Integral models of Shimura varieties

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April 9, 2011

Motivation

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Here is a prototypical theorem:

Theorem (Igusa, 1959)

The scheme M_n can be compactified by a smooth projective scheme M_n^* over $\mathrm{Spec}(\mathbb{Z}[1/n])$, in such a way that $M_n^* \backslash M_n$ is an étale cover of $\mathrm{Spec}(\mathbb{Z}[1/n])$

Abelian schemes with real multiplication

- K a totally real number field of degree g over $\mathbb{Q}$ with ring of integers R.
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Rapoport (1978) considered a moduli problem of abelian varieties of dimension g, with multiplication by R, plus extra data. He showed that this is representable by an algebraic space $\mathscr{M}^R$ over $\mathrm{Spec}(\mathbb{Z}[\zeta_n][1/n])$. Over the complex numbers, $\mathscr{M}^R(\mathbb{C}) \cong \mathbb{H}^g/\Gamma(n)$. Where

$$\Gamma(n) := \mathrm{Ker}(SL(2,R) \to SL(2,R/nR))$$

acts on each $\mathbb{H}$ via a separate embedding

$$SL(2,K) \hookrightarrow SL(2,\mathbb{R}).$$

Rapoport's moduli problem

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- An expectation was that $\overline{\mathscr{M}^R}$ would be smooth and proper over $\mathrm{Spec}(\mathbb{Z}[\zeta_n][1/n])$, but this turned out not to be true. This would have implied that the geometric fibres of $\mathscr{M}^R$ are irreducible.

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- An expectation was that $\overline{\mathscr{M}}^R$ would be smooth and proper over $\mathrm{Spec}(\mathbb{Z}[\zeta_n][1/n])$, but this turned out not to be true. This would have implied that the geometric fibres of $\mathscr{M}^R$ are irreducible.
- To prove the desired properties of $\mathscr{M}^R$, Deligne and Pappas (1994) introduced another moduli space $\mathscr{M}$ in which $\mathscr{M}^R$ is open and dense at every fibre. They then glued $\mathscr{M}$ and $\overline{\mathscr{M}}^R$ along $\mathscr{M}^R$ to get a proper (but not smooth) scheme over $\mathrm{Spec}(\mathbb{Z}[\zeta_n][1/n])$.

L-polarised abelian schemes with real multiplication by R

Let L be an invertible R-module with a choice of orientation on $L \otimes_{R,\sigma} \mathbb{R}$ for each $\sigma : K \hookrightarrow \mathbb{R}$.

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In addition, a *level n structure* is an isomorpism $(R/nR)^2 \xrightarrow{\sim} A_n$ over S.

Moduli of L-polarised abelian schemes with RM

The functor that associates to a scheme S, the isomorphism classes of L-polarised abelian schemes over S with real multiplication by R and level n structure, is representable by an algebraic space $\mathscr{M}_n^L$ over $\mathrm{Spec}(\mathbb{Z}[1/n])$.

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Rapoport's space $\mathscr{M}_n$ is one of the connected components of $\mathscr{M}_n^L$, for $L = D^{-1}$, the inverse different of R over $\mathbb{Z}$.

The non-smooth locus of $\mathscr{M}_n^L$

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Using this local model, other moduli spaces $\mathscr{N}_i$ of filtrations which stratify $\mathscr{N}$, and explicit charts for those $\mathscr{N}_i$, Deligne-Pappas describe the non-smooth locus of M_n^L.

The non-smooth locus of $\mathscr{M}_n^L$

Theorem (1994 Deligne-Pappas)

Let Δ be the discriminant of R over $\mathbb{Z}$. The scheme $\mathscr{M}_n^L$ is smooth over Spec$(\mathbb{Z}[1/n\Delta])$, flat of relative complete intersection over Spec$(\mathbb{Z}[1/n])$, and for p prime to n, $p|\Delta$, the non-smooth locus of $\mathscr{M}_n^L$ in characteristic p has codimension 2 in the fibre at p.

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The fibres of $\mathscr{M}_n^L \to Spec(\mathbb{Z}[1/n])$ are normal.

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This corollary in turn implies that $\mathscr{M}_n^L \to \mathrm{Spec}(\mathbb{Z}[1/n])$ factors through a scheme T which is étale over $\mathrm{Spec}(\mathbb{Z}[1/n])$.

The geometric fibres of $\mathscr{M}_n^L \to T$ are irreducible.

Another angle: the supersingular locus

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For a moduli problem, the supersingular points have to be defined separately. In the following, we will call an isogeny supersingular, if either its source or target are supersingular.

The supersingular locus: the modular curve

Let $M_{n,p}$ be the moduli scheme of p-isogenies between elliptic curves with level-n structure, and M_n as before. Then we have:

$$
\begin{array}{ccc}
 & M_{n,p} & \\
\swarrow & & \searrow \\
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Modulo p, the Frobenius and the Verschiebung provide sections for the projections.

$$\begin{array}{ccc} & M_{n,p} \otimes \mathbb{F}_p & \\ {\scriptstyle Fr} \nearrow \swarrow & & \searrow \nwarrow {\scriptstyle Ver} \\ M_n \otimes \mathbb{F}_p & & M_n \otimes \mathbb{F}_p \end{array}$$

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$$M_n \longleftarrow M_{n,p} \longrightarrow M_n \qquad\qquad E \longmapsfrom [\phi : E \to F] \longmapsto F$$

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$$M_n \otimes \mathbb{F}_p \underset{\longleftarrow}{\overset{Fr}{\longrightarrow}} M_{n,p} \otimes \mathbb{F}_p \underset{\longrightarrow}{\overset{Ver}{\longleftarrow}} M_n \otimes \mathbb{F}_p$$

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It is a theorem of Deligne and Rapoport that the images of *Fr* and *Ver* inside $M_{n,p} \otimes \mathbb{F}_p$ are transveral, with intersection exactly the supersingular points of $M_{n,p}$.

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Hellmuth Stamm in 1997 studied this situation in higher dimensions, and generalized this phenomenon to Hilbert-Blumenthal varieties for $g = 2$.

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For the special case when $g = 2$, Stamm gives a description of the global structure of supersingular abelian varieties inside the fibre at p.

Let N denote the fibre at p of the moduli space of D^{-1}-polarized abelian schemes with RM (considered by Rapoport), and let N^{ss} be its supersingular points.

Theorem (Stamm (1997))

- *The* $\mathbb{F}_p$ *scheme* N^s *is purely one-dimensional*

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- *The set of these irreducible components are a union of two subsets which are interchanged under the action of* $Gal(\mathbb{F}_{p^2}/\mathbb{F}_p)$.

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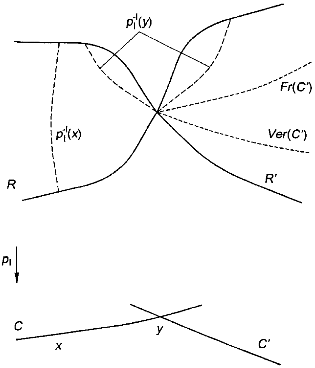
- *The irreducible components of $N^s \otimes \mathbb{F}_{p^2}$ are isomorphic to $\mathbb{P}^1_{\mathbb{F}_{p^2}}$.*
- *The set of these irreducible components are a union of two subsets which are interchanged under the action of $Gal(\mathbb{F}_{p^2}/\mathbb{F}_p)$.*
- *The components in each subset are disjoint. Those that intersect, do it transversally, and at one point, defined over $\mathbb{F}_{p^2}$. Every $\mathbb{F}_{p^2}$ point of N^s is such an intersection.*

The following picture is from page 5 of Stamm's paper:



Some other work on the supersingular include:

- Work Chai-Fu Yu on Hilbert-Blumenthal 4-folds. Mass formula for supersingular abelian surfaces.
- Work of Goren and Bachmat on the non-ordinary locus.
- Work of Ke-Zheng Li, Frans Oort on moduli of supersingular varieties