EXERCISES FOR HIGHER ALGEBRA I, FALL 2009

NOTE: although I do my best to make sure no typos occur, they still do occur sometimes. If you think there's an error, please discuss it with me. You may be right.

(1) Let R be a division ring. Prove that every module over R is free. You will need to use Zorn's lemma:

Recall that a partially order set (=poset) S is a set with a relation $x \leq y$ defined between some pairs of elements $x, y \in S$, such that: (i) $x \leq x$; (ii) $x \leq y$ and $y \leq x$ implies $x = y$; (iii) $x \leq y, y \leq z \Rightarrow x \leq z$. A chain in S is a subset $T \subset S$ such that for all t, t' in T, either $t \leq t'$ or $t' \leq t$. We say that a chain has an upper bound if there's an element $s \in S$ (we don't require $s \in T$) such that $s \geq t$ for all $t \in T$. Zorn's lemma states for a non-empty poset S that if every chain in S has an upper bound than S has a maximal element, namely an element $s_0 \in S$ such that if $s \in S$ and $s \geq s_0$ then $s = s_0$ (note that we do not require that $s_0 \geq s$ for all $s \in S$). If you have never seen Zorn's lemma in action, try to use it to prove that any ring R has a maximal left ideal. Take S to be the set of ideals $I \neq R$ of R with the partial order $I \leq J$ if $I \subseteq J$.

(2) Analyze the structure of the rings $\mathbb{Q}[G]$, $\mathbb{C}[G]$, where G is the cyclic group $\mathbb{Z}/n\mathbb{Z}$.

(3) We define a ring $R[S^{-1}]$ as follows: consider symbols $\frac{r}{s}$ where $r \in R$ and $s \in S$ and define a relation:

$$\frac{r_1}{s_1} \sim \frac{r_2}{s_2} \quad \Longleftrightarrow \quad \exists t \in S \quad t(r_1 s_2 - r_2 s_1) = 0.$$

Prove that this is an equivalence relation. Prove that the operations

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{r_1 s_2 + r_2 s_1}{s_1 s_2}, \qquad \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2},$$

make $R[S^{-1}]$ into a commutative ring and that the natural map

$$R \to R[S^{-1}], \qquad r \mapsto \frac{r}{1}$$

is a ring homomorphism. Find its kernel. Give examples when the kernel is trivial and when the kernel is not trivial.

(4) Let I be an ideal of R then $I[S^{-1}]$ is an ideal of $R[S^{-1}]$, which is the ideal generated by I in $R[S^{-1}]$. Conversely, if $\varphi : R \to R[S^{-1}]$ is the natural map and J is an ideal of $R[S^{-1}]$ then $\varphi^{-1}(J)$ is an ideal of R. Prove that $(\varphi^{-1}(J))[S^{-1}] = J$ and if $I \cap S = \emptyset$ then $\varphi^{-1}(I[S^{-1}]) \supseteq I$.

Prove that if I is a prime ideal and $S = R - I$ then these constructions provide a bijection between the prime ideals of R contained in I and the prime ideals of $R[S^{-1}]$. In particular, $R[S^{-1}]$ is a local ring whose maximal ideal is $I[S^{-1}]$.

(5) Let S be a multiplicative set and $0 \to M_1 \to M_2 \to M_3 \to 0$ an exact sequence of R-modules. Prove that the sequence $0 \to M_1[S^{-1}] \to M_2[S^{-1}] \to M_3[S^{-1}] \to 0$ is also exact.

(6) Let $\mathbf{C}$ be a category. An object A of $\mathbf{C}$ is called initial (resp. final) if for every object B there is a unique morphism $A \to B$ (resp., $B \to A$). Prove that if $\mathbf{C}$ has an initial (resp. final) object then it is unique, up to unique isomorphism. Give examples of categories $\mathbf{C}$ such that:
 (a) $\mathbf{C}$ has an initial object and doesn't have a final object.
 (b) $\mathbf{C}$ doesn't have an initial object and has a final object.
 (c) $\mathbf{C}$ doesn't have an initial object and doesn't have a final object.
 (d) $\mathbf{C}$ has an initial object and has a final object, but they are non-isomorphic.
 (e) $\mathbf{C}$ has object that is both initial and final. (Such an object is sometimes called a zero object.)

(7) Let n be a positive integer and k a field. Consider the category $\mathbf{V}$ whose objects are finite dimensional k-vector spaces V, equipped with n-linear maps $T_1, \dots, T_n : V \to V$ that commute with each other. A morphism

$$H : (V; T_1, \dots, T_n) \to (V'; T'_1, \dots, T'_n)$$

is a linear map $H : V \to V'$ such that $H \circ T_i = T'_i \circ H$.

Prove that this category is equivalent to the category $\mathbf{M}$ whose objects are n-tuples of commuting $m \times m$-matrices $(M_1, \dots, M_n)$ and a morphism

$$H : (m; M_1, \dots, M_n) \to (m'; M'_1, \dots, M'_n)$$

is an $m' \times m$ matrix H such that $HM_i = M'_i H$ for all i.

Prove further that these categories are equivalent to the category of modules over $k[x_1, \dots, x_n]$ that are finite dimensional k-vector spaces. To which data does $k[x_1, x_2]/(x_1^2, x_2^2)$ correspond?

(8) Let R be a ring and define a ring R^{op} to be the same underlying abelian group of R, but where multiplication is defined by $a * b := ba$, where ba is the product of b and a in R.
 (a) Prove that R^{op} is a ring.
 (b) Prove that the category of left R-modules ${}_R\mathbf{Mod}$ is equivalent to the category of right R^{op}-modules.
 (c) Bonus question: Give an example of a ring that is not isomorphic to its opposite ring.

(9) Let $\mathbf{K}$ be a category and define the opposite category $\mathbf{K}^{\mathrm{op}}$ to be the category $\mathbf{K}$ with the same objects and $\mathrm{Mor}_{\mathbf{K}^{\mathrm{op}}}(A, B) := \mathrm{Mor}_{\mathbf{K}}(B, A)$. We define $f \circ g$ in $\mathbf{K}^{\mathrm{op}}$ to be $g \circ f$ as performed in $\mathbf{K}$. Prove that $\mathbf{K}^{\mathrm{op}}$ is a category and note that $(\mathbf{K}^{\mathrm{op}})^{\mathrm{op}} = \mathbf{K}$.

Prove that if A is an initial (resp. final) object then A is a final (resp. initial) object in $\mathbf{K}^{\mathrm{op}}$. Prove that if $\mathbf{K}$ has products (resp. coproducts) then $\mathbf{K}^{\mathrm{op}}$ has coproducts (resp. products).

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(10) Prove that $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ is an infinite group.

(11) Prove that in the category of linearly ordered sets co-products $A\coprod B$ need not exists, but that co-products $A\coprod B$ exists in the category of posets.
(12) Prove that injective limits exist in the category Sets.
(13) For all $m|n$ we have $\frac{1}{m}\mathbb{Z}/\mathbb{Z} \hookrightarrow \frac{1}{n}\mathbb{Z}/\mathbb{Z}$. Prove that this gives a direct system and $\varinjlim_{n} \frac{1}{n}\mathbb{Z}/\mathbb{Z} \cong \mathbb{Q}/\mathbb{Z}$.
(14) Let I be a poset (the index set). I is called directed if $\forall i,j \in I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$.

Given an injective system $\{M_i, f_{ij}\}$ of R-modules indexed by a directed set I, let us say that $m_i \in M_i$ is equivalent to $m_j \in M_j$ if there k such that $i \leq k, j \leq k$ and $f_{ik}(m_i) = f_{jk}(m_j)$. Check that this defines an equivalence relation on the disjoint union of the M_i and denote an equivalence class by $[m_i]$. Give the equivalence classes a structure of an R-module by

$$r[m_i] = [rm_i], \qquad [m_i] + [n_j] = [f_{ik}(m_i) + f_{jk}(m_j)],$$

where k is any element such that $i \leq k, j \leq k$. Show that this is well-defined and that this R-module is isomorphic to $\varinjlim_{I} M_i$.
(15) Let k be a field. Prove that $\varprojlim_{n} k[t]/(t^n) \cong k[[t]]$.
(16) The projective limit of $\ldots \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to \{0\}$ is denoted $\mathbb{Z}_p$. Prove that $\mathbb{Z}_p$ is a commutative integral domain containing $\mathbb{Z}$. Define a metric on $\mathbb{Z}$ by

$$d(m,n) = p^{-\mathrm{val}_p(m-n)},$$

where, for $x \in \mathbb{Z}$, we let,

$$\mathrm{val}_p(x) = \text{ highest power of } p \text{ dividing } x.$$

Define also a valuation on $\mathbb{Z}_p$ by $\mathrm{val}_p((m_i)) = \max\{i : m_i = 0\}$. Show that this extends the definition of val_p on $\mathbb{Z}$ and that $d(x,y) = p^{-\mathrm{val}_p(x-y)}$ is a metric on $\mathbb{Z}_p$. Prove that $\mathbb{Z}_p$ is the completion of $\mathbb{Z}$ relative to this metric. Prove that $\mathbb{Z}_p$ is compact.
(17) Let F : **TopSp** $\to$ **Set** be the forgetful functor. Show that F has both a left adjoint and a right adjoint.
(18) Let (F,G), $F : C \to D, G : D \to C$, be an adjoint pair. Suppose that A, B are object in D having a product $A\Pi B$. Prove that $G(A\Pi B)$ is a product for $G(A), G(B)$. Similarly, F takes co-products to co-products.
(19) Let R be a commutative ring and I_1, I_2 two ideals of R. Prove that

$$R/I_1 \otimes R/I_2 \cong R/(I_1 + I_2).$$

(20) Prove that $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \cong \{0\}$.
(21) Let V/k be a finite dimensional vector space over a field k of characteristic different from 2. The tensor algebra $T(V)$ of V is the direct sum

$$\oplus_{I=0}^{\infty} V^{\otimes i} = k \oplus V \oplus (V \otimes_k V) \oplus (V \otimes_k V \otimes_k V) \oplus \ldots$$

It is a k-vector space, being a direct sum of k-vector spaces, of infinite dimension. It has a well-defined multiplication induced by

$$v_1 \otimes \cdots \otimes v_a \ \cdot \ w_1 \otimes \cdots \otimes w_b = v_1 \otimes \cdots \otimes v_a \otimes w_1 \otimes \cdots \otimes w_b.$$

This makes $T(V)$ into a (non-commutative, in general) ring containing k. This ring is graded: $T(V) = \oplus_{i=0}^{\infty} V^{\otimes i}$, the weight i elements are $V^{\otimes i}$, and we have under multiplication $V^{\otimes i} \times V^{\otimes j} \to V^{\otimes i+j}$.

Let now $q : V \to k$ be a quadratic form, i.e., the pairing $B(x, y) = q(x+y) - q(x) - q(y)$ is bilinear. Let I be the two-sided ideal of $T(V)$ generated by all expressions of the form $v \otimes v - q(v)$ where v runs over V. The Clifford algebra is defined to be

$$\mathrm{Cliff}(V, q) = T(V)/I.$$

There is a natural map $V \to \mathrm{Cliff}(V, q)$ which is injective.

(a) Prove that $\mathrm{Cliff}(V, q)$ has a natural $\mathbb{Z}/2\mathbb{Z}$-grading. Namely, we can write $\mathrm{Cliff}(V, q) = \mathrm{Cliff}(V, q)^0 \oplus \mathrm{Cliff}(V, q)^1$ such that under multiplication $\mathrm{Cliff}(V, q)^i \times \mathrm{Cliff}(V, q)^j \subseteq \mathrm{Cliff}(V, q)^{i+j \pmod 2}$. In particular, $\mathrm{Cliff}(V, q)^0$ is an algebra, called the even Clifford algebra.

(b) Prove that $\mathrm{Cliff}(V, q)$ has the following universal property. Any map f of k-vector spaces from V to a k-algebra W, $f : V \to W$, such that $f(v)^2 = q(v)$ for all $v \in V$, factors uniquely through $\mathrm{Cliff}(V, q)$:

$$\begin{tikzcd} V \arrow[r, "can"] \arrow[dr, "f"'] & \mathrm{Cliff}(V, q) \arrow[d, "\exists!"] \\ & W \end{tikzcd}.$$

(c) Deduce from (b) an isomorphism $\mathrm{Cliff}(V, q) \to \mathrm{Cliff}(V, q)^{op}$, which is the identity on V, between the Clifford algebra and its opposite algebra. Deduce that $\mathrm{Cliff}(V, q)$ has an anti-automorphism ϵ, whose effect on generators is $\epsilon(v_1 \cdots v_m) = v_m \cdots v_1$, where $v_i \in V$.

(d) Show that the map $V \to V$, $v \mapsto -v$, induces an automorphism of V as well. We denote it by σ. Write the action of σ on elements $v_1 \cdots v_m$ as above. Let $\alpha = \sigma \circ \epsilon = \epsilon \circ \sigma$.

(e) Show that if $x \in \mathrm{Cliff}(V, q)$ is invertible so is $\alpha(x)$ and that x acts on $\mathrm{Cliff}(V, q)$ by "twisted conjugation"

$$y \mapsto c_x(y) := xy\alpha(x)^{-1}.$$

The invertible elements $x \in \mathrm{Cliff}(V, q)$ with the property that $c_x(V) \subseteq V$ form a group, called the Clifford group. Prove that indeed this is a group. Denote it by Γ. One can prove (but this is not easy, and not required here) that there's an exact sequence,

$$1 \to k^* \to \Gamma \to O(V, q) \to 1.$$

This forms the first step in constructing the Spin group and the Spin representation, which play a role in physics.

(f) Suppose that $\dim(V) = n$. Prove that $\dim(\mathrm{Cliff}(V, q)) = 2^n$. This may prove a little difficult. Try doing first the cases $n = 1, 2$. In general, show that V has an orthogonal basis: namely, a basis $x_1, \dots, x_n$ such that $B(x_i, x_j) = 0$ for $i \neq j$. Note that since q is allowed to be identically zero we cannot expect to find a basis such that $q(x_i) = 1$ or even not equal to zero. Show in general that $x_1^{a_1} \cdots x_n^{a_n}$, where $a_i \in \{0, 1\}$ is a spanning set for $\mathrm{Cliff}(V, q)$ as a k-vector space. The difficult part is to show it's independent (although you can probably do it).

(22) Suppose that G is a finite group, $H < G$ a subgroup and that $(\sharp H, char(k)) = 1$. Let W be a representation of H. Given $f : G \to W$, which is an element of $\mathrm{Ind}_H^G(W)$, show that its image under the isomorphism $\mathrm{Ind}_H^G(W) \cong k[G] \otimes_{k[H]} W$ is $\frac{1}{\sharp H}\sum_{g\in G} g \otimes f(g)$.

(23) Give examples showing that $\mathrm{Hom}_R(A, -)$ and $\mathrm{Hom}_R(-, A)$ are not right exact. Prove that they are both left exact.

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(24) Let R be a commutative ring and M_1, M_2 projective R-modules. Give two proofs that $M_1 \otimes_R M_2$ is a projective R-module.

(25) Let M be a projective R-module. Prove that there is free R-module F such that $M \oplus F$ is free.

(26) Let R be a commutative ring and M an R-module. Let $\mathfrak{p}$ be a prime ideal and denote by $R_\mathfrak{p}$ and $M_\mathfrak{p}$ the localizations of R and M in the multiplicative set $S = R - \mathfrak{p}$, respectively. Prove that M is zero if and only if each $M_\mathfrak{p}$ is zero. In fact, show that it is enough to let $\mathfrak{p}$ run over maximal ideals.

(27) With the notation above, prove that localization at $\mathfrak{p}$ (meaning: in S) is an exact functor from the category of R-modules to the category of $R_\mathfrak{p}$-modules.

(28) Assume that M is a finitely presented R-module. Prove that M is a projective R-module if and only if each $M_\mathfrak{p}$ is a projective $R_\mathfrak{p}$-module. One says that "projective is a local property". (Hint: show that Hom and localization commute in a suitable sense.)

(29) Show that projective doesn't imply injective; that injective doesn't imply projective.

(30) Show that $\mathbb{Q}/\mathbb{Z}$ is injective but not flat; Show that $\mathbb{Z} \oplus \mathbb{Q}$ is flat but is not injective or projective.

(31) Let R be a principal ideal domain (for example, a field). Show that R is an injective R-module if and only if R is divisible (for every non-zero $r \in R$ multiplication by r is surjective map $R \to R$).

(32) Let R be an integral domain. Prove that R is a field if and only if R is both injective and projective R-module.

(33) Let $B \in \mathbf{Mod}_R$ and let its character module B^* be defined as

$$B^* = \mathrm{Hom}_\mathbb{Z}(B, \mathbb{Q}/\mathbb{Z}).$$

It is a module in ${}_R\mathbf{Mod}$. Prove that

$$0 \to A \to B \to C \to 0$$

is exact, if and only if

$$0 \to C^* \to B^* \to A^* \to 0$$

is exact.

Guidance: the direction $\Rightarrow$ is easy if you use what you should.... For the other direction, first argue that it's enough to prove that a diagram $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ with $\mathrm{Ker}(\alpha^*) = \mathrm{Im}(\beta^*)$ is exact (for any A, B, C, α, β). In order to show that, first prove the following lemma

Lemma: If Q is a non-trivial abelian group and $q \in Q$ is non-zero, then there is a homomorphism $h : Q \to \mathbb{Q}/\mathbb{Z}$ such that $h(q) \neq 0$.

Suppose that $\mathrm{Im}(\alpha)$ is not contained in $\mathrm{Ker}(\beta)$ and choose an $a \in A$ such that $\beta\alpha a \neq 0$. Apply the lemma to get $h : C \to \mathbb{Q}/\mathbb{Z}$ such that $h(\beta\alpha a) \neq 0$. Proceed to derive a contradiction.

After proving that $\mathrm{Im}(\alpha) \subset \mathrm{Ker}(\beta)$ assume that there is a $b \in \mathrm{Ker}(\beta) - \mathrm{Im}(\alpha)$. Use the lemma to construct $h : B/\mathrm{Im}(\alpha) \to \mathbb{Q}/\mathbb{Z}$ such that $h(b + \mathrm{Im}(\alpha)) \neq 0$.

The context of this exercise is the following theorem (which you are NOT required to prove):

Theorem: $B \in \mathbf{Mod}_R$ is flat if and only if $B^* \in_R \mathbf{Mod}$ is injective.

Note that the case $B = R$ is a consequence of a result we proved in class.

(34) Let $\{M_i, f_{ij}\}$ be an inverse system in ${}_R\mathbf{Mod}$ and B a left R-module. Prove that

$$\mathrm{Hom}_R(B, \varprojlim M_i) \cong \varprojlim \mathrm{Hom}_R(B, M_i).$$

(35) Let $\{M_i, f_{ij}\}$ be a direct system in ${}_R\mathbf{Mod}$ and B a left R-module. Prove that

$$\mathrm{Hom}(\varinjlim M_i, B) \cong \varprojlim \mathrm{Hom}(M_i, B).$$

If B is a right R-module, prove that

$$B \otimes_R \varinjlim M_i \cong \varinjlim (B \otimes_R M_i).$$

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(36) Let $f : M \to M$ be a surjective homomorphism of R-modules. Prove that if M is noetherian then f is an isomorphism.

(37) Prove that the following rings are not noetherian: (i) $\mathbb{C}[t^{1/n} : n \in \mathbb{N}_{>0}]$. (Note that $\mathbb{C}[t^{1/n} : n \in \mathbb{N}_{>0}] = \varprojlim_m \mathbb{C}[t^{1/m!}]$ and so a direct limit of noetherian rings need not be noetherian. (ii) The ring of continuous functions $f : [0, 1] \to \mathbb{R}$. (iii) The ring $\mathbb{C}[x, x^2y, x^3y^2, \dots, x^iy^{i-1}, \dots]$. (Note that this is a subring of $\mathbb{C}[x, y]$ and so a subring of a noetherian ring need not be noetherian.)

(38) Show that the number of generators for ideals in $\mathbb{C}[x, y]$ is not bounded. Namely that for every $n \in \mathbb{N}$ there is an ideal of $\mathbb{C}[x, y]$ that cannot be generated by less than n elements.

(39) Let $P = \cup_{k=1}^{\infty} \frac{1}{p^k}\mathbb{Z}/\mathbb{Z}$. Prove that P is an artinian but not noetherian $\mathbb{Z}$-module.

(40) Let R be a simple R-module. Prove that R is a division ring. (Remark: don't forget that for $a \neq 0$ you need to find b such that $ab = ba = 1$.)

(41) For a ring R define, $\tilde{J}(R)$ to be the intersection of all maximal right ideals of R. In analogy to $J(R)$ it would be the collection of all element $x \in R$ such that for every $r \in R$ the element $(1 - xr)$ has a right inverse. Prove that $\tilde{J}(R) = J(R)$. Guidance: For $z \in J(R)$ use that $1 - z$ has a left inverse and right it as $(1 - z')(1 - z) = 1$. What can you say about z' and therefore about $1 - z'$?

(42) Show that in Nakayama's lemma the condition that the module is finitely generated is necessary. Hint: take R to be a local ring.

(43) Let R be a commutative ring. Let $nil(R)$ the collection of nilpotent elements of R. Show that $nil(R)$ is an ideal of R and that $nil(R) \subseteq J(R)$. Show that $nil(R)$ need not be equal to $J(R)$. Show that if R is artinian then $nil(R)$ is a nilpotent ideal (and in fact, $nil(R) = J(R)$), but this need not be the case of a general ring R.

(44) Prove the Hopkins-Levitzky theorem: Every left artinian ring is left noetherian.
Suggestion: consider the series $R \subseteq J(R) \subseteq J(R)^2 \subseteq \cdots \subseteq J(R)^m = \{0\}$. Let $J = J(R)$.

(a) Show that $R/J(R)$ is a semi-simple artinian ring.
(b) Show that each J^i/J^{i+1} is an artinian R/J-module.
(c) Show that each J^i/J^{i+1} has a finite length as an R/J-module, hence as an R-module.
(d) Put everything together to conclude that R is an R-module of finite length. Use the fact explained in class to finish the proof.

(45) Let k be a field. Prove that every commutative finite dimensional k-algebra is a finite sum of field extensions of k.

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(46) Let G be a finite group and V a finite dimensional complex representation of G, $\rho : G \to \mathrm{GL}(V)$. Prove that there is a hermitian form which is G-invariant:

$$\langle \rho(g)v, \rho(g)w \rangle = \langle v, w \rangle.$$

Conclude that if $U \subset V$ is a subrepresentation then also $U^\perp$ is. Conclude that $\mathbb{C}[G]$ is semi-simple.

(47) Let k be an (algebraically closed) field and G a finite group. The trivial representation of G, $G \to \mathrm{GL}_1(k)$, $g \mapsto 1$ for all g, is an irreducible representation. Find the corresponding maximal ideal of $k[G]$.

(48) Let k be an (algebraically closed) field and G a finite abelian group such that $\mathrm{char}(k) \nmid \sharp G$, $\rho : G \to \mathrm{GL}_n(k)$ a representation. Prove, directly, that ρ is diagonalizable; there is a basis relative to which every $\rho(g)$ is a diagonal matrix. Conclude that every irreducible representation of G is one dimensional.

(49) Use the isomorphism $S_3 \cong D_3$ to find another realization of the irreducible two dimensional representation of S_3.

(50) Find all irreducible representations of D_4, including a model for each such representation. (Note that D_4 has a normal subgroup of order 2 such that the quotient is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.) Write the character table of D_4.

(51) Let $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the non-abelian quaternion group of order 8. (Here $ij = k = -ji, i^2 = j^2 = k^2 = -1$.) Find the conjugacy classes of Q and the number of irreducible representations and their dimension. Find a model for each irreducible representation. Write the character table of Q.

(52)
- Let G be a finite group and (V, ρ) a finite dimensional representation of G over an algebraically closed field k such that $char(k) \nmid \ \sharp\, G$. Let $V^G = \{v \in V : \rho(g)v = v, \forall g \in G\}$ be the subspace of fixed vectors. Prove from first principles that

$$\dim_k(V^G) = \frac{1}{\sharp\, G} \sum_{g \in G} \chi_\rho(g).$$

(Hint: consider the operator $\frac{1}{\sharp\, G} \sum_{g \in G} \rho(g)$.)
- In the same setting, let V and W be irreducible representations of G. Prove that $\mathrm{Hom}_{k[G]}(V, W)$ is one dimensional if $V \cong W$ and 0-dimensional otherwise. (You may want to use Schur's lemma and perhaps Morita's equivalence + Artin-Weddernburn).
- Now, use the calculation of the character of $\mathrm{Hom}_k(V, W)$ and the previous parts of the question to conclude that if χ, ψ, are irreducible characters then

$$\langle \chi, \psi \rangle = \begin{cases} 1 & \chi = \psi \\ 0 & \text{else} \end{cases}$$

(53) Let G be a finite group acting on a finite non-empty set S. Use the formula for $\dim(V^G)$ to deduce the Cauchy-Frobenius formula (also knows as Burnside's lemma) that states that the number of orbits of G in S is equal to $\frac{1}{\sharp G}\sum_{g\in G} I(g)$, where $I(g)$ is the number of fixed points of g in S.

(54) Deduce from the previous question that if $|S| = n > 1$ and G acts transitively on S then there is an element $g \in G$ without fixed points. Let $G_0 = \{g \in G : g \text{ has no fixed point in S}\}$. It is a subset of G (but usually not a subgroup). Let

$$c_0 = \sharp\, G_0/\sharp\, G.$$

Jordan proved that $c_0 \geq 1/\sharp\, G$. Here we prove the stronger result (a result of Cameron-Stewart) that $c_0 \geq 1/\sharp S$.

To prove that construct the vector space on the basis S and let χ be the character of the representation of G on that space. First prove that

$$\frac{1}{\sharp\, G}\sum_{g\in G} \chi^2(g) \geq 2.$$

(Which representation is lurking in the background?...) Then prove that theorem on c_0 by arguing that

$$\sum_{g\in G}(\chi(g)-1)(\chi(g)-n) \leq n\, \sharp G_0$$

and continuing to examine this inequality.

(55) Find the character table of A_4. If χ is an irreducible character of S_4 calculate its restriction to A_4 as a sum of irreducible characters of A_4. Finally, is the 3-dimensional representation of A_4, coming from its action on the tetrahedron, irreducible?

(56) Let G_1, G_2 be finite groups. Prove (under the usual conditions on k) that every irreducible representation of $G_1 \times G_2$ is isomorphic to the tensor product $V_1 \otimes V_2$, where V_i is an irreducible representation of G_i and in fact, letting V_1 range over the irreducible representations of G_1 and V_2 range over the irreducible representations of G_2 we get every irreducible representation of $G_1 \times G_2$ once. (Suggestion: do first the last part and count how many irreducible representations you get this way.)

(57) Consider the group $\mathbb{Z}/2\mathbb{Z} \times S_3$ - a non abelian group of order 12. Write its character table. Compare it with the character table of A_4 to deduce it is not isomorphic to A_4 (although it is not hard to see that directly either).