Algebra II MATH 251 **Instructor: Dr. E. Goren.**

Assignment 9

To be submitted by March 28, 12:00

1. Recall the matrices from Assignment 8:

$$\begin{pmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix} \qquad \begin{pmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{pmatrix}$$

Consider them as matrices of complex numbers. For each matrix N, considered as a linear transformation T, find the Primary Decomposition, i.e., the factorization of the minimal polynomial, the kernels of the factors, and for each kernel a matrix representation of T; the total basis for the space (a union of the bases for the invariant spaces) and the matrix representing T with respect to it (You may refer to your calculations for assignment 8).

2. (A) Let S and T be commuting linear maps from a vector space V to itself. Let λ be an eigenvalue of T and let E_λ be the corresponding eigenspace. Prove that E_λ is S invariant. Conclude that if T is diagonalizable with eigenvalues $\lambda_1, \dots, \lambda_r$, and therefore

$$V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_r},$$

we may decompose S as

$$S = S_1 \oplus \cdots \oplus S_r,$$

where $S_i : E_{\lambda_i} \longrightarrow E_{\lambda_i}$.

(B) Assume that both S and T are diagonalizable. Prove now that there exists a basis of V in which both S and T are diagonal.

3. Let A be an $n \times n$ matrix over an algebraically closed field such that A^2 is diagonalizable. Prove that if A is a non-singular matrix then also A is a diagonalizable, and provide an example showing this condition is necessary.

4. Recall the matrices of rotating the plane $\mathbb{R}^2$ by an angle θ

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Prove that this matrix usually cannot be diagonalized over $\mathbb{R}$ but is diagonalizable over $\mathbb{C}$.

5. Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a *distance preserving* transformation: $\|Tv\| = \|v\|, \forall v \in \mathbb{R}^n$.

(1) Prove that T preserves the inner product: $\langle Tv, Tw \rangle = \langle v, w \rangle, \forall v, w, \in \mathbb{R}^n$.

(2) Let A be the matrix representing T with respect to the standard basis. Prove that T is an orthogonal transformation if and only if ${}^tAA = Id$.

(3) Prove that the collection of all distance preserving linear transformations of $\mathbb{R}^n$ is a group; it is called the *orthogonal group* and denoted $O_n(\mathbb{R})$. We call such a transformation T an *orthogonal transformation*.

(4) A *reflection* of $\mathbb{R}^n$ is a transformation of the following form: Let W be a subspace of dimension $n-1$ and $W^\perp$ its orthogonal. Then $\mathbb{R}^n = W \oplus W^\perp$. Define a linear map by the identity on W and multiplication by -1 on $W^\perp$. This is called the *reflection through* W. Characterize reflections in terms of their eigenvalues.

(5) Show that every reflection is an orthogonal transformation.

Bonus question - 20 points.

1. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be an orthogonal transformation. Prove that T is a product of at most two reflections.

2. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be an orthogonal transformation. Prove that either T or UT, where U is a suitable reflection, are a rotation of $\mathbb{R}^3$ with respect to a suitable axis of rotation. Prove that $O_3(\mathbb{R})$ is generated by reflections.