Algebra II MATH 251 **Instructor: Dr. E. Goren.**

Assignment 7

To be submitted by February 28, 12:00

1. Let W be the subspace of $\mathbb{F}^4$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) defined by the equation $x_1 + x_2 + x_3 + x_4 = 0$. Find the orthogonal projection of $(1, 0, 0, 0)$ on W.
Remark: I would allow here solutions that use Maple (or other software) if you include a print out of the commands you used.

2. Let V be a vector space with inner-product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$.

(1) Prove the parallelogram law
$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2.$$

(2) We say that u and v are perpendicular (or orthogonal) and write $u \perp v$ if $\langle u, v \rangle = 0$. Prove Pythagoras's Theorem: if u and v are perpendicular then
$$\|u\|^2 + \|v\|^2 = \|u + v\|^2.$$

3. Let $a < b$ be real numbers. Show that the function
$$\langle f, g \rangle = \int_a^b f(x)g(x)dx$$
defines an inner product of $\mathbb{R}[x]_n$.[1] Compute the norm of the vector $f(x) = 1 + x + x^2$ in the case $(a, b) = (0, 1)$ and in the case $(a, b) = (0, 2)$.

4. Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be a matrix of complex numbers.

(1) Prove that the function
$$\langle (x_1, x_2), (y_1, y_2) \rangle = (x_1, x_2) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \overline{y_1} \\ \overline{y_2} \end{pmatrix},$$
is an inner product on $\mathbb{C}^2$ if and only if a and d are positive real numbers, $c = \bar{b}$ and $ad - b\bar{b} > 0$.

(2) In the case of the matrix $\left(\begin{smallmatrix} 1 & 1+i \\ 1-i & 5 \end{smallmatrix}\right)$ compute $\langle (1, 2), (3, 4) \rangle$ and $\|(2, 5i)\|$.

(3) Find an orthonormal basis for $\mathbb{C}^2$ with the inner product defined by the matrix $\left(\begin{smallmatrix} 1 & 1+i \\ 1-i & 5 \end{smallmatrix}\right)$.

5. Perform the Gram-Schmidt process for the basis $\{1, x, x^2\}$ to $\mathbb{R}[x]_3$ with respect to the inner product
$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx.$$

---

[1] This is also true for $\mathbb{C}[x]_n$ if we define $\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx$. Note that in this case one can do the integration formally because we are dealing with polynomials. Thus, for example,
$$\langle x^2, 1 + ix \rangle = \int_a^b (x^2 - ix^3)\, dx = (x^3/3 - ix^4/4)|_a^b = (b^3/3 - ib^4/4) - (a^3/3 - ia^4/4).$$

6. Given an inner product, define the angle θ between two vectors by

$$\cos(\theta) = \frac{\langle u, v\rangle}{\|u\| \cdot \|v\|}.$$

This determined θ up to a sign, as one may expect. Show that in the case of $u, v \in \mathbb{R}^2$, with the standard inner product, this agrees with the geometric definition.

(Bonus question - up to 10 %)

Latin squares: A Latin square is a square matrix of order n in which each row and each column are permutations of $\{1, 2, \ldots, n\}$. Such matrices are important for group theory, experimental designs and linear algebra; there are many open questions. It is a hard theorem that there are more than $(n!)^{2n}/n^{n^2}$ Latin squares of order n (this is more than exponential in n).

Let $\mathbb{F}$ be a finite field with q elements. Fix a line ℓ in the projective plane $\mathbb{P}^2(\mathbb{F})$ and three distinct points x, y, z on it. Enumerate the lines through x by $\ell_1, \ldots, \ell_{q+1}$; the lines through y by $m_1, \ldots, m_{q+1}$ and the lines through z by $n_1, \ldots, n_{q+1}$. We may assume that the lines ℓ_{q+1}, m_{q+1} and n_{q+1} are all equal and are the original line on which x, y, z lie.

Define a matrix (a_{ij}) as follows. Consider the point of intersection of ℓ_i with m_j and the unique line between it and n. This line is some z_k. Put $a_{ij} = k$. Show that this is a well defined matrix that is a Latin square of order q.

Gram-Schmidt on Maple.

One can perform the Gram Schmidt process on Maple.

with(LinearAlgebra):

w1 := $\langle 2, 1, 0, -1\rangle$;

w2 := $\langle 1, 0, 2, -1\rangle$;

w3 := $\langle 0, -2, 1, 0\rangle$;

orthobasis :=GramSchmidt([w1,w2,w3]);

$$\text{orthobasis} := \begin{bmatrix} 2 & 0 & 2/3 \\ 1 & -1/2 & -4/3 \\ 0 & 2-1/3 & \\ -1 & -1/2 & 0 \end{bmatrix}$$