Math 235: Assignment 1 Solutions

1.1: For $n \in \mathbb{N}$ not zero, let $A_n = [-n, n]$ (The closed interval in $\mathbb{R}$ containing all real numbers x satisfying $-n \leq x \leq n$). It is easy to see that we have the chain of inclusion $A_1 \subset A_2 \subset A_3 \subset ...$, and therefore

$$\bigcup_{n=1}^{N} A_n = A_N = [-N, N]$$

$$\text{and } \bigcap_{n=1}^{N} A_n = A_1 = [-1, 1].$$

1.2: Notice that the consecutive closed intervals, $[n, n+1]$ and $[n+1, n+2]$ do in fact overlap (This is because we have taken the *closed* interval. What if we had taken the *open* intervals instead?) We conclude that the *finite* union is

$$\bigcup_{n=1}^{N} [n, n+1] = [1, N+1].$$

Since any real number $x \geq 1$ will eventually lie inside the interval $[1, N+1]$ for some sufficiently large $N \in \mathbb{N}$, we conclude

$$\bigcup_{n=1}^{\infty} [n, n+1] = [1, \infty).$$

(Note we have the open bracket ")" on the right side of the above interval. Why?). Similarly, the consecutive open intervals $(n, n+2)$ and $(n+1, n+3)$ also overlap, so

$$\bigcup_{n=1}^{N} (n, n+2) = (1, N+2)$$

$$\therefore \bigcup_{n=1}^{\infty} (n, n+2) = (1, \infty).$$

($\therefore$ means "therefore")

1.3: It is helpful to say in words what each A_n is. A_1 is the set of all natural numbers, $\mathbb{N}$. A_2 is the set of all *squares* of natural numbers. A_3 is all *cubes*, etc. Of course, a square of a natural number is in particular a natural number, so $A_2 \subseteq A_1$. Likewise for cubes, 4th powers, etc. We conclude $A_1 \supseteq A_n$ for all $n \in \mathbb{N}$,

$$\therefore \bigcup_{n=1}^{\infty} A_n = A_1 = \mathbb{N}.$$

Again it is helpful to ask in words, what does $\bigcap_{n=1}^{\infty} A_n$ mean? Since each A_n is the subset of natural numbers consisting of nth powers, their intersection must be all natural numbers x such that x is a square, a cube, a 4th power, a 5th, etc. There are only 2 such numbers:

$$\bigcap_{n=1}^{\infty} A_n = \{0, 1\}.$$

(Challenge: Can you find the smallest natural number $x \neq 0, 1$ such that x is a square, a cube, a 4th power, ..., and a 10th power? What if we replace 10 with some arbitrarily large N^{th} power?)

2.1: Consider the sets $A = B = C = \{1\}$. Then $(A\backslash B)\backslash C$ is the empty set ϕ, while $A\backslash(B\backslash C) = A\backslash\phi = A$. This counterexample is enough to show that the formula in 2.1 *does not* hold (There are of course many more interesting counterexamples that show this as well, but any *one* is sufficient).

2.2: Note that no amount of examples will suffice to prove that this formula *does* hold, hence we need to get down to the nitty-gritty and construct a proof. In general, when proving two sets are equal (say X and Y), a good place to start is to try to show that if $x \in X$ then x must also be in Y and vice versa. This will show that X is a *subset* of Y and Y is a subset of X, which is possible only if $X = Y$. We proceed in this manner:

Assume $x \in (B\backslash A) \cup (A\backslash B)$.

$\Rightarrow$ either x is in $B\backslash A$ or $x \in A\backslash B$ (Note that the word 'or' in mathematics includes the possibility of both events occurring unless explicitly stated otherwise, although in this case that is in fact not possible (Why?). Also $\Rightarrow$ means 'this implies')

Case 1: $x \in B\backslash A$.

Then $x \in B$ but $x \notin A$. $\therefore x \in A \cup B$ but $x \notin A \cap B \Rightarrow x \in (A \cup B)\backslash(A \cap B)$.

Case 2: $x \in A\backslash B$.

Repeat the exact same argument as case 1, but reverse the letters A and B (This is because the equation is 'symmetric' with respect to the sets A and B).

Hence in either case, $x \in (A \cup B)\backslash(A \cap B)$. Since x was an arbitrary element of $(B\backslash A) \cup (A\backslash B)$, we conclude that $(A \cup B)\backslash(A \cap B) \supseteq (B\backslash A) \cup (A\backslash B)$ (Make sure you understand why). It remains to show the reverse inclusion.

Assume $y \in (A \cup B)\backslash(A \cap B)$.

$\Rightarrow y \in A \cup B$ but $y \notin A \cap B$. Hence y is in either A or B, but not both for in that case, y would also be in their intersection (Again make sure you understand this conclusion).

Case 1: $y \in A$ and $y \notin B$.

$\Rightarrow y \in A\backslash B$

$\Rightarrow y \in (B\backslash A) \cup (A\backslash B)$.

Case 2: $y \in B$ and $y \notin A$.

Again by the *symmetry* in the argument we can follow case 1, just switching A and B. We conclude $(B\backslash A) \cup (A\backslash B) \supseteq (A \cup B)\backslash(A \cap B)$. Since we have now proved both inclusions, we can conclude $(B\backslash A) \cup (A\backslash B) = (A \cup B)\backslash(A \cap B)$.

3: We assume the P.O.I. and prove the statement "every non-empty set of natural numbers contains a minimal element". Actually, we will prove the *contrapositive* of this statement, namely "Every set of natural numbers without a minimal element is necessarily empty" (Note that these two claims are *logically equivalent*, that is to say one statement is true if and only if the other is. Proof by contrapositive is

a common tactic in mathematics, so it is important that you understand what it means and how it works). Let $X \subset \mathbb{N}$ be a subset that does not have a minimal element, and consider the following statement about the natural numbers:

$$P(n): \text{ No natural number less then or equal to } n \text{ lies in } X.$$

We would like to prove this statement is true for all natural numbers. We will proceed by induction.

Base case: $n = 0$.
P(0) says, albeit in a rather long winded fashion, that $0 \notin X$. Could 0 lie in X? Well if it did, then it would have to be the minimal element, since 0 is the smallest natural number. Since X has no minimal element, we conclude that $0 \notin X$ and therefore P(0) is true.
Induction step: Assume that P(k) is true for some $k \in \mathbb{N}$. Does it follow that P($k + 1$) is also true?
P($k + 1$) says that no natural number less then or equal to $k + 1$ lies in X. By assumption, P(k) is true, and therefore no natural number less then or equal to k lies in X. Hence P($k + 1$) is false if and only if $k + 1 \in X$. Again, we wonder if this is possible. If $k + 1 \in X$ and no natural number strictly less then k lies in X (Because we are assuming P(k) is true), then we must conclude that $k + 1$ would be the minimal element of X. Since X has no minimal element, we conclude this is not the case, therefore P($k + 1$) is true.

By the Principle of Induction, we conclude that P(n) holds for all $n \in \mathbb{N}$. We claim that this shows X is necessarily empty. For if X was non-empty, then there would be some $m \in \mathbb{N}$ such that $m \in X$. But then P(m) would be false, contradictory to what we just proved (Make sure you understand why here). Hence X is empty, so we have proved the contrapositive to our original claim, which as mentioned earlier, proves our original statement is also true.

4:

Base case: $n = 1$.
The LHS of the equation is 1^3. The RHS is 1^2. Clearly equality holds in this case. (LHS and RHS are shorthand for 'left hand side' and 'right hand side')
Induction step: Assume our equation holds for some $n = k$. Does this imply it holds for $n = k + 1$?
Working with the LHS of our equation, we have

$$1^3 + 2^3 + ... + k^3 + (k+1)^3 = (1^3 + 2^3 + ... + k^3) + (k+1)^3$$

$$= (1 + 2 + ... + k)^2 + (k+1)^3,$$

where the last inequality comes from our inductive assumption. Using the equality

$$1 + 2 + ... + k = \frac{(k)(k+1)}{2}$$

from the notes and then factoring, we get

$$(1 + 2 + ... + k)^2 + (k+1)^3 = \frac{(k)^2(k+1)^2}{4} + (k+1)^3$$

$$= \frac{(k+1)^2(k^2+4k+4)}{4} = \frac{(k+1)^2(k+2)^2}{4}.$$

Once again we use the formula from the notes and finally conclude that

$$\frac{(k+1)^2(k+2)^2}{4} = (\frac{(k+1)(k+2)}{2})^2 = (1+2+...+k+(k+1))^2.$$

This proves the induction step and hence the formula holds for all $n \in \mathbb{N}$.

5.1: Many different surjective functions exist, we list 4 below in set notation:

$$f_1 := \{(1,a),(2,b),(3,b),(4,c)\}$$
$$f_2 := \{(1,a),(2,a),(3,b),(4,c)\}$$
$$f_3 := \{(1,c),(2,a),(3,c),(4,b)\}$$
$$f_4 := \{(1,b),(2,c),(3,c),(4,a)\}.$$

Note that the set notation for a function means, for example $f_1(1) = a$, $f_1(2) = b$, etc.

5.2: Again, there are more then 4 injective functions from B to A, but here are 4:

$$g_1 := \{(a,1),(b,2),(c,3)\}$$
$$g_2 := \{(a,1),(b,3),(c,2)\}$$
$$g_3 := \{(a,4),(b,3),(c,1)\}$$
$$g_4 := \{(a,3),(b,1),(c,4)\}.$$

5.3: Recall that a function sends each element of its domain (In this case A) to a unique element in its range (In this case B). Since B has precisely 3 elements, that means when constructing a function $f : A \to B$, for every $x \in A$ there are 3 possibilities for $f(x)$. Since A has 4 elements, we conclude there are $3 \times 3 \times 3 \times 3$ possible combinations, or 81 different functions from A to B.

5.4: A surjective function must hit every element in its range at least once. Since $|A| = 4$ and $|B| = 3$ we conclude that any surjective function hits one element of B twice, and the other two exactly once. There are 3 ways to choose the element that will be hit twice, so it suffices to find the number of functions that hit say a twice and the other two elements once, and then multiply that number by 3. How many functions is this? Well there are $\binom{4}{2} = 6$ different ways to choose 2 elements of A that will hit a, and once we have made that choice, there is still the question of where to send the remaining two elements. But we know there are only 2 possibilities for that choice, since b and c both need to be hit exactly once. Hence there are $6 \times 2 = 12$ different surjective functions that hit a twice, and therefore $12 \times 3 = 36$ different surjective functions altogether.

5.5: Suppose $f : A \to B$ is an injective function. Injective means that each element of B is the f-image of *at most* one element of A. Hence the set $f^{-1}(B)$ has at most 3 elements total (Since in particular it has size at most equal to B). But B is the range space of f, so $f^{-1}(B) = A$ which of course has 4 elements. This is a contradiction and thus no such f exists.

5.6: Following the argument in 5.3, for each element y of B, there are 4 possible images, and each function send y to exactly one of them. Hence there are $4 \times 4 \times 4 = 64$ different functions.

5.7: We argue similarly to 5.5 to show there does not exist a surjective function $f : B \to A$. Indeed assuming otherwise, then since f is surjective, every element of A is in its image so in particular, the set $f^{-1}(A)$ has at least 4 distinct elements (Namely, one element that gets sent to 1, one that gets sent to 2, etc. Why must these elements be distinct?) But $f^{-1}(A) = B$ which has only 3 elements and $\therefore$ no such f exists.

5.8: Similarily to 5.4, we reason combinatorially. Suppose $f : B \to A$ is some injective function. There are 4 possible place f could send a. Once we have made the choice of where to send a, there are now 3 possible choices of where to send b (Because $f(a) \neq f(b)$). Likewise, after choosing where to send a and b, there are now only 2 possible choice for the image of c. Hence there are $4 \times 3 \times 2 = 24$ total possibile injective functions.