ASSIGNMENT 7 - MATH235, FALL 2009

Submit by 16:00, Monday, November 2

1. Is the given polynomial irreducible:
   (1) $x^2 - 3$ in $\mathbb{Q}[x]$? In $\mathbb{R}[x]$?
   (2) $x^2 + x - 2$ in $\mathbb{F}_3[x]$? In $\mathbb{F}_7[x]$?

2. Find the rational roots of the polynomial $2x^4 + 4x^3 - 5x^2 - 5x + 2$.

3. Recall that for the ring $\mathbb{Z}$ a complete list of ideals is given by (0), (1), (2), (3), (4), $(5), \dots,$ where (n) is the principal ideal generated by n, namely, $(n) = \{na : a \in \mathbb{Z}\}$. Find the complete list of ideals of the ring $\mathbb{Z} \times \mathbb{Z}$.

4. Let R be a ring and let I and J be two ideals of R.
   (1) Prove that $I \cap J$ is an ideal of R, where

$$I \cap J = \{r : r \in I, r \in J\}$$

   (the intersection of the sets). It is called the intersection of the ideals I and J.
   (2) Prove that

$$I + J = \{i + j : i \in I, j \in J\}$$

   is an ideal of R. It is called the sum of the ideals I and J.
   (3) Find for every two ideals of the ring $\mathbb{Z}$ their sum and intersection.

5. Let $\mathbb{F}$ be a field. Prove that the ring $M_2(\mathbb{F})$ of 2×2 matrices with entries in $\mathbb{F}$ has no non-trivial (two-sided) ideals. That is, every ideal is either the zero ideal or $M_2(\mathbb{F})$ itself.
(Note: there is also a notion of a one-sided ideal that we don't discuss in this course. The ring $M_2(\mathbb{F})$ has a non-trivial one sided ideal. The notion of one-sided ideals is usually studied in MATH570, MATH571).

6. Consider the ring $\mathbb{Z}[\sqrt{3}]$ defined as the subring of $\mathbb{C}$ consisting of the expressions

$$\{a + b\sqrt{3} : a, b \in \mathbb{Z}\}$$

   (1) Prove that this is indeed a subring of $\mathbb{C}$, namely, that it is closed under addition, multiplication and contains $0, 1$.
   (2) For each of the following subsets of $\mathbb{Z}[\sqrt{3}]$ determine if it's an ideal. If so, show it is a principal ideal and find a generator.
      (a) $\{5a + b\sqrt{3} : a, b \in \mathbb{Z}\}$.

(b) $\{2a + 2b\sqrt{3} : a, b \in \mathbb{Z}\}$.
(c) $\{2a + 15b + (5a + 2b)\sqrt{3} : a, b \in \mathbb{Z}\}$.

7. Which of the following subsets if a subring of $M_2(\mathbb{C})$?

(1) $\left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{C} \right\}$.

(2) $\left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbb{C} \right\}$.

(3) $\left\{ \begin{pmatrix} a & b \\ 0 & \bar{a} \end{pmatrix} : a, b \in \mathbb{C} \right\}$.

(4) $\left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{R} \right\}$.

(5) $\left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, d \in \mathbb{C}, b \in \mathbb{R} \right\}$.