March 20, 2008

Quiz 3, MATH 251, Winter 2008

Time: 16:30 - 18:00.

Answer the following questions. Write clearly and precisely, citing accurately results you are using. Explain your calculations!

(1) Let W be the subspace of $\mathbb{R}^4$ defined by the equations

$$x_1 + x_3 = 0, x_1 + 2x_2 + 3x_3 = 0.$$

Answer the following questions:

(a) Find a basis for W.

(b) Find a basis for $W^\perp$ (relative to the usual inner product on $\mathbb{R}^4$).

(c) Find the orthogonal projection of a general vector (y_1, y_2, y_3, y_4) on W.

(d) Find the distance of the vector $(1, 1, 1, 1)$ from W.

Proof. (a) W are the solutions to the homogenous system given by the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 0 \end{pmatrix},$$

which is row equivalent to

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

The rank is 2 so the dimension of W is $4-2 = 2$. Since $(0, 0, 0, 1), (-1, -1, 1, 0)$ are two independent vectors in W, this is a basis.

(b) Observe that $(1, 0, 1, 0), (1, 2, 3, 0)$ are in $W^\perp$. (This is a general principle: if W are the solutions to the homogenous system corresponding to the rows of a matrix A, then the rows of A span $W^\perp$.) The dimension of $W^\perp$ is $4-\dim(W) = 2$. Since the vectors are independent they form a basis for $W^\perp$.

(c) We perform Gram-Schmidt on $(0, 0, 0, 1), (-1, -1, 1, 0)$ to get an orthonormal basis for W. The vectors are already orthogonal and we easily find the orthonormal basis $(0, 0, 0, 1), \frac{1}{\sqrt{3}}(-1, -1, 1, 0)$.

The projection of (y_1, y_2, y_3, y_4) on W is given by $\langle (y_1, y_2, y_3, y_4), (0, 0, 0, 1)\rangle \cdot (0, 0, 0, 1) + \langle (y_1, y_2, y_3, y_4), \frac{1}{\sqrt{3}}(-1, -1, 1, 0)\rangle \cdot \frac{1}{\sqrt{3}}(-1, -1, 1, 0)$, which is equal to $(0, 0, 0, y_4) + \frac{y_3-y_2-y_1}{3}(-1, -1, 1, 0) = (\frac{-y_3+y_2+y_1}{3}, \frac{-y_3+y_2+y_1}{3}, \frac{y_3-y_2-y_1}{3}, y_4)$.

(d) The projection of $(1, 1, 1, 1)$ is $(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}, 1)$. The distance is $\|(1, 1, 1, 1) - (1/3, 1/3, -1/3, 1)\| = \|(2/3, 2/3, 4/3, 0)\| = \frac{\sqrt{24}}{3}$.

$\square$

(2) Consider the matrix

$$A = \begin{pmatrix} 5 & 12 & 3 \\ -1 & -2 & -1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Answer the following:

(a) What is the characteristic polynomial of A?
(b) What are the eigenvalues of A (hint: they are small integers)?
(c) Find a basis for each of the eigenspaces of A.
(d) Find a matrix M such that $D = M^{-1}AM$ is diagonal.
(e) Calculate the $(2, 3)$ entry of A^{10}.

Proof. (a) The characteristic polynomial is easily calculated using the block structure and is $(t^2 - 3t + 2)(t - 2) = (t - 1)(t - 2)^2$.

(b) Calculated this way, obviously the eigenvalues are $1, 2$.

(c) The eigenspace E_1 corresponding to 1 is the solutions of

$$I - A = \begin{pmatrix} -4 & -12 & -3 \\ 1 & 3 & 1 \\ 0 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

A basis is given by $(-3, 1, 0)$.
For the eigenvalue 2 the eigenspace E_2 is the solutions to

$$\begin{pmatrix} -3 & -12 & -3 \\ 1 & 4 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 4 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A basis is given by $(-1, 0, 1), (-4, 1, 0)$.

(d) Let M be the matrix

$$\begin{pmatrix} -3 & -4 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $M^{-1}AM$ is diagonal, equal to $\operatorname{diag}(1, 2, 2)$. Here

$$M^{-1} = \begin{pmatrix} 1 & 4 & 1 \\ -1 & -3 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(e) It follows that

$$A^{10} = \begin{pmatrix} -3 & -4 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^{10} & 0 \\ 0 & 0 & 2^{10} \end{pmatrix} \begin{pmatrix} 1 & 4 & 1 \\ -1 & -3 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} -3 & -4 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} * & * & 1 \\ * & * & -2^{10} \\ * & * & 2^{10} \end{pmatrix}$$
$$= \begin{pmatrix} * & * & * \\ * & * & -1023 \\ * & * & * \end{pmatrix}.$$

□

(3) Prove the Cayley-Hamilton theorem: *Let* $\mathbb{F}$ *be a field and* $A \in M_n(\mathbb{F})$*, a square matrix of size* n *with coefficients in* $\mathbb{F}$*, then* A *satisfies its characteristic polynomial, that is,* $\Delta_A(A) = 0$.
Guidance: You may supply a proof different from the one given in class. Recall that the proof given in class was short given a certain "technical lemma". If you choose to follow the proof given in class then cite precisely the technical lemma - there is no need to prove it - and continue to prove the theorem.

Proof. As in the notes. □