Algebra II MATH 251 **Instructor: Dr. E. Goren.**

Assignment 11

To be submitted by April 11, 12:00

1. Find a basis in which the following matrix is in a Jordan canonical form. The minimal polynomial is $(t-3)^3$.

$$\begin{pmatrix} 4 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 \\ -3 & -3 & 3 & 0 & 0 \\ 6 & 6 & 0 & 3 & 0 \\ -7 & -7 & 0 & 0 & 3 \end{pmatrix}.$$

2. Find an orthogonal matrix P such that $P^{-1}AP$ is diagonal, where

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Furthermore, as in the Spectral Theorem, find orthogonal projections ϵ_i and write them relative to the standard basis so that $A = \lambda_1\epsilon_1 + \cdots + \lambda_r\epsilon_r$.

3. Let $A = \left(\begin{smallmatrix} 2 & i \\ i & 2 \end{smallmatrix}\right)$. Verify that A is normal. Is it Hermitian? Unitary? Find a unitary matrix M such that M^*AM is diagonal.

4. Prove that T is a normal operator on V if and only if $\|T(v)\| = \|T^*(v)\|$ for all $v \in V$.

5. Let T be a normal operator. Prove that T and T^* have the same kernel and image. Provide an example showing this may fail (usually does, in fact) for non-normal operators.

6. Let $V = \mathbb{R}[x]_n$ be the space of polynomials with real coefficients of degree less than n. Let b be a real number and define

$$\phi_b : V \longrightarrow \mathbb{R}, \quad \phi_b(f) = f(b).$$

Prove that ϕ_b is a linear functional.

Let $a_1, \ldots, a_n$ be any n distinct real numbers. Prove that $\mathcal{C} = \{\phi_{a_1}, \ldots, \phi_{a_n}\}$ is a basis for V^t - the dual vector space of V. Find a basis $\mathcal{B}$ for V such that $\mathcal{C}$ is its dual basis.

Bonus question (20 points (10 for each direction...)): Let $A = (a_{ij})$ be a hermitian matrix of size $n \times n$. For every $1 \le m \le n$ let A_m be the matrix $(a_{ij})_{1 \le i,j \le m}$. Prove that A is positive definite if and only if $\det(A_m) > 0,\ m = 1, 2, \ldots, n$.