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QUIZ 3 – ALGEBRA I, MATH235, FALL 2007

PART A. (50 points for answering correctly 5 out of 6).

(1) Which of the following functions is a ring homomorphism from $\mathbb{R}[x] \to \mathbb{R}$? (circle all correct answers)

(a) The map taking $f(x) = a_n x^n + \cdots + a_1 x + a_0$ to a_0.

(b) The map taking $f(x) = a_n x^n + \cdots + a_1 x + a_0$ to $a_1 + a_0$.

(c) The map taking $f(x) = a_n x^n + \cdots + a_1 x + a_0$, $a_n \neq 0$, to a_n and taking the zero polynomial to zero.

(d) The map taking $f(x) = a_n x^n + \cdots + a_1 x + a_0$ to $a_0 - a_1 + a_2 - \cdots + (-1)^n a_n$.

The correct answers are (a) and (d). They are special cases of evaluation homomorphisms. The first is by substituting 0 for x, the second is by substituting -1 for x.

(2) Circle all correct statements:

(a) Every ideal of $\mathbb{Z}$ is principal.

(b) Every ideal of $\mathbb{Z}[x]$ is principal.

(c) Every ideal of $\mathbb{Q}$ is principal.

(d) Every ideal of $\mathbb{Q}[x]$ is principal.

(e) Every ideal of $\mathbb{Q}[x, y]$ is principal.

(f) Every ideal of $\mathbb{Q}[x]/(x^2 + x + 1)$ is principal.

(g) Every ideal of $\mathbb{Q}[x]/(x^2)$ is principal.

The correct answers are (a), (c), (d), (f), (g). For (a) and (d) these are theorems we proved. In case (c) and (f) we deal with fields ($x^2 + x + 1$ is irreducible over $\mathbb{Q}$) and a field $\mathbb{F}$ has only two ideals (0) and $(1) = \mathbb{F}$ (because any non-zero ideal contains a unit and so is equal to $\mathbb{F}$). Case (g) (which is just the ring $\mathbb{Q}[\epsilon]$) was analyzed in class. The cases (b) and (e) were discussed in the notes, tutorials and/or assignments.

(3) Let $\sigma = (1234)$, $\tau = (15)$ be permutations in the symmetric group S_5. Then $\sigma^2\tau\sigma^{-1}$ is
  (a) (12453).
  (b) (12)(45).
  (c) (12534).
  (d) (134)(25).

The correct answer is (12534).

(4) The maximal order an element of S_7 can have is
  (a) 7.
  (b) 12.
  (c) 14.
  (d) 8.
  (e) 10.

The answer is 12, realized for example by (1234)(567).

(5) Which is of the following is the correct statement? (only one statement is correct)
  (a) The ring $\mathbb{R}[x]/(x^2+1)$ is isomorphic to $\mathbb{C}$, the isomorphism taking x to i.
  (b) The ring $\mathbb{R}[x]/(x^2+1)$ is isomorphic to $\mathbb{R}\times\mathbb{R}$, the isomorphism taking x to $(0,1)$.
  (c) The ring $\mathbb{R}[x]/(x^2+1)$ is isomorphic to $\mathbb{R}$, the isomorphism taking x to 0.
  (d) The ring $\mathbb{R}[x]/(x^2+1)$ is isomorphic to the ring of dual numbers $\mathbb{R}[\epsilon]$, the isomorphism taking x to ϵ.

The correct statement is (a). We did it in class and it's in the notes too.

(6) The number of solutions for the equation $x^2+x-4\equiv 0 \pmod{17\cdot 19}$ is
  (a) 1.
  (b) 0.
  (c) 2.
  (d) 4.

The answer is 2. The discriminant is 17 and so there is one solution modulo 17. On the other hand $17 \equiv 36 \pmod{19}$ and so there are two solutions modulo 19. The number of solutions $\pmod{17\cdot 19}$ is $1\times 2 = 2$.

PART B. (50 points) Answer the following question in the exam notebook. Make sure you quote explicitly and precisely the results you are using in your proof.

Let $\mathbb{F}_5 = \mathbb{Z}/5\mathbb{Z}$ be the field of 5 elements. Prove the following:

(1) The polynomial $x^2 - 2$ is irreducible in $\mathbb{F}_5$.
(2) The ring $L = \mathbb{F}_5[x]/(x^2 - 2)$ is a field of 25 elements.
(3) The polynomial $t^2 - 2$ has a root in L. What is it?
(4) Prove that the polynomial $f(t) = t^2 + t - 3$ is also irreducible over $\mathbb{F}_5$. Find a root of f in L (write it as $ax + b$ for some $a, b \in \mathbb{F}_5$).

The polynomial $x^2 - 2$ is irreducible in $\mathbb{F}_5$ because 2 is not a square in $\mathbb{F}_5$. In fact, the squares are $0, 1, 4$.

We proved that if $\mathbb{F}$ is a field and $h(x)$ is an irreducible (non-constant) polynomial then $\mathbb{F}[x]/(h(x))$ is a field and so in particular L is a field. We further proved that if h has degree n, then any element of $\mathbb{F}[x]/(h(x))$ is represented by a unique polynomial class of degree less than n. In particular, if $\mathbb{F}$ has q elements then $\mathbb{F}[x]/(h(x))$ has q^n elements. Applying it to our situation we find that L has $5^2 = 25$ elements.

A root of the polynomial $t^2 - 2$ in L is x. Indeed, $x^2 - 2$ is zero in L. (Another root is $-x$).

The discriminant of $t^2 + t - 3$ is $1 + 4 \cdot 3 = 13 \equiv 3 \pmod 5$, which is not a square in $\mathbb{F}_5$ (see above). Since a quadratic polynomial is reducible if and only if it has a root, and the roots of $f(t)$ are $2^{-1}(-1 \pm \sqrt{3})$, if $f(t)$ is reducible then 3 is a square in $\mathbb{F}_5$ and we know that's not the case.

To show 3 is a square in L, we solve the equation $(ax + b)^2 = 3$ for $a, b \in \mathbb{F}_5$. One finds that $a^2x^2 + 2abx + b^2 = 2abx + 2a^2 + b^2 = 3$. It follows that $2ab = 0$ and so that either $a = 0$ or $b = 0$. If $a = 0$ it follows that $b^2 = 3$, which does not have a solution. Thus, $b = 0$ and we find that $2a^2 = 3$ and we can take $a = 2$. Thus $3 = (2x)^2$ in L. The solutions for $f(t)$ are then $2^{-1}(-1 \pm \sqrt{3}) = 3(-1 \pm 2x) = 2 \pm x$.