Chern numbers and Hilbert Modular Varieties

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A Topological Point of View

Given a topological group G, F_G : **Top** $\rightarrow$ **Set** via

$F_G(X) = \{$Isomorphism classes of principal G-bundles on $X\}$
$F_G(f) = f^*$ (The map sending a bundle to its pullback induced by f).

We define a *characteristic class* of principal G-bundles as any natural transformation of F_G with a cohomology functor.

Examples (Fibers, G, Cohomology functor)

- Whitney Classes w_m ($\mathbb{R}^n$, $GL_n(\mathbb{R})$, $H^m(\,\cdot\,, \mathbb{Z}/2\mathbb{Z}), m \leq n$)
- Euler Class e ($\mathbb{R}^n$, $GL_n(\mathbb{R})^+$, $H^n(\,\cdot\,, \mathbb{Z})$)
- Chern Classes c_m ($\mathbb{C}^n$, $GL_n(\mathbb{C})$, $H^{2m}(\,\cdot\,, \mathbb{Z}), m \leq n$)
- Pontrjagin Classes p_m ($\mathbb{R}^n$, $GL_n(\mathbb{R})$, $H^{4m}(\,\cdot\,, \mathbb{Z}), m \leq n/2$)

Chern Classes

Definition ($\zeta := E \xrightarrow{\pi} B$ an arbitrary $\mathbb{C}^n$-bundle)

Chern classes c_m can be defined inductively by $c_n(\zeta) = e(\zeta_{\mathbb{R}})$ and $c_m(\zeta) = \pi_0^{*-1} c_i(\zeta_0)$, where ζ_0 is a canonically defined bundle whose base space is E_0 (E with the trivial section removed) and whose fiber over (x, v) looks like $\mathbb{C}^n/(\mathbb{C} \cdot v)$.

Alternative Definition (ζ smooth, B paracompact)

Any such bundle ζ has a connection ∇ and hence a curvature tensor $K_\nabla \in C^\infty(\wedge^2 \tau_{\mathbb{C}}^* \otimes \mathrm{Hom}(\zeta, \zeta))$. K_∇ can be described locally as an $n \times n$ matrix of 2-forms. It is a theorem that the polynomial $\det(I + tK_\nabla)$ is independent of ∇ and abusing notation we have

$$\det(I + tK_\nabla) = \sum_{r=0}^{n} (2\pi i)^r c_r(\zeta) t^r.$$

An Algebraic Approach

Chern classes have an alternative characterization in algebraic geometry. Recall the *Chow ring* $A(X)$ of a non-singular quasi-projective variety X of dimension n over an algebraically closed field K.

Given a locally free sheaf $\mathcal{E}$ of rank r on X, let $\mathbf{P}(\mathcal{E})$ be the associated projective bundle over X with projection π. Then the line bundle $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ produces $\zeta \in A^1(\mathbf{P}(\mathcal{E}))$ and $A(\mathbf{P}(\mathcal{E}))$ becomes a free graded $A(X)$-module generated by $\{\zeta^i\}_{i=0}^{r-1}$. For $i = 1, 2, \ldots, r$, define the i^{th} *Chern class* $c_i(\mathcal{E})$ via the relation

$$\sum_{i=1}^{r} (-1)^i \pi^* c_i(\mathcal{E}).\zeta^{r-i} = -\zeta^r.$$

When X is a nonsingular projective algebraic variety over $\mathbb{C}$, these two definitions "agree" via Poincaré duality and Chow's theorem identifying the Chow ring of X with its nonsingular cohomology ring.

Properties of Chern Classes

- *Chern polynomial* $c_t(\mathcal{E}) := \sum_{i=0}^{r} c_i(\mathcal{E})t^i$
- $c_t(\mathcal{E} \oplus \mathcal{F}) = c_t(\mathcal{E}) \cdot c_t(\mathcal{F})$
- $c_t(\epsilon) = 1$, where ϵ is the trivial bundle of any dimension
- $c_t(\check{\mathcal{E}}) = c_{-t}(\mathcal{E})$, where $\check{\mathcal{E}}$ is the dual bundle of $\mathcal{E}$.
- If D is a divisor, then $c_t(\mathcal{L}(D)) = 1 + tD$
- The *splitting principle*

Examples ($c_t(X)$ is the Chern polynomial for the *tangent bundle* over X)

- $c_t(\mathbb{A}^n_K) = 1$
- $c_t(\mathbb{P}^n_K) = (1 + Ht)^{n+1}$, where H is the divisor of a hyperplane
- If $X \subset \mathbb{P}^n$ is a complete intersection of hypersurfaces of degrees $n_1, \ldots, n_k$ then $c_t(X) = (1 + Ht)^{n+1} \prod(1 + n_i Ht)^{-1} \in A(X)$

Why do we care?

Why Chern Classes?

- For paracompact base spaces, every characteristic class defined for $\mathbb{C}^n$-bundles with coefficient group $\mathbb{Z}$ is a unique polynomial in the Chern classes
- Chern classes provide obstructions to embeddings of manifolds / smooth varieties (Ex. No abelian surface $\hookrightarrow \mathbb{P}^3_{\mathbb{C}}$)
- Self intersection formula

Chern numbers of compact complex manifolds: For such an M of dimension n, fix a bundle $\mathcal{E}$ and consider n-cycles (n-forms) arising from intersections (cup products) of the $c_i(\mathcal{E})$. Taking degree maps (integrating over M) gives rise to these important invariants.

Chern numbers

Information contained in Chern numbers

- Arithmetic genus $p_a(X)$
- Geometric genus $p_g(X)$
- Hirzebruch-Riemann-Roch formula for holomorphic Euler characteristic $\chi(\mathcal{E}) = \deg(ch(\mathcal{E}).td(\mathcal{T}))_n$
- Intersection numbers of divisors
  (Ex. $c_1^n[X] = (-1)^n deg(\prod_1^n K)$, for K the canonical divisor of complex n-manifold X)
- Various cohomology group dimensions
  (Ex. $c_n[X] = e = \sum(-1)^i \text{rank } H^i(X, \mathbb{Z})$, X as above)

The Chern numbers of a complex projective manifold (ie. of its tangent sheaf) also uniquely determine its *complex cobordism* class. This is the focus of the works of Thomas and Vasquez.

Hilbert Modular Varieties

Fix totally real field $\mathcal{F}$ of degree n with Hilbert modular group G. Let $\Gamma = G$ or some torsion-free arithmetic subgroup of finite index. Let $\hat{Y}_\Gamma$ be the minimal compactification of $\mathcal{H}^n/\Gamma \subset \mathbb{P}^1(\mathbb{C})^n$.

- By Hironaka there exists minimal resolutions $Z_\Gamma \to \hat{Y}_\Gamma$
- When $n = 2$, well known canonical construction of Z_Γ due to Hirzebruch
  - For a cusp of type (M, V), decompose $\mathbb{R}^2_+$ into simplicial cones spanned by boundary points of M_+. Glue together via V. Distinct copies correspond to distinct nonsingular rational curves lying over cusp in resolution.
  - Similar for quotient singularities. $H =$ isotropy subgroup of x, H acts on $\mathbb{C}^2$ via complex rotations (type of H) which identifies neighbourhood of 0 in $H\backslash\mathbb{C}^2$ with neighbourhood of x. Again reduce to finding bases of a lattice M as vertices of simplicial cones. Each cone gives a copy of $\mathbb{P}^1$ lying over 0.

Classification of Hilbert Modular Surfaces

An important problem is to classify the Z_Γ for all Γ. For $n = 2$, a lot of progress has been made on this problem.

Explicit descriptions for certain fields / subgroups exist [6], [7]:

- $\mathcal{F} = \mathbb{Q}(\sqrt{13}), \Gamma =$ principal congruence subgroup of 2, Z_Γ is isomorphic to a blow up of the minimal resolution of

$$\left\{ (x_0 : \ldots : x_4) \in \mathbb{P}^4(\mathbb{C}) \mid \sum_{i=0}^{4} x_i = 0, \sum_{i=0}^{4} x_i^5 = \frac{5}{12} \sum_{i=0}^{4} x_i^2 \sum_{j=0}^{4} x_j^3 \right\}$$

- $\mathcal{F} = \mathbb{Q}(\sqrt{6}), \Gamma =$ principal congruence subgroup of unique prime lying over 2, Z_Γ is isomorphic to the minimal resolution of singularities of a certain double covering of $\mathbb{P}^2(\mathbb{C})$.

One major result for $\mathcal{F}$ a real quad. field of discriminant D, $\Gamma = SL_2(\mathcal{O}_{\mathcal{F}})$ is the complete understanding of how the corresponding Hilbert modular surface ($Z_\Gamma(D)$) fits into the Enriques-Kodaira rough classification of surfaces, which classifies a surface X in terms of its Hodge numbers $h^{i,j} = \dim H^j(X, \Omega^i)$.

- Such a surface is regular and simply connected, hence is either rational, or admits a minimal model that is a K-3, honestly elliptic, or of general type
- One can use the Chern numbers of $Z_\Gamma(D)$ to determine the classification. For example, if D is a prime congruent with 1 mod 4, the arithmetic genus completely determines the surface type:

$$P_a = \frac{c_1^2 + c_2}{12}[Z_\Gamma(D)] - 1 \begin{cases} = 1 \Leftrightarrow Z_\Gamma(D) \text{ is rational} \\ = 2 \Leftrightarrow Z_\Gamma(D) \text{ is a blown up } K\text{-3} \\ = 3 \Leftrightarrow Z_\Gamma(D) \text{ is a blown up elliptic} \\ \geq 4 \Leftrightarrow Z_\Gamma(D) \text{ is of general type.} \end{cases}$$

- More careful analysis shows that the surface type of general $Z_\Gamma(D)$ can be determined by the Chern numbers $c_1^2[Z_\Gamma(D)] = K.K$ and $c_2[Z_\Gamma(D)] = e$ along with numerical estimates and the intersection numbers of various curves (which are determined by the Chern classes of said curves).
- This is convenient, as the Chern numbers of these surfaces are readily calculable:
  - $c_2[Z_\Gamma(D)] = 2\zeta_{\mathcal{F}}(-1) + \frac{1}{2}a_2(Z_\Gamma(D)) + \frac{2}{3}a_3(Z_\Gamma(D))$ for $D \geq 13$
  - $c_1^2[Z_\Gamma(D)] = 4\zeta_{\mathcal{F}}(-1) - l_0^- - \frac{a_3^+(Z_\Gamma(D))}{3}$ for $D \geq 13$,

  where a_i is the number of quotient singularities of type $(i, 1, *)$, a_3^+ the number of quotient singularities of type $(3, 1, 1)$, l_0^- the number of curves arising from minimal resolution of cusps of $\mathcal{H} \times \mathcal{H}^-$

Final Result of the Classification of $Z_\Gamma(D)$

- Rational for $D \in \{5, 8, 12, 13, 17\}$
- Blown-up K-3 for $D \in \{21, 24, 28, 29, 33, 37, 40, 41\}$
- Blown-up honestly elliptic for $D \in \{44, 53, 57, 61, 65, 73, 85\}$
- General type otherwise.

3 Dimensional Hilbert Modular Varieties

When $n > 2$, much less is know about the structure of the Z_Γ.

- Ehlers [1] generalizes Hirzebruch method for resolving singularities however for a cusp singularity (M, V), there is no canonical decomposition of $(M \otimes \mathbb{R})_+$ into appropriate simplicial cones.
- Thomas and Vasquez [2] have explicitly constructed resolutions for a certain family of cubic fields using this method. The surfaces lying over the cusps are usually fairly simple (ex. projective planes, ruled surfaces, Hirzebruch surface, etc).

In [3], [4], Thomas and Vasquez classify the varieties Z_Γ up to complex cobordism, which by Milnor amounts to calculating the Chern numbers

$$c_1^3[Z_\Gamma], c_1c_2[Z_\Gamma], c_3[Z_\Gamma].$$

(Note that these numbers are for a fixed resolution Z_Γ.)

Calculating the Chern numbers

Each of the Chern numbers for such a Z_Γ reflects an interesting geometric invariant: $c_1^3[Z_\Gamma] = -K^3, c_1c_2[Z_\Gamma] = 24p_a, c_3[Z_\Gamma] = e$.

The arithmetic genus p_a is actually a birational invariant and hence independent of resolution. It's calculation follows from the following theorem of Hirzebruch and Vigneras:

Theorem

Let $\mathcal{F}$ be a totally real number field of degree n which contains a unit of norm -1, and let Γ be a subgroup of G of modular type. Then

$$p_a(Y_\Gamma) = \frac{1}{2^n}\left[[G:\Gamma]\cdot\zeta_{\mathcal{F}}(-1) + \sum_{r\geq 2} a_r(\Gamma)\cdot\frac{r-1}{r}\right]$$

The values of the other two invariants are more delicate, and depend explicitly on the resolution chosen. Let $\xi = \{E_1, \ldots, E_h\}$ denote a set of non-singular varieties resolving each cusp x_i of Y_Γ. Set $K^3(\xi) = \sum_1^h K^3[E_i]$ and $e(\xi) = \sum_1^h e[E_i]$. The following theorem is the main result of [4].

Theorem (Thomas and Vasquez)

Let Γ be a subgroup of modular type for a totally real cubic number field $\mathcal{F}$. Let Z_Γ be a non-singular Hilbert modular variety for Γ, with cusp resolutions ξ as above. Then

$$\begin{aligned} K^3[Z_\Gamma] &= a_2/2 + a_3/2 + 6a_7/7 + 7a_9/6 + K^3(\xi) - 12d\zeta_{\mathcal{F}}(-1), \\ e[Z_\Gamma] &= 5a_2/2 + 25a_3/6 + 117a_7/14 + 187a_9/18 + e(\xi) + 2d\zeta_{\mathcal{F}}(-1). \end{aligned}$$

Here $a_r = a_r(\Gamma)$, and $d = [G : \Gamma]$.

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