Algebra II, MATH 251 **Instructor: Dr. E. Goren.**

Assignment 4

To be submitted by February 7, 12:00

1. Let $W = \{(x, y, z, w) : x + y + z + w = 0, x - y + z - w = 0\}$, $U_1 = \{(x, x, x, x) : x \in \mathbb{R}\}$ be subspaces of $\mathbb{R}^4$. Find a subspace $U \supset U_1$ such that $\mathbb{R}^4 = U \oplus W$. Let T be the projection of $\mathbb{R}^4$ on U along W. Write the matrix representing T with respect to the standard basis.

2. Let V and W be any two finite dimensional vector spaces over a field $\mathbb{F}$. Let $T : V \longrightarrow W$ be a linear map. Prove that there are bases of V and W such that with respect to those bases T is represented by a matrix composed of 0's and 1's only. If T is an isomorphism, prove that with respect to suitable bases it is represented by the identity matrix.

3. Let V be a vector space of dimension n and let W be a vector space of dimension m, both over the same field $\mathbb{F}$. Prove that $\mathrm{Hom}(V, W) \cong M_{m\times n}(\mathbb{F})$ as vector spaces over $\mathbb{F}$. Here $M_{m\times n}(\mathbb{F})$ stands for matrices with n columns and m rows with entries in $\mathbb{F}$.

4. Consider the transformation that rotates the plane $\mathbb{R}^2$ by angle θ counter-clockwise. Write this transformation as a matrix in the standard basis. Write it also as a matrix with respect to the basis $(1, 1), (1, 0)$.

5. Explain how we may view the complex numbers $\mathbb{C}$ as a vector space of dimension 2 over $\mathbb{R}$. Find a basis for $\mathbb{C}$ over $\mathbb{R}$. Show that multiplication by i is a linear transformation with respect to the structure of a vector space over $\mathbb{R}$. Write a two-by-two matrix A representing it and prove that $A^2 + I = 0$ (here I is the two-by-two identity matrix).

6. Write the following permutation σ as a product of disjoint cycles. For each cycle determine its sign and its order. Determine the sign and order of the permutation σ itself.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 6 & 4 & 5 & 1 & 2 & 7 & 9 & 8 \end{pmatrix}$$

7. For the groups S_3 and S_4 write explicitly all the odd transformation and all the even transformations (make a list!).

Bonus question - up to 20 points. What can you say on the maximal order of an element of the symmetric group S_n?