Algebra II MATH 251 **Instructor: Dr. E. Goren.**

Assignment 2

To be submitted by January 24, 12:00

1. Let V be the set of all sequences of complex numbers $(a_0, a_1, a_2, \dots)$ satisfying

$$a_n = a_{n-1} + a_{n-2}, \quad \forall n \geq 2.$$

Show that V has a natural structure of a vector space over $\mathbb{C}$. Find its dimension and a basis.

2. Let V be an n-dimensional vector space over a field $\mathbb{F}$. Let $T = \{t_1, \dots, t_m\} \subset V$ be a linearly independent set. Let $W = \mathrm{Span}(T)$. Prove:

$$\dim(W) = m.$$

3. Let W be a subspace of a vector space V of dimension n. Let $\{t_1, \dots, t_m\}$ be a basis for W. Prove that there exist vectors $\{t_{m+1}, \dots, t_n\}$ in V such that $\{t_1, \dots, t_m, t_{m+1}, \dots, t_n\}$ is a basis for V.

4. Let V_1, V_2 be finite dimensional vector spaces over a field $\mathbb{F}$. Prove that

$$\dim(V_1 \oplus V_2) = \dim(V_1) + \dim(V_2).$$

5. Consider $V := \mathbb{R}[t]_n$, the vector space of polynomials of degree $< n$ with real coefficients. Let

$$r_1 < r_2 < \cdots < r_n$$

be any real numbers. Show that for every i there exists a unique polynomial f_i in V that vanishes at all the r_j except for r_i where it obtains the value 1. Give an explicit formula for f_i. Show that

$$f_1, f_2, \dots, f_n$$

comprise a basis for V.

6. Let $\mathcal{B} = \{(1,1), (1,5)\}$ and $\mathcal{C} = \{(2,1), (1,-1)\}$ be bases of $\mathbb{R}^2$. Find the change of basis matrices ${}_{\mathcal{B}}M_{\mathcal{C}}$ and ${}_{\mathcal{C}}M_{\mathcal{B}}$ between the bases $\mathcal{B}$ and $\mathcal{C}$. Let $v = \left(\begin{smallmatrix} 8 \\ 28 \end{smallmatrix}\right)$ with respect to the standard basis. Find $[v]_{\mathcal{B}}$ and $[v]_{\mathcal{C}}$.

7. Let $\mathbb{F}$ be a finite field with q elements.
   (1) Show that the kernel of the ring homomorphism

   $$\mathbb{Z} \longrightarrow \mathbb{F}$$

   defined by $n \mapsto n \cdot 1 = 1 + \cdots + 1$ (n times) is of the form $p\mathbb{Z}$ for some prime p. Conclude that we may assume that $\mathbb{F} \supseteq \mathbb{Z}/p\mathbb{Z}$ for some prime p.
   (2) Prove that $\mathbb{F}$ is a vector space of finite dimension over $\mathbb{Z}/p\mathbb{Z}$ and if this dimension is n then $\mathbb{F}$ has p^n elements[1].

Bonus question (= 20%). Let $\mathbb{F}$ be a finite field of q elements. Let $V = \mathbb{F}^n$ and let C be a code (= a subspace) of dimension k, hence having q^k elements. Let d be the minimal Hamming weight of a non zero element of C. Prove that

$$d \leq n - k + 1.$$

[1]Note: at this point you've proven that every finite field has cardinality p^n for some prime p.