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INTRODUCTION TO MODULAR FORMS

Introduction

Modular forms arose in association to the elliptic funcions in the early 19th century.
Nowadays the Galois rapresentation associated to modular forms play a central role in the modern Number Theory. A goal in Number Theory is to understand the finite extensions of $\mathcal{Q}$, and by Galois Thepry this is equivalent to understand the absolute Galois Group $G_{\mathcal{Q}} = Gal(\mathcal{Q}'/\mathcal{Q})$. Now, we can say that we can know the group if we know its representation, which is classified by the degrees. By Class Field Theory we have a precise understanding of the representations of deg 1, or characters. Now, when we explore outside the domain of Class Field Theory, the Galois representations associated to a modular forms are the first one we encounter.
A modular form is a certain kind of holomorphic function on the upper half plane $H = \{\tau | Im\tau > 0\}$, which we view simultaneusly as a complex manifold and as a Riemannian manifold equipped with hyperbolic metric $y^{-2}(dx^2 + dy^2)$.
In brief, an holomorphic function $f(\tau)$ on H is a modular form if it transform in a certain way under a subgroup of $SL_2(\mathbb{R})$.

Example 1

Let V be a vector space of finite dimension n, endowed with an invariant mesaure μ. We denote by V' the dual of V.

Definition 1. If f is a rapidly decreasing smooth function on V, we can define the ***Fourier Transform*** f' of f as:

$$f'(y) = \int_V e^{-2\pi i <x,y>} f(x)\mu(x)$$

Let now Γ be the lattice in V, Γ' its dual.

Proposition 1. *Let $v = \mu(V/\Gamma)$, we get:*

$$\sum_{x\in\Gamma} f(x) = \frac{1}{v}\sum_{y\in\Gamma'} f'(y)$$

After replacing μ by $v^{-1}\mu$, we can assume that $\mu(V/\Gamma) = 1$. Moroever, by taking a basis $e_1, ..., e_n$ for Γ, we can identify V with $\mathbb{R}^n$, Γ with $\mathbb{Z}^n$ and μ with $dx_1, ..., dx_n$. Thus we have: $V' = \mathbb{R}^n$, $\Gamma' = \mathbb{Z}^n$ and we are reduced to the classical **Poisson Formula**.

Remark 1. Consider V endowed with a simmetric bilinear form $x.y$ positive and non degenerate, we can identify V with V'. Then we can associate to the lattice Γ the following function defined on $\mathbb{R}$:

$$\Theta_\Gamma(t) = \sum_{x\in\Gamma} e^{-2\pi t x.x}$$

Proposition 2. *We have:*

$$\Theta_\Gamma(t) = t^{-n/2} v^{-1} \Theta_{\Gamma'}(t^{-1})$$

with $v = \mu(V/\Gamma)$ volume of the lattice.

Proof. Let $f = e^{-\pi x.x}$, we have that $f' = f$, in fact if we choose an ortonormal basis of V and we identify V with $\mathbb{R}^n$, and μ becomes $dx = dx_1 \dots dx_n$ and $f = e^{-\pi(x_1^2+\cdots+x_n^2)}$. Then, since the Fourier Transform of $e^{\pi x^2}$ is $e^{\pi x^2}$ we have done.
Now, using *Proposition* 1 on f and the lattice $t^{1/2}\Gamma$ we get the formula to be proved.

□

We are now going to consider the pair (V, Γ) with these two properties:

- The dual Γ' of Γ is equal to Γ
- $x.x \equiv 0 \mod 2 \qquad \forall x \in \Gamma$

Now, let $m \geq 0$ integer, $r_\Gamma(m) := \#\{x \in \Gamma \mid x.x = 2m\}$. It is easy to see that r_Γ is bounded by a polynomial in m. This shows that the series with integer coefficients:

$$\sum_{m=0} r_\Gamma(m) q^m$$

converges for $\mid q \mid < 1$.
Thus, one can define a function Θ_Γ on H

$$\Theta_\Gamma(\tau) = \sum_{x\in\Gamma} r_\Gamma(m) q^m$$

we have:

$$\Theta_\Gamma(\tau) = \sum_{x\in\Gamma} q^{\frac{(x.x)}{2}} = \sum_{x\in\Gamma} e^{\pi i \tau (x.x)}$$

Proposition 3. *We have:*

$$\Theta_\Gamma(-1/\tau) = (i\tau)^{n/2}\Theta_\Gamma(\tau)$$

Proof. Since the two sides are analytic in τ it is suffice to prove this formula when $\tau = iz$, with $z \in R$, $z > 0$. We have:

$$\Theta_\Gamma(iz) = \sum_{x\in\Gamma} e^{-\pi z(x.x)} = \Theta_\Gamma(z)$$

Similarly,

$$\Gamma(-1/iz) = \Theta_\Gamma(z^{-1})$$

Then, by *Proposition* 2 with $v = 1$, $\Gamma = \Gamma'$ we can conclude. □

Now, since $8|n$ we can rewrite the relation as:

$$\Theta_\Gamma(-1/z) = z^{n/2}\Theta_\Gamma(z)$$

and Θ_Γ is a modular form of weight $n/2$.

Definitions

Given H and $SL_2(\mathbb{R})$ we can make $SL_2(\mathbb{R})$ act on $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ in this way:

$$gz = \frac{az+b}{cz+d} \qquad for \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R}), \qquad z \in \mathbb{C}^*$$

We get:

$$Im(gz) = \frac{Im(z)}{|cz+d|^2}$$

i.e. H is stable under the action of $SL_2(\mathbb{R})$. We have that the element $-\mathbb{1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in GL_2(\mathbb{R})$ acts trivially on H, then we can consider that it is the projective special linear group over $\mathbb{R}$ which operates.

Definition 2. $G = SL_2(\mathbb{R})/\mp 1$ is the ***Modular Group***.

Let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, S, T in G.

Theorem 1. *The group G is generated by S and T.*

We can now consider the subset D of H formed of all points z such that $|z| > 1$ and $|Re(z)| \leq \frac{1}{2}$:

$$D = \{z = x + iy : |z| > 1, |x| \leq \frac{1}{2}\}$$

It is possible to show that D is a fundamental domani for the action og G on H. More precisely:

Theorem 2. *(1) $\forall z \in H, \exists g \in G : \quad gz \in D$*

(2) Suppose that two distinct point $z, z' \in D$ are congruent mod G. Then: $re(z) = \mp\frac{1}{2}$ and $z = z' + 1$ or $|z| = 1$ and $z' = -\frac{1}{z}$

(3) Let $z \in D$ and let $Stab(z) = \{g | g \in G, gz = z\}$ the stabilizer of z in G. We get $Stab(z) = 1$ except in the following three cases:

- *$z = i$, in which case $Stab(z)$ is the group of order 2 generated by S*
- *$z = e^{2\pi i/3}$, in which case $Stab(z)$ is the group of order 3 generated by ST*
- *$z = e^{\pi i/3}$, in which case $Stab(z)$ is the group of order 3 generated by TS*

Corollary 1. *By (1) and (2) follows that the canonical map from D to H/G is surjective and its restriction to the interior of D is injective.*

We can now state the first definition:

Definition 3. Let k be an integer, we say that f is ***weakly modular of weight 2k*** if f is meromorphic on H and:

$$f(z) = (cz + d)^{-2k} f = \left(\frac{az + b}{cz + d}\right) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

Proposition 4. *Let f be meromorphic on H, f is weakly modular of weight $2k$ if and only if it satisfies the two relations:*

(a) $f(z + 1) = f(z)$

(b) $f(-1/z) = z^{2k} f(z)$

Definition 4. A weakly modular function is a ***Modular Function*** if it is meromorphic at infinity. Moroever, we say that a modular function is of ***level*** N if it is a meromorphic funtion on H invariant under $\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a \equiv d \equiv 1, b \equiv c \equiv 0 \mod N \right\}$

Definition 5. A modular function which is holomorphic everywhere is called a ***Modular Form***, if suche a form is zero at infinity it is called a ***cusp form***.

A modular form of weight $2k$ is thus given by a series:

$$f(z) = \sum_{n=0} a_n q^n = \sum_{n=0} a_n e^{2\pi i n z}$$

which converges for $|q| < 1$ and verifies the identity (b) above. It is a cusp form if $a_0 = 0$

We can define modular forms also by means of lattices in a vector space.
Let Γ be a lattice in V, $M = \{(w_1, w_2) \in \mathbb{C}^* : Im(w_1/w_2) > 0\}$.

Proposition 5. *Two elements in M define the same lattice if and only if they are congruent mod $SL_2(\mathbb{Z})$.*

If R is the set of lattices of $\mathbb{C}$, we can identify it with the quotient of M by the action of $SL_2(\mathbb{Z})$. Make now $\mathbb{C}*$ act on R sending Γ to $\lambda\Gamma$ for $\lambda \in \mathbb{C}*$, then the quotient $M/\mathbb{C}*$ is identified with H by sending (w_1, w_2) to $z = w_1/w_2$ and this identification transforms the action of $SL_2(\mathbb{Z})$ on M into that of $G = SL_2(\mathbb{Z})/\{\mp 1\}$ on H. So, by pasing to the quotient, we get that an element of H/G can be identified with a lattice of $\mathbb{C}$ defined uo to an homothety.
So let F be a function on R with complex values, let $k \in \mathbb{Z}$, we say that F is of weight 2k if:

$$F(\lambda\Gamma) = \lambda^{-2k} F(\Gamma) \qquad \forall \Gamma, \forall \lambda \in \mathbb{C}^*$$

Let F be such a function, if $(w_1, w_2) \in M$, we denote by $F(w_1, w_2)$ the value of F on the lattice $\Gamma(w_1, w_2)$, and we can rewrite the formula above as:

$$F(\lambda w_1, \lambda w_2) = \lambda^{-2k} F(w_1, w_2)$$

Then $\exists f$ function on H such that:

(*) $f(w_1, w_2) = w_2^{-2k} f(w_1/w_2)$

Since F is invariant by $SL_2(\mathbb{R})$ we see that f satisfies the identity:

(**) $f(z) = (cz+d)^{-2k} f = \left(\frac{az+b}{cz+d}\right) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

Conversely, if $(**)$ holds, by $(*)$ we can obtain a function F on R of weight 2k.
In conclusion, we can identify modular function of weight 2k with some lattice function of weight 2k.

Example 2 - Eisenstein series

Lemma 1. *Let Γ be a lattice in $\mathbb{C}$. The series $\sum'_{\gamma \in \Gamma} \frac{1}{|\gamma|^\sigma}$ is convergent for $\sigma > 0$, where we denote with $\sum'$ the summation over all the non zero elements.*

Let k be an integer, $k > 1$. If Γ is a lattice of $\mathbb{C}$ we put: $G_k(\Gamma) = \sum'_{\gamma \in \Gamma} \frac{1}{|\gamma|^{2k}}$. By the *Lemma* above we know that the series converges absolutely. Using the definition given in the case of lattices, we can view G_k as a function on M given by:

$$G_k(w_1, w_2) = \sum_{m,n}{}' \frac{1}{(mw_1 + nw_2)^{2k}}$$

So we get that the function on H is:

$$G_k(z) = \sum_{m,n}{}' \frac{1}{(mz + n)^{2k}}$$

Proposition 6. *Let k be an integer, $k \geq 1$. The Eisenstein series $G_k(z)$ is a modular form of weight 2k*

Proof. The above arguments show that $G_k(z)$ is weakly modular of weight 2k. We have to show that it is also everywhere holomorphic.
First suppose that $z \in D$, where D is the fundamental domain. Then we get:

$$|mz + n|^2 = m^2 z\bar{z} + 2mnRe(z) + n^2 \geq M^2 - mn + n^2 = |m\rho - n|^2$$

By the Lemma above the series $\sum' \frac{1}{|m\rho - n|^{2k}}$ is convergent. This shows that the series $G_k(z)$ converges normally in D, thus also (applying the result to $G_k(g^{-1}z)$ with $g \in G$) in each of the transforms gD of D by G. Since these cover H, we see that G_k is holomorphic in H. It remains to see that G_k is holomorphic at infinity. This amount to proving that G_k has a limit for $Im(z) \to \infty$. But one may suppose that z

remains in the fundamental domain D; in view of the uniform convergence in D, we can make the passage to the limit term by term. The terms: $\frac{1}{(mz+n)^{2k}}$ relative to $m \neq 0$ give 0, the others give $\frac{1}{n^{2k}}$. Thus:

$$\lim . G_k(k) = \sum{}' \frac{1}{n^{2k}} = s \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = 2\zeta(2k)$$

So in particular $G_k(\infty) = 2\zeta(2k)$ □

Moroever, its Fourier expansion is:

$$G_k(z) = 2\zeta(k)\left(1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n\right)$$

where: $q = e^{2\pi i z}$, $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$, and where $B_k \in \mathbb{Q}$ is the k-th Bernoulli number. If we normalize the Eisenstein serie by getting:

$$E_k(z) = \frac{G_k(z)}{2\zeta(k)}$$

Then the Fourier expanzion if $E_k(z)$ has rational coefficient and constant term 1. For example, the Fourier expansion of the first two non zero Eisenstein series are:

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n$$

$$E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n$$

So we have a modular form of weight 2k and it is not a cusp form.

Remark 2. The Eisenstein series are the special case $m = 0$ of the *Poincaré series* $P_{m,k}$ defined by:

$$P_{m,k}(z) = \sum_{\gamma \in \Gamma_\infty - \Gamma} \frac{1}{j(\gamma, z)^k} \exp(2\pi i m \gamma(z))$$

For $m > 0$ and $k \geq 3$ the Poincaré series are cusp forms of weight 2k.

Example 3

We denote by:

$$g_2 = 60 G_4 = \frac{4\pi^4}{3} E_4$$

$$g_3 = 140 G_6 = \frac{8\pi^6}{27} E_7$$

$$\Delta := g_2^3 - 27 g_3^2 = \frac{(2\pi)^{12}}{1728}(E_4^3 - E_6^2)$$

It follows that Δ is a modular form of weight 12, and that $\Delta \neq 0$ in H. Moroever, the q-expansions for the E_k's show that Δ vanishes at ∞, so Δ is a cusp form. Δ has integral fourier coefficient:

$$\Delta(z) = q \Pi_{n \geq 1}(1 - q^n)^{24} = \sum_{n \geq 1} \tau(n) q^n$$

and this defines the Ramanujan function $\tau(n)$ (it can be shown that $tau(n) \in \mathbb{Z} \quad \forall n \in \mathbb{Z}$, and that $\tau(nm) = \tau(n)\tau(m)$). Using Δ we can define the *j- invariant modular form*, which is the modular function of weight 0 defined by:

$$j(z) = 1728 \frac{g_2^3}{\Delta} = 1728 \frac{E_4^3}{E_4^3 - E_6^2}$$

It is holomorphic in H (because $\Delta \neq 0$) and has a simple pole at ∞.

Example 4

Let Θ be the Jacobi theta function of the first example:

$$\Theta(\tau) = \sum_{n\in\mathbb{Z}} e^{i\pi\tau n^2}$$

Then:

$$\Theta(2\tau)^2 = \sum_{a,b\in\mathbb{Z}} q^{a^2+b^2}$$

is a modular form of weight 1 and level 4. This is an instance of a very general contruction involving rings of integers and quadratic fields, such as $\mathbb{Z}[i]$.
Suppose $\alpha \in \mathbb{Z}[i]$ is non zero, and:

$$\chi : (\mathbb{Z}[i]/(\alpha)^x \longrightarrow \mathbb{C}^x$$

is an isomorphism. Assume that $\chi(i) = 1$, extend χ to a multiplicative function on $\mathbb{Z}[i]$ by declaring it to be 0 on elements which are not primes to α. It is a result of Hecke that the series:

$$\Theta_\chi(\tau) = \frac{1}{4}\sum_{a,b\in\mathbb{Z}} \chi(a+bi)q^{a^2+b^2}$$

is a modular form of weight 1 and level $4|\alpha|^2$, and if x is non trivial, then Θ_χ is a cusp form.

Example 5

The abelian group $(\mathbb{Z}[i]/8\mathbb{Z}[i])^x$ has generators: $3, 5, i, 1 + 2i$ with order: $2, 2, 4, 4$ respectively.
Let $\chi : (\mathbb{Z}[i]/8\mathbb{Z}[i])^x \longrightarrow \mathbb{C}^x$ be the unique homomorphism which is trivial on the first three generators and which sends $1 + 21$ to i. Then Θ_χ is a modular form of weight 1.
For p prime, the p-th coefficient in the Fourier expansion of Θ_χ is:

$$a_p(\Theta_\chi) = \begin{cases} \chi(a+bi) + \chi(a-bi) & \text{if} \quad p \equiv 1 \mod 4, \quad p = a^2+b^2 \\ 0 & \text{if} \quad p \equiv 3 \mod 4, \quad \text{or} \quad p = 2 \end{cases}$$

Now, if $p \equiv 1 \mod 4$ we can write $p = a^2 + b^2$ with a odd and b even. A short calculation shows that:

$$a_p(\Theta_\chi) = \begin{cases} 2 & \text{if} \quad 8|b \\ -2 & \text{if} \quad 4|b \quad \text{but} \quad 8 \nmid b \\ 0 & \text{if} \quad 4 \nmid b \end{cases}$$

Referring back to the example showed in the previous talk about two dimensional Artin representation over $\mathbb{Q}$, we find that $\forall p$ odd prime we have the following relation:

$$a_p(\Theta_\chi) = tr\rho(Frob_p)$$

for the Galois representation $\rho : Gal(\mathbb{Q}'/\mathbb{Q}) \longrightarrow GL_2(\mathbb{C})$ with:

$$tr\rho(Frob_p) = \begin{cases} 2 & \text{if} \quad p = a^2+64b^2 \\ -2 & \text{if} \quad p = a^2+16b^2, \quad b \quad \text{odd} \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

and the equation stated above hints an extraordinary relation between modular forms and Galois rapresentation of $Gal(\mathbb{Q}'/\mathbb{Q})$.

Example 6

More in general, if $Q : \mathbb{Z}^r \longrightarrow \mathbb{Z}$ is any positive defined integer-valued quadratic form in r variables, r even, then:

$$\Theta_Q(\tau) = \sum_{x\in\mathbb{Z}^r} q^{q(x)}$$

is a modular form of weight $r/2$ on some group $\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{Z}) | c \equiv 0 \mod N \right\}$ with some character $\chi \mod N$, i.e.

$$\Theta_Q(\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(d)(c\tau + d)^{r/2}\Theta_Q(\tau) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$$

The integer N is the level of $\mathbb{Q}$ and it is determined as follow write $Q(x) = \frac{1}{2}x^t Ax$ where A is an even symmetric $r \times r$ matrix (i.e. $A = (a_i a_j), a_{ij} = a_j \in 2\mathbb{Z})$); then N is the smallest NA^{-1} is again even. The character χ is given by $\chi(d) = \left(\frac{D}{d}\right)$ with $D = (-1)^{r/2} det A$.
For example, if we take:

- $Q(x_1, x_2) = x_1^2 + x_2^2$, $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $N = 4$, $\chi(d) = (-1)^{\frac{d-1}{2}}$
- The two quadratic forms:
$$Q_1(x_1, x_2) = x_1^2 + x_1 x_2 + 6x_2^2$$
$$Q_2(x_1, x_2) = 2x_1^2 + x_1 x_2 + 3x_2^2$$
have level $N = 23$ and character $\chi(d) = \left(\frac{-23}{d}\right) = \left(\frac{d}{23}\right)$.

The sum $\Theta_{Q_1}(\tau) + 2\Theta_{Q_2}(\tau)$ is an Eisenstein series: $3 + 2\sum_{n=1}(\sum_{d|n} \chi(d))q^n$ of weight 1 and level 23, and the difference: $\Theta_{Q_1} - \Theta_{Q_2}$ is two times the cusp form $q\Pi_{n=1}(1-q^n)(1-q^{23n})$ the 24-th root of $\Delta(\tau)\Delta(23\tau)$. If we want modular forms on the full modular group $PSL_2(\mathbb{Z})$, then we must have $N = 1$ as the level $\mathbb{Q}$, i.e. the even symmetric matrix A must be unimodular.

References

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