Algebra II MATH 251 **Instructor: Dr. E. Goren.**

Assignment 6

To be submitted by February 21, 12:00

1. (A) Let W be a k-dimensional subspace of $\mathbb{F}^n$. Prove that there are $n-k$ linear equations such that W is the solutions to that homogenous system.

(B) Let $W_1 = \mathrm{Span}(\{(1,1,0,0),(0,1,1,-1)\})$ and $W_2 = \mathrm{Span}(\{(3,2,1,0),(4,4,2,-1)\})$. Find a system of homogeneous linear equations such that W_1 is their solutions. The same for W_2. Find then a basis for $W_1 \cap W_2$, and a basis for $W_1 + W_2$. (Note that you are not allowed to make any assumption on the field over which these equations are given.)

2. (i) Compute the rank of the following system of linear equations with real coefficients, by finding the reduced echelon form.

$$\begin{gathered} 5x_1 + x_2 + 3x_3 + 2x_4 + x_5 = b_1 \\ x_1 + x_3 + x_4 = b_2 \\ 7x_1 + 2x_2 + 3x_3 + x_4 + 2x_5 = b_3 \\ -x_1 - x_2 + x_3 + 2x_4 - x_5 = b_4 \end{gathered}$$

(ii) Prove that at least for one of the vectors $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ we can't solve the system.

3. We proved that the determinant of an $n \times n$ matrix $A = (a_{ij})$ is non-zero if and only if the columns of A, $\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}, \dots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{pmatrix}$, form a basis for $\mathbb{F}^n$.

Generalize this as follows. Let $A = (a_{ij})$ be an $m \times n$ matrix. Prove that rank(A) $= k$ if and only if there exists a $k \times k$ sub-determinant of A that is not zero, and every $(k+1) \times (k+1)$ sub-determinant is zero.

By a $k \times k$ sub-determinant we mean the following. We choose k columns $j_1 < j_2 < \dots < j_k$ among the n columns (so k is assumed to be $\leq n$), and we choose k rows $i_1 < i_2 < \dots < i_k$ among the m rows (so k is assumed to be also $\leq m$ as well). We then look at the $k \times k$ matrix $(a_{i_\ell j_m})_{\ell, m = 1, \dots, k}$. Its determinant is what we call a $k \times k$ sub-determinant.

<u>Example</u>: Consider the matrix

$$\begin{pmatrix} 3 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{pmatrix}$$

We know it has rank 2. We note that the only 3×3 determinant is 0. It has nine 2×2 sub-determinants. Some of them are

$$\det\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}, \ \det\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}, \det\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}, \ \det\begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}.$$

The determinants are $1, -1, -2, 3$. Note that over every field the first determinant is non-zero. This shows that the matrix has rank 2. Note also that over the field $\mathbb{Z}/2\mathbb{Z}$ the before last determinant is zero, and over $\mathbb{Z}/3\mathbb{Z}$ the last determinant is zero. This shows that if some $k \times k$ sub-determinant is zero this need not hold for other $k \times k$ sub-determinants.

4. Give another proof of the following theorem by applying Exercise 3 and properties of determinants.
Theorem Let A be an $m \times n$ matrix. Then

$$\mathrm{rank}_c(A) = \mathrm{rank}_r(A).$$

5. Let $\mathbb{F}$ be a field. We define the *projective plane* $\mathbb{P}^2(\mathbb{F})$ as a set whose points are the lines through the origin in $\mathbb{F}^3$.

We define a *line* in $\mathbb{P}^2(\mathbb{F})$ to be the image of a plane through the origin in $\mathbb{F}^3$, i.e., a line is a collection of points in $\mathbb{P}^2(\mathbb{F})$ corresponding to the lines contained in a given plane in $\mathbb{F}^3$.

Prove that through every two distinct points in $\mathbb{P}^2(\mathbb{F})$ there is a unique line. Prove that every two distinct lines intersect at a unique point. [1]

Bonus - up to 10%. If $\mathbb{F}$ is a finite field with q elements, prove that there q^2+q+1 points in $\mathbb{P}^2(\mathbb{F})$ and the same number of lines. Prove that every line contains $q+1$ points.

The picture below is an example of a projective plane. Which one? What does the picture portray precisely?

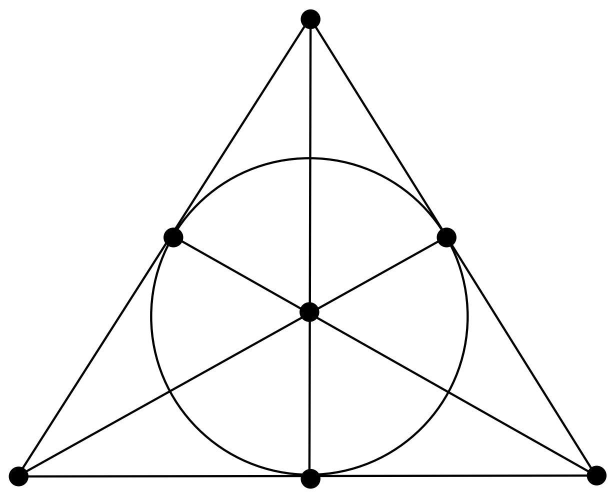

FIGURE 1. A Projective Plane.

[1]This defines a geometry - the projective plane - in which there are points, lines and we know when a point lies on a line, but there are no parallel lines. This geometry has duality in the following sense. Let us call now each line a "tniop" and each point an "enil". Lets say that a tniop lies on an enil if the original line contains the point. The statement, "through every two points there is a unique line" becomes the statement "every two enils intersect at a unique tniop"; the statement, "every two lines intersect at a unique point" becomes the statement "through every two tniops there is a unique enil".