Quiz 3 – Higher Algebra I, Math 570, Fall 2008

Date: November 17, 2008.
Time: 10:30 - 12:00.
Instructor: Prof. Eyal Goren.

Instructions: Answer as many questions as you can, but not at the cost of writing untidy solutions. You can answer a question assuming other questions (if needed!).

For a ring R we let $J(R)$ denote its Jacobson radical. $[0,1]$ denotes $\{x \in \mathbb{R} : 0 \leq x \leq 1\}$. For a ring R we say that R has no zero-divisors if the equation $xy = 0$ implies $x = 0$ or $y = 0$. For a field $\mathbb{F}$ and a finite group G, $\mathbb{F}[G]$ is the group ring of G.

1. (15 points) Prove Nakayama's lemma:
Let R be a ring and M a finitely generated left R-module such that $J(R)M = M$ then $M = 0$.

2. (15 points) Show that the condition that M is finitely generated in Nakayama's lemma is necessary.

3. (15 points) Let R be the ring of continuous functions $f : [0,1] \to [0,1]$. Prove that R is neither artinian nor noetherian. Do not use the Hopkins-Levitzky Theorem.

4. (20 points) Let R be a ring and

$$0 \to M_1 \to M_2 \to M_3 \to 0,$$

an exact sequence of left R-modules. Prove that if M_1 and M_3 are noetherian then so is M_2.

5. (25 points) Let $\mathbb{F}$ be a field with p elements, p prime and G a cyclic group of order p, say $G = \{1, x, \ldots, x^{p-1}\}$. Show that $\mathbb{F}[G]$ has a nilpotent element. Conclude that $\mathbb{F}[G]$ is not semi-simple.

6. (20 points) Let k be a field and let R be a semisimple, finite dimensional k-algebra with no zero divisors. Prove that R is a division ring.