The letter $\mathbb{F}$ stands for a field and $\mathbb{F}[t]_n$ denotes the vector space of polynomials of degree less than n with coefficients in $\mathbb{F}$. We use $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ to denote the integers, rational, real and complex numbers, respectively. For p a prime number, we use $\mathbb{F}_p$ to denote the field of p elements ($= \mathbb{Z}/p\mathbb{Z}$).

If A is a matrix A^t denotes it transpose; the ij element of A^t is the ji element of A. We use Δ_A and m_A to denote the characteristic and minimal polynomials of A, respectively. We denote by $\mathrm{rank}_r(A)$ (resp., $\mathrm{rank}_c(A)$) the row-rank (resp., column-rank) of A.

The maximal grade for this exam is 100 points. It is then averaged with the midterm and the assignments to give the final grade for the course, according to the formula given at the beginning of the semester.

Part I. (28 points)
Instructions: Answer all questions. Do not provide proofs or justification.

1. The characteristic polynomial of the matrix of rational numbers $A = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 10 & 11 \\ 1 & 7 & 7 \end{pmatrix}$ is

   a) $t^3 - t^2 + 61t + 5$.
   b) $t^3 - 20t^2 + 41t + 16$.
   c) $t^3 + 20t^2 + 41t + 16$.

2. Let A be a square matrix. The minimal polynomial of A is equal to the minimal polynomial of A^t:
   a) Always.
   b) Sometimes.
   c) Never.

3. Let x, y, z be real numbers. The following is a basis for $\mathbb{R}^3$:

$$\left\{ \begin{pmatrix} 1 \\ x \\ 0 \end{pmatrix}, \begin{pmatrix} y \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ x+y \\ x+z \end{pmatrix} \right\}$$

.
   a) True.
   b) True if and only if x, y and z are all non-zero.
   c) True if and only if $x^2 y \neq -xyz$.

4. The matrix $A = \begin{pmatrix} 1 & 7 & 0 & 1 \\ 2 & 3 & 1 & 1 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 5 & 7 \end{pmatrix}$ is invertible:

   a) Over $\mathbb{F}_7$.
   b) Over $\mathbb{F}_3$.
   c) Over both $\mathbb{F}_3$ and $\mathbb{F}_7$.
   d) None of the above.

5. Let A be a matrix with characteristic polynomial $t^3(t-1)^4$ and minimal polynomial $t^x(t-1)^y$. The Jordan blocks for A are determined uniquely if (choose *all* correct answers):
a) $x = 1$ and $y = 1$.
b) $x = 2$ and $y = 1$.
c) $x = 1$ and $y = 2$.
d) $x = 1$ and $y = 3$.
e) $x = 2$ and $y = 2$.

6. Which of the following matrices is diagonalizable over $\mathbb{C}$ (choose *all* correct answers):
a) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
b) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.
c) $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

7. Is there a linear map $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ such that $\dim(\mathrm{Im}(T)) = 3$ and $\dim(\mathrm{Im}(T^2)) = 1$?
a) Yes.
b) No.

Part II. (72 points)

Instruction: Answer 3 out of the following 4 questions. In each question you have to answer all parts. If you answer more than 3 questions, cross out one of them clearly, else I will pick 3 questions arbitrarily.

1.

(1) Let $T : V \longrightarrow W$ be a linear transformation between two finite dimensional vector spaces. Prove that

$$\dim(\mathrm{Ker}(T)) + \dim(\mathrm{Im}(T)) = \dim(V).$$

(2) Let W be the image of the linear map $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given in the standard bases by the matrix

$$\begin{pmatrix} 5 & 0 & 1 \\ -8 & 1 & -1 \\ 5 & 0 & 1 \end{pmatrix}.$$

Find a 3×3 matrix in the standard basis of $\mathbb{R}^3$ that represents the orthogonal projection of $\mathbb{R}^3$ on the subspace W.

2.

(1) Formulate the Laplace rules of developing a determinant by a row and by a column. Define the adjoint matrix $\mathrm{adj}(A)$ of a square matrix A. Prove that

$$A \cdot \mathrm{adj}(A) = \det(A) \cdot \mathrm{Id}.$$

(2) Find the primary decomposition of $\mathbb{R}^3$ (a suitable decomposition of $\mathbb{R}^3$ and the resulting representation of the operator in block form) for the operator given by the matrix

$$\begin{pmatrix} -2 & -5 & 5 \\ 1 & 2 & -3 \\ 0 & 0 & 3 \end{pmatrix}.$$

3.

(1) Let U be a nilpotent operator on $\mathbb{R}^{10}$ such that $m_U(t) = t^5$, $\dim \operatorname{Ker}(U) = 3$ and $\dim \operatorname{Ker}(U^2) = 6$. What are the possibilities for the Jordan form of U? What if we are only given that $m_U(t) = t^5$ and $\dim \operatorname{Ker}(U^2) = 6$? Prove your answers.

(2) Let T be a normal operator. Prove that T is self adjoint if and only if its eigenvalues are real.

(3) Let V be a vector space. Formulate Steinitz's Substitution Lemma. Assuming Steinitz's Substitution Lemma and that V has a finite basis B, prove that any other basis of V has the same cardinality as B.

4.

(1) Find the Jordan canonical form of the matrix

$$A = \begin{pmatrix} 3 & 1 & 3 \\ -4 & -1 & -6 \\ 0 & 0 & 1 \end{pmatrix}.$$

Use it to find A^{15} explicitly.

(2) Let T be a normal operator on a finite dimensional vector space. Prove that if λ and μ are distinct eigenvalues of T then $E_\lambda \perp E_\mu$. You need to prove all the properties of normal operator your proof uses.

Good Luck !!