- *Non-Compact Type* if $B|_{\mathfrak{p}}$ positive-definite (if and only if s_x is a Cartan involution).
- *Euclidean Type* if $B|_{\mathfrak{p}} = 0$.

Theorem

Every symmetric space M can be decomposed into a product

$$M = M_c \times M_{nc} \times M_e$$

where the factors are of compact, non-compact and Euclidean types respectively.

Dual Symmetric Pairs

Studying modular forms on G/K requires constructing interesting vector bundles. In the non-compact case this is done via an embedding into a projective variety. We shall now work towards obtaining such an embedding.

Definition

Given a Riemannian symmetric space M with associated Lie algebra $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, we define the dual Lie algebra (for the pair $(\mathfrak{g}, \mathfrak{k})$) to be:

$$\mathfrak{g}^* = \mathfrak{k} + i\mathfrak{p} \subset \mathfrak{g} \otimes \mathbb{C}.$$

If $\mathfrak{g}$ was compact (resp non-compact, resp Euclidean) type then $\mathfrak{g}^*$ is non-compact (resp compact, resp Euclidean) type.

One typically can associate to this dual Lie algebra an associated Lie real group $\breve{G} \subset G_{\mathbb{C}}$ such that $K \subset \breve{G}$ and symmetric space $\breve{G}/K$.

For the remainder of this talk, G/K will be a Hermitian symmetric space of the non-compact type with $\breve{G}/K$ the dual symmetric space of the compact type.

Embedding into the Compact Dual

Theorem

Let U be the center of K and $\mathfrak{u}$ be its Lie algebra. The action of $\mathfrak{u}_{\mathbb{C}}$ on $\mathfrak{g}_{\mathbb{C}}$ decomposes $\mathfrak{g}_{\mathbb{C}}$ into three eigenspaces: the 0-eigenspace $\mathfrak{k}_{\mathbb{C}}$ and two others we shall denote $\mathfrak{p}^+$ and $\mathfrak{p}^-$. The Lie algebra $\mathfrak{k} \otimes \mathbb{C} + \mathfrak{p}^-$ is then parabolic. Moreover

$$(\mathfrak{k} \otimes \mathbb{C} + \mathfrak{p}^-) \cap \mathfrak{g} = \mathfrak{k}.$$

Theorem

Let $P^- \subset G_{\mathbb{C}}$ be the parabolic subgroup associated to $\mathfrak{k} \otimes \mathbb{C} + \mathfrak{p}^-$. Then:

$$G/K \hookrightarrow G_{\mathbb{C}}/P^- \simeq \check{G}/K$$

This is an open immersion of G/K into the complex projective variety $G_{\mathbb{C}}/P^-$,which is isomorphic to the compact dual $\check{M} = \check{G}/K$. The maps are induced by the inclusions $G \hookrightarrow G_{\mathbb{C}}$ and $\check{G} \hookrightarrow G_{\mathbb{C}}$

We have that $G_{\mathbb{C}}/P^-$ is a “generalized flag manifold”.

The O(2,n) Case

Definition

Let $(V, x.y)$ be a rational quadratic space of signature $(2, n)$. We define the *Grassmannian* by

$$Gr(V) = \{\text{positive-definite planes in } V(\mathbb{R})\}$$

and the *quadric* by

$$\mathcal{Q} = \{v \in P(V(\mathbb{C})) \text{ with } X.X = 0 \text{ and } X.\overline{X} > 0\}.$$

The group $G = \mathrm{PSO}(2, n)(\mathbb{R})$ acts transitively on positive-definite planes in $V(\mathbb{R})$ and thus on $Gr(V)$. Likewise it acts transitively on $\mathcal{Q}$. The kernal of these actions is $K = \mathrm{PS}(\mathrm{O}(2) \times \mathrm{O}(n))$. Moreover, $Gr(V) \simeq \mathcal{Q}$.

Removing the conditions 'positive-definite' equivalently $X.\overline{X} > 0$ we shall obtain the compact dual and the map $M \hookrightarrow \check{M}$.

The O(2,n) Case (Lie Algebra)

Fix a plane $x \in Gr(V)$. Define $\tilde{s_x}$ to be the map of $V(\mathbb{R})$ which acts as the identity on x and as -1 on $x^\perp$. This gives a map s_x on $Gr(V)$ which lifts to an involution of $\mathrm{PSO}(2,n)(\mathbb{R})$ via conjugation by $\tilde{s_x}$. One can then check that

$$\mathrm{PSO}(2,n)(\mathbb{R})^{s_x} = \mathrm{PS}(\mathrm{O}(2) \times \mathrm{O}(n)) = \mathrm{Stab}_G(x) = K.$$

The Lie algebra of G is

$$\mathfrak{g} = \mathrm{Lie}(G) \simeq \left\{ \left(\begin{smallmatrix} A & U \\ U^T & C \end{smallmatrix} \right) | A, C \text{ skew-symmetric} \right\}$$

with $\mathfrak{g}$ decomposing into

$$\mathfrak{k} = \left\{ \left(\begin{smallmatrix} A & 0 \\ 0 & C \end{smallmatrix} \right) \in \mathfrak{g} \right\} \text{ and } \mathfrak{p} = \left\{ \left(\begin{smallmatrix} 0 & U \\ U^T & 0 \end{smallmatrix} \right) \right\}.$$

The Killing form on $\mathfrak{p}$ is given by $B|_{\mathfrak{p}}(U_1, U_2) = \mathrm{Tr}(U_1 U_2^T)$.

Identifying $\mathfrak{p} = \mathfrak{p}_x$ with the tangent space at $x \in Gr(V)$ the Killing form induces a G-invariant Riemannian metric on $Gr(V)$.

The O(2,n) Case (The Dual)

The Compact Dual dual group is $\check{G} = \mathrm{PSO}(2+n)(\mathbb{R})$. Its Lie algebra is given by:

$$\mathfrak{g}^* = \mathrm{Lie}(\check{G}) \simeq \left\{ \begin{pmatrix} A & U \\ -U^T & C \end{pmatrix} | A, C \text{ skew-symmetric} \right\}.$$

with $\mathfrak{p}'$ the subspace given by $\left\{ \begin{pmatrix} 0 & U \\ -U^T & 0 \end{pmatrix} \right\}$.

The group $\check{G}$ has a natural transitive action on $\{\text{planes in } V(\mathbb{R})\}$. The stabilizer of the plane x for ths action of $\check{G}$ is again K.

The inclusion of $Gr(V)$ into $\{\text{planes in } V(\mathbb{R})\}$ thus realizes the embedding of G/K into the compact dual $\check{G}/K$.

The boundary components of $Gr(V)$ in this larger space come from isotropic subspaces, we remark that G acts transitively on these.

Locally Symmetric Spaces

Definition

Let $M = G/K$ be a symmetric space and Γ be a discrete subgroup of G then $X = \Gamma\backslash M$ is a *locally symmetric space*.
(One is often interested in the cases where Γ is 'torsion free' and has finite covolume so that X is more manageable)

For the case of the orthogonal group, let L be a full lattice in V then $\mathrm{SO}_L \subset G = \mathrm{SO}_V(\mathbb{R})$ is discrete and and has finite co-volume. We may thus consider the locally symmetric space $X = \mathrm{SO}_L \backslash G/K$.

In general there exists a natural compactification $\overline{X}$ which may be realized by adjoining to M its 'rational' boundary components in $\check{M}$. In the orthogonal case, these boundary components correspond to rational isotropic subspaces.

Definition

Let $\mathcal{Q}$ be the image of $M = G/K$ in some projective space embedding of $\check{M} = G_{\mathbb{C}}/P^-$ and let $\tilde{\mathcal{Q}}$ be the cone over $\mathcal{Q}$. A modular form f for Γ of weight k on M can be thought of as any of the equivalent notions:

1. A section of $\Gamma\backslash(\mathcal{O}_{\check{M}}(-k)|_M)$ on $\Gamma\backslash M$.
2. A function on $\tilde{\mathcal{Q}}$ homogeneous of degree k which is invariant under the action of Γ.
3. A function on $\mathcal{Q}$ which transforms with respect to the k^{th} power of the factor of automorphy under Γ.

To be a *meromorphic* (resp. *holomorphic*) modular form we require that f extends to the boundary and that it be meromorphic (resp. holomorphic). One may also consider forms which are holomorphic on the space but are only meromorphic on the boundary.

The End

Thank you.