Algebra II MATH 251 **Instructor: Dr. E. Goren.**

Assignment 10

To be submitted by April 4, 12:00

1. Let $v_1, \dots, v_n$ be a basis for V and let $T : V \longrightarrow V$ be a linear transformation. For every $1 \leq i \leq n$ we define a polynomial P_i as follows: Let r be the first integer so that $v_i, Tv_i, \dots, T^r v_i$ are linearly dependent. Then for suitable scalars $b_0, \dots, b_r$ we have $b_r T^r v_i + \dots, b_1 T v_i + b_0 = 0$. We may assume without loss of generality that $b_r = 1$. Let $P_i(t) = t^r + b_{r-1}t^{r-1} + \cdots + b_1 t + b_0$ (the b_j depend on i but we do not make this dependence explicit in the notation). Let $P = \mathrm{lcm}\{P_1, \dots, P_n\}$. Prove that P is the minimal polynomial of T.

2. Prove that the matrix of complex numbers

$$A = \begin{pmatrix} 1 & 1 & 5 & 0 & 0 \\ 7 & 1 & 3 & 0 & 0 \\ 2 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

is diagonalizable. Note: you don't need to diagonalize it to prove it !!

3. Determine the possibilities for the Jordan canonical form of a matrix A with characteristic polynomial $\Delta_A(t) = (t-1)^6(t-2)^4(t-4)^5$ and minimal polynomial $m_A(t) = (t-1)^3(t-2)^2(t-4)$.

4. Find the Jordan canonical form of the matrix

$$A = \begin{pmatrix} 5 & 9 & -2 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 5 & 9 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix}.$$

5. It is known that a differentiable function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ has a

- **maximum** at a point P if $\partial f/\partial x = \partial f/\partial y = 0$ at P and the 2×2 symmetric matrix

$$-\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

is positive definite;

- **minimum** at a point P if $\partial f/\partial x = \partial f/\partial y = 0$ at P and the 2×2 symmetric matrix

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

is positive definite;

- **saddle point** at P if the 2×2 symmetric matrix

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

  has one negative eigenvalue and one positive eigenvalue.

If P is either a maximum, minimum or saddle point, we call it a simple critical point. Determine the nature of the simple critical point of the following functions at the origin $(0,0)$

$$f(x,y) = 2x^2 + 6xy + y^2, \quad f(x,y) = x\sin(x) - \cos(y) - xy.$$

(You may view the graphs and rotate them in Maple using
plot3d(2*xˆ 2+6*x*y+yˆ 2, x= -4..4, y = -4..4);
plot3d(x*sin(x) - cos(y) -x*y, x= -4..4, y = -4..4, numpoints=3000);).

Remark. This criterion can be generalized to functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$. If all the first partials vanish at a point P and the matrix of mixed derivatives is positive definite (resp. negative definite) at P, then the function has a minimum (resp. maximum) at P.

6. Find a formula for the general term of the sequence

$$0, 1, 4, 12, 32, 80, \ldots$$

$(a_{n+2} = 4a_{n+1} - 4a_n)$.

Note: The point here would be that the matrix A that comes out is not diagonalizable. Still you should find some matrix $D = MAM^{-1}$ simple enough so that the trick $A^N = M^{-1}D^N M$ is still useful.