Algebraic groups I

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k = field char. 0, $\bar{k}$ = alg. closure

A linear algebraic group is a Zariski – closed subgroup of $GL_n(\bar{k})$ for some n.

Examples:

① $G = GL_n$ $\quad \underbrace{B = \left\{\begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix}\right\}}_{\text{Borel}}$ $\quad \underbrace{U = \left\{\begin{pmatrix} 1 & \cdots & * \\ & \ddots & \vdots \\ & & 1 \end{pmatrix}\right\}}_{\text{unipotent}}$ $\quad \underbrace{T = \left\{\begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}\right\}}_{\text{Torus}}$ $\quad \mathbb{G}_m := GL_1$

② q = symm. bilinear form, $q = (q_{ij})_{1 \le i,j \le N}$ $\quad q(x) := \frac{1}{2} q(x,y)$

$SO(q) = \{ M \in GL_N \mid M q\, {}^t M = q,\ \det(M) = 1_N \}$. E.g. $q = \begin{pmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{pmatrix}$, $N = 2n+1$

$q(x) = \frac{1}{2}(x_1 x_{2n+1} + x_2 x_{2n} + \cdots + x_n x_{n+2} + x_{n+1}^2)$

$SO \supseteq T = \left\{\begin{pmatrix} t_1 & & & & & \\ & t_2 & & & & \\ & & \ddots & & & \\ & & & t_n, 1, t_n^{-1} & & \\ & & & & \ddots & \\ & & & & & t_1^{-1} \end{pmatrix}\right\} \subseteq B = \left\{\begin{pmatrix} * & & * \\ & \ddots & \\ & & * \end{pmatrix} \cap SO\right\}$

G algebraic group: $G\times G \xrightarrow{m} G$, $\iota: G\to G$, $\{*\} \xrightarrow{e} G$

$\Rightarrow \bar{k}[G]$ = Hopf algebra = $\bar{k}$-algebra equipped w. $\bar{k}$-alg. homomor.

$m^*: \bar{k}[G] \longrightarrow \bar{k}[G] \otimes_{\bar{k}} \bar{k}[G]$, $\iota^*: \bar{k}[G] \to \bar{k}[G]$, $e^*: \bar{k}[G] \longrightarrow \bar{k}$ s.t. .....

A homomorphism $f: G\to H$, is a morphism of $\bar{k}$-varieties and of groups.

$\Leftrightarrow$ homom. $\bar{k}[H] \longrightarrow \bar{k}[G]$ of Hopf algebras.

A character of G: $\chi: G \longrightarrow \mathbb{G}_m$ corresponds to

$\bar{k}[t,t^{-1}] \longrightarrow \bar{k}[G]$, $t \longmapsto f$ and

$m^*(t) = t\otimes t \longmapsto f\otimes f = m^*(f)$ such f are rare and called group-like elements.

If $g\in G$ we get an inner automorphism $\mathrm{Int}_g(x) = g x g^{-1}$.

Non-abelian cohomology

G = topological group acting continuously on a discrete group M.

$H^0(G, M) = M^G := \{ m \in M : gm = m, \forall g \in G \}$

$H^1(G, M) = \{ \zeta : G \to M \mid \zeta(ab) = \zeta(a) \cdot a\zeta(b) \} / \sim$ (cont's fcn's)

$\zeta \sim \xi$ if $\exists\, m \in M$ s.t. $\zeta(a) = m^{-1} \cdot \xi(a) \cdot am$, $\forall a \in G$.

$0 \to A \to B \to C \to 0$ an exact sequence of G-groups

$$0 \to A^G \to B^G \to C^G \xrightarrow{\delta} \underbrace{H^1(G, A) \to H^1(G, B) \to H^1(G, C)}_{\text{pointed sets}} \left(\to H^2(G, A) \right)$$

if $A \subseteq \mathrm{Center}(B)$

Given $c \in C^G$ lift to $b \in B$: $\zeta(g) = b^{-1} \cdot gb$

Forms: Sp. G is defined over k.

A k-form of G is an algebraic group H over k s.t. $H \cong G$ over $\bar{k}$.

Let $\Gamma = Gal(\bar{k}/k)$. $G \xrightarrow{f} H$, $\forall \sigma \in \Gamma$ $G \xrightarrow{{}^\sigma f} H$ and $f^{-1} \circ {}^\sigma f \in Aut_{\bar{k}}(G)$.

$H \rightsquigarrow \zeta_H : \Gamma \longrightarrow Aut(G)$, $\zeta_H(\sigma) = f^{-1} \circ {}^\sigma f \in H^1(G, Aut_{\bar{k}}(G))$

$\exists$ bijection of pointed sets: Forms of $G \longleftrightarrow H^1(G, Aut_{\bar{k}}(G))$.

If H corresponds to ζ, $H(k) = G(\bar{k})^\Gamma$,

where Γ acts on $G(\bar{k})$ by

$$\tau * g = \zeta(\tau)(\tau(g)).$$

The compact real form of $GL_n(\mathbb{C})$:

Cartan involution

$\Gamma = Gal(\mathbb{C}/\mathbb{R})$, $\zeta(cc) = (g \longmapsto {}^t g^{-1})$, $U_n :=$ corr. group

$U_n(\mathbb{R}) = \{ g \in GL_n(\mathbb{C}) : g = {}^t \bar{g}^{-1} \} = \underbrace{\{ g \mid g g^* = 1 \}}_{\text{compact in complex topology.}} =$ Unitary group.

Theorem: Any reductive* group has a unique compact real form.

Example: $G = \mathbb{G}_m$, $\mathrm{Aut}_{\mathbb{C}}(\mathbb{G}_m) = \{\pm 1\}$, compact form $\mathbb{T}$ s.t. $\mathbb{T}(\mathbb{R}) = \{z \in \mathbb{C}^\times \mid z\bar{z} = 1\} = \mathbb{C}^1$

$\mathbb{T} \cong SO_2 = \left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a^2+b^2=1\right\}$. $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mapsto a+bi \in \mathbb{T}$.

Example: $1 \to \mathbb{G}_m \to GL_n \to PGL_n \to 1$ $\quad PGL_n(\bar{k}) = GL_n(\bar{k})/\bar{k}^\times$ <u>is</u> affine algebraic group.

$$1 \to k^\times \to GL_n(k) \to PGL_n(k) := PGL_n(\bar{k})^\Gamma \to H^1(\Gamma, \bar{k}^\times) \to H^1(\Gamma, GL_n(\bar{k})) \to H^1(\Gamma, PGL_n(\bar{k})) \to H^2(\Gamma, \bar{k}^\times)$$

$= \{1\}$ (Hilbert's 90)

The Brauer group

$\Longrightarrow PGL_n(\bar{k})^\Gamma = GL_n(k)/k^\times$

As $PGL_n(\bar{k}) = \mathrm{Aut}_{\bar{k}\text{-alg.}}(M_n(\bar{k}))$, $H^1(\Gamma, PGL_n(\bar{k})) =$ central simple algebras of rank n^2

Jordan decomposition

Any $g \in GL_N(\bar{k})$ has a unique decomposition

$$g = g_s \cdot g_u \qquad g_s = \text{semi simple}, \quad g_u = \text{unipotent}, \quad g_s g_u = g_u g_s$$

semi-simple := diagonalizable, unipotent $(g - 1_N)^N = 0$.

$$\begin{pmatrix} t_1 & u \\ & t_2 \end{pmatrix} \nearrow \begin{pmatrix} t_1 & u \\ & t_2 \end{pmatrix} \quad \text{if } t_1 \neq t_2, \text{ or } u = 0$$

$$\begin{pmatrix} t_1 & u \\ & t_2 \end{pmatrix} \searrow \begin{pmatrix} t_1 & 0 \\ & t_1 \end{pmatrix}\begin{pmatrix} 1 & u/t_1 \\ & 1 \end{pmatrix} \quad \text{if } t_1 = t_2.$$

- if $g \in H$ so are g_s, g_u.
- if $f: G \to H$ hom. $f(g_s) = f(g)_s$, $f(g_u) = f(g)_u$.

Tori

A torus T over k is a form of $\mathbb{G}_m^N$, for some N. Classified by

$$H^1(\Gamma, GL_N(\mathbb{Z})) = \mathrm{Hom}(\Gamma, GL_N(\mathbb{Z}))/\sim \text{conjugation}$$

$$X^*(T \otimes_k \bar{k}) := \mathrm{Hom}(T \otimes_k \bar{k}, \mathbb{G}_m) \cong \mathrm{Hom}(\mathbb{G}_m^N, \mathbb{G}_m) \cong \mathbb{Z}^N$$

With Galois action $({}^\sigma\chi)(t) = \sigma(\chi(\sigma^{-1}(t)))$

There is an anti-equivalence of categories b/w tori over k and free $\mathbb{Z}$-modules of finite rank equipped with a Galois action.

A linear action of $\mathbb{G}_m^N$ is simultaneously diagonalizable: $\mathbb{G}_m^N \longrightarrow GL(V)$, $V = \bigoplus_{\alpha \in X^*(\mathbb{G}_m^N)} V_\alpha$.

Theorem: All maximal tori are conjugate in G. Their common dimension is called rank(G).

Example: $\mathrm{rank}(GL_N) = N$, $\mathrm{rank}(SO_{2n+1}) = n$.

Solvable groups

A linear algebraic grp that is solvable in the usual sense ($\{1\} \subseteq G_1 \subseteq \dots \subseteq G_t = G$, G_i/G_{i-1} abelian $\forall i$)

* Kolchin–Lie: $G \subseteq GL_N$ solvable $\Longrightarrow$ G can be conjugated into B.
  Equivalently: when G acts on $\mathbb{P}^n$, it has a fixed point.

* Borel's fixed point theorem: G solvable $\Longrightarrow$ when G acts on any complete algebraic variety it has a fixed point.

A maximal connected solvable subgroup is called a Borel subgroup.

* Every torus is contained in a Borel.

* If G is reductive*, all Borel subgroups are conjugate.

For a group G let

$R(G)$ = maximal connected normal solvable subgroup.

$R_u(G)$ = " " " unipotent " .

G is called <u>semi-simple</u> if $R(G)=\{1\}$. $(SL_N, SO_{2n}, SO_{2n+1}, Sp_{2n})$

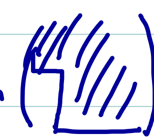
G is called <u>reductive</u> if $R_u(G)=\{1\}$. $(GL_N, GSp_{2n}, GSpin_{2n}, GSpin_{2n+1}, \dots)$

Any G has a decomposition

$$G = H \ltimes R_u(G)$$, H = max'l reductive group

$G/R_u(G)$ reductive, $G/R(G)$ semisimple

G reductive $\Rightarrow$ $G/$center is semi-simple.

<u>Chevalley</u>: G is reductive, $H < G$, $\exists\, G \to GL_n$ such that $H = G \cap$ 

G is reductive $\iff \forall \rho: G \to GL_N$, $\mathbb{C}[\rho(G)]$ is semi-simple. ($\mathbb{C}[\rho(G)]$ = linear span of $\rho(G)$)

$\iff$ Any linear representation of G is a direct sum of irreducible representations.

<u>Example</u>: $G_a \cong U_2 = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right\}$ is not reductive.

Parabolic Subgroups

A subgroup P of a connected algebraic group is called <u>parabolic</u> if G/P is projective.

P is parabolic $\iff$ P contains a Borel subgroup B.

<u>Example</u>: A <u>flag</u> in $\bar{k}^n$ is a collection of subspaces
$$F = (0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_a = \bar{k}^n)$$

The type of F is $\underline{d} = \{d_i\}$, $d_i = \dim(F_i)$, so $0 < d_1 < d_2 < \cdots < d_a = n$.

The space of all flags of type $\underline{d}$ is a projective variety $\mathcal{F}_{\underline{d}}$ on which GL_n acts transitively. Let P be a stabilizer of a flag. Then
$$G/P \cong \mathcal{F}_{\underline{d}} \quad \text{and } P \text{ is parabolic.}$$
For example, if $F_i = \mathrm{Span}\{e_1, \dots, e_{d_i}\}$ then $P = \begin{pmatrix} \boxed{n_1} & & /// \\ & \boxed{n_2} & \\ 0 & & \ddots \\ & & \boxed{n_a} \end{pmatrix}$ $\qquad n_i := d_i - d_{i-1}$

Consider $V = k^N$ with $q = \begin{pmatrix} & & 1 \\ & ⋰ & \\ 1 & & \end{pmatrix}$. An isotropic flag

$$F = \left(\{0\} \subsetneq F_1 \subsetneq \cdots \subsetneq F_a \right)$$

is a sequence of subspaces of V s.t. $q(x,y)|_{F_i} \equiv 0$. The space $\mathcal{F}_{\underline{d}}$ of isotropic flags of type $\underline{d}$ is a projective variety on which SO_q acts transitively, by Witt's theorem.

Let P be a stabilizer of a flag, then $SO_q / P \cong \mathcal{F}_{\underline{d}}$ and so P is a parabolic subgroup. Any parabolic arises this way.

For a standard choice $\underline{P} = \begin{pmatrix} \boxed{n_1} & * & * & * & * & * \\ & \ddots & * & * & * & * \\ & & \boxed{n_a} & * & * & * \\ & & & \boxed{/ / /} & * & * \\ & & & & \boxed{n_a} & * \\ & & & & & \ddots \\ & & & & & & \boxed{n_1} \end{pmatrix} \cap SO_q$

Algebraic Groups II

G reductive group over $\bar{k}$, alg. closed field of char. 0

Lie algebra:

$\mathrm{Lie}(G) = T_{G,1}$ = tangent space to G at 1 can be identified with the invariant derivations of G. Namely, a section of the tangent bundle $T_G \to G$ that is invariant under left-multiplication:

$l_g : G \to G$, $l_g(x) = gx$ induces $l_{g*} : T_G \to T_G$.

The identification of $\mathrm{Lie}(G)$ with derivation gives it a bracket operation

$$[X, Y] = X \circ Y - Y \circ X$$

$G \longrightarrow \mathfrak{g} := \mathrm{Lie}(G)$ is a functor. In particular,

$\mathrm{Int}_g : G \to G$ induces $\mathrm{Ad}_{\mathfrak{g}}(g) : \mathrm{Lie}(G) \to \mathrm{Lie}(G)$ giving the adjoint representation

$$\mathrm{Ad} : G \longrightarrow \mathrm{Aut}(\mathrm{Lie}(G)) \cong GL_N(\bar{k})$$

The fundamental example:

$G = GL_N(\bar{k})$, $\mathfrak{g} = M_N(\bar{k})$

$$Ad(g)(X) = gXg^{-1}.$$

For $H \subseteq G$, $\mathfrak{h} = Lie(H)$ is found by 1^{st} order approx. to the equations defining H.

* $q = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$, $O(q) = \{ M\, {}^tM = 1_N \}$. Write $M = 1 + M'$ then

$$(1+M')\,{}^t(1+M') = 1 + M' + {}^tM' + \cancel{M' \cdot {}^tM'} = 1$$

$$\mathfrak{o} = \mathfrak{so} = \{ M \in M_N(\bar{k}) : M + {}^tM = 0 \}$$

* $SL_N = \{ M \in GL_N \mid \det M = 1 \}$. Write $M = 1 + M' = 1 + (a_{ij})$. Then

$$\det(1+M') \equiv 1 + tr(M') \mod \square$$

$$\mathfrak{sl}_N = \{ M \in M_n(\bar{k}) \mid tr(M) = 0 \}$$

$* \quad T \cong \left\{ \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_N \end{pmatrix} \,\middle|\, t_i \in \bar{k}, \ \prod_i t_i \neq 0 \right\}$

$\mathfrak{h} = \mathrm{Lie}(T) \cong \left\{ \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_N \end{pmatrix} : t_i \in \bar{k} \right\}$

Over $\mathbb{C}$: $\quad \exp : \mathfrak{h} \longrightarrow T$, $\exp(t) = \sum_{n \geq 0} \frac{t^n}{n!}$, a surjective group homomorphism.

T is "universally semisimple" so

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha, \quad \Phi = \Phi(G,T) = \left\{ \alpha \in X^*(T) \setminus \{0\} \,\middle|\, \mathfrak{g}_\alpha \neq \{0\} \right\}.$$

$\Phi \subseteq X^*(T) \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ a real v.sp. equipped with a canonical inner product (induced from the Killing form on $\mathfrak{g}$).

Φ is a root system ($\Phi = -\Phi$, $\forall \alpha \in \Phi \quad s_\alpha(\Phi) = \Phi$ where $s_\alpha(v) = v - \frac{2(\alpha, v)}{(\alpha,\alpha)}\alpha, \ldots$)

A Weyl chamber W = connected component of $X^*(T)_{\mathbb{R}} \setminus \bigcup_{\alpha \in \Phi} \alpha^{\perp}$.
It determines an ordering of the roots

$$\alpha > 0 \text{ if } (\alpha, v) > 0 \quad \forall v \in W.$$

A positive root is called simple if it is not a non-trivial sum of other positive roots.

$\Delta \subseteq \Phi$ the set of simple roots. Any root is $\sum_{\delta_i \in \Delta} a_i \delta_i$, a_i all pos. or all neg.

Standard parabolic subgroups

A choice of a Borel $B \supseteq T$ $\longleftrightarrow$ choice of a Weyl chamber W.

$$B \longrightarrow \mathfrak{b} = \mathrm{Lie}(B) = \sum_{\alpha \geq 0} \mathfrak{g}_\alpha \supseteq \mathfrak{u} = \sum_{\alpha > 0} \mathfrak{g}_\alpha$$

$B = T \cdot U$, $\quad \mathrm{Lie}(U) = \mathfrak{u}$.

Given a subset $\Theta \subseteq \Delta$, let $S_\Theta = \left(\bigcap_{\alpha \in \Theta} \mathrm{Ker}(\alpha)\right)^\circ$ = torus of rank $\mathrm{rk}(G) - \#\Theta$. Define

$$P_\Theta = Z(S_\Theta)\cdot U \quad \text{a standard parabolic subgroup.}$$

<u>Theorem</u>: $\exists$ bijection Parabolic sgps / conjugation $\xleftrightarrow{1:1}$ std. parabolics

and $\exists$ $2^{\#\Delta}$ such std parabolics.

<u>Example</u>: $G = GL_N$. $T = \left\{ \begin{pmatrix} t_{11} & & \\ & \ddots & \\ & & t_{nn} \end{pmatrix} : \prod t_{ii} \neq 0 \right\}$, $\lambda_i(t) := t_{ii}$.

$X^*(T) = \bigoplus_{i=1}^{n} \mathbb{Z}\cdot \lambda_i$. $\mathfrak{g} = M_n$ with basis $E_{ij} = \begin{array}{cc} & j \\ \left(\begin{array}{c} \\ \end{array} \right. & \left. \begin{array}{c} \vdots \\ * \cdots \end{array} \right) \end{array} i$

$t E_{ij} t^{-1} = \frac{t_{ii}}{t_{jj}} \cdot E_{ij} = (\lambda_i - \lambda_j)(t)\cdot E_{ij}$.

$\Phi = \{ \lambda_i - \lambda_j \mid i \neq j \}$. $B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ & & * \end{pmatrix} \right\}$, $\mathfrak{b} = \mathrm{Lie}(B) = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ & & * \end{pmatrix} \right\} = \mathrm{Span}\{ E_{ij} \mid i \leq j \}$

$\mathfrak{b}$ induces the ordering $\lambda_i \geqslant \lambda_j \iff i \leq j$.

Thus: $\Phi^+ = \{\lambda_i - \lambda_j : i<j\} \supseteq \Delta = \{\lambda_1-\lambda_2, \lambda_2-\lambda_3, \dots, \lambda_{n-1}-\lambda_n\}$

n=2 $\quad \emptyset, \quad S_\emptyset = T = Z(T), \quad P = B.$

$\Theta = \{\lambda_1-\lambda_2\}, \quad S_\Theta = \left\{\begin{pmatrix} t_{11} & \\ & t_{11}\end{pmatrix}\right\}, \quad Z(S_\Theta) = GL_2, \quad P = GL_2.$

n=3 $\quad \Theta = \{\lambda_1-\lambda_2\}, \quad S_\Theta = \left\{\begin{pmatrix} t_{11} & & \\ & t_{11} & \\ & & t_{33}\end{pmatrix}\right\}, \quad Z(S_\Theta) = \begin{pmatrix} GL_2 & \\ & t_{33}\end{pmatrix}, \quad P = \left\{\begin{pmatrix} * & * \\ & \boxed{*}\end{pmatrix}\right\}$

$\Theta = \{\lambda_2-\lambda_3\}, \quad \dots\dots \quad P = \left\{\begin{pmatrix} * & * \\ & \boxed{*}\end{pmatrix}\right\}.$

And also B $(\longleftrightarrow \emptyset)$ and GL_3 $(\longleftrightarrow \Delta)$.

Weights:

G semisimple and fix $T \subseteq B$ a max'l torus contained in a Borel subgroup.

Let $\rho: G \to GL(V)$ a linear representation then

$$V = \bigoplus_{\alpha \in X^*(T)} V_\alpha, \quad V_\alpha = \{v \in V \mid \rho(t)(v) = \alpha(t)\cdot v, \ \forall t \in T\}.$$

The weights of ρ are $\{\alpha : V_\alpha \neq \{0\}\}$.

The weight lattice Λ_w is the minimal subgroup of $X^*(T)$ containing all weights of all linear representations of G.

$$\Lambda_w \underset{\substack{\text{finite}\\ \text{index}}}{\supseteq} \text{root lattice} = \Lambda_r = \mathbb{Z}\text{-span of } \Phi.$$

If V is irreducible $\exists!$ maximal weight α in the weights of V and it determines the representation. Thus say $V \cong U_\alpha$. (The set of α serving as highest wt. vectors is $\Lambda_w \cap W$). The wts appearing in U_α are congruent to α mod Λ_r

Given any repr'n W of G (e.g., $V^{\otimes n} \otimes (V^*)^{\otimes m}$, $\Lambda^a V, \ldots$) one can decompose W into irreducible representations "by hand".
Find a maximal weight α of W. Then $W \cong U_\alpha \oplus W'$ as G is reductive. Repeat!

Hilbert's invariants theorem

As before, G is a reductive group and $\rho: G \longrightarrow GL(V)$ a linear representation.

$\mathrm{Sym}(V^*)$ = ring of polynomial functions on V.
Indeed, if $\{e_1, \ldots, e_n\}$ is a basis for V and $\{x_1, \ldots, x_n\}$ the dual basis so any $v \in V$, $v = \sum_i x_i(v) \cdot e_i$, then $\mathrm{Sym}(V^*) = \bar{k}[x_1, \ldots, x_n]$.

$R := \mathrm{Sym}(V^*)$, G acts on R by substitutions,

$$(g * f)(v) = f(g^{-1} v).$$

The fundamental problem of invariant theory is to give a presentation for the $\bar{k}$-algebra R^G = the G-invariant polynomial fcns on V.

<u>Hilbert's theorem:</u> R^G is a finitely generated $\bar{k}$-algebra.

<u>Example:</u>

$G = SL_2(\bar{k})$ acting on symmetric bilinear forms $q = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$ by

$$g * q = g\, q\, {}^t g .$$

$\det q = ad - b^2 = \mathrm{disc.}(q)$ is an invariant. Is it the only one? Are the invariants sufficient to classify (separate) the orbits? What is the nature of the map

$$\mathrm{Spec}(R) \longrightarrow \mathrm{Spec}(R^G) \quad \text{corresponding to} \quad V \longrightarrow V /\!/ G \ ?$$

This, when V is replaced by an arbitrary variety is the subject of <u>Geometric Invariant Theory.</u>

$\mathrm{Sym}\, V^* = \bigoplus_{d \geq 0} \mathrm{Sym}^d V^*$ (homo' poly. of degree d)

G acts on each $\mathrm{Sym}^d V^*$, which is a finite dim'l repr'n. Decompose! Collect!

$R = \bigoplus_{d \geq 0} R[d]$, $R = \bigoplus_{\alpha} R_\alpha$, R_α = isotypical component of type α.

$f = \sum_\alpha f_\alpha$ isotypical components; each f_α = sum of homog. poly. all in R_α.

Reynolds operator: $f \longmapsto f^\natural$ = trivial repr'n component. Thus,

$R^G = \{ f^\natural : f \in R \}$

Properties: k-linear ✓ and $(\varphi f)^\natural = \varphi f^\natural \quad \forall \varphi \in R^G$.

Enough to show that for $f \in R[d]_\alpha$. Then, $\varphi . R[d]_\alpha \cong \mathbb{C}\varphi \otimes_{\mathbb{C}} R[d]_\alpha$,

by $\varphi \otimes f \longmapsto \varphi . f$. This is an isomorphism of G-modules and r.h.s. clearly of type α.

$$g*(\varphi\otimes f) = \varphi\otimes(g*f) = \varphi\otimes f\circ g^{-1} \longmapsto \{v \mapsto \varphi(v)\, f(g^{-1}v)\} = g*(\varphi f)(v). \quad \checkmark$$

<u>Thus:</u> $\natural : R \to R^G$ is a homomorphism of R^G-modules.

Let $R^G_+ = \bigoplus_{d>0} R[d]^G$. Consider the ideal $R\cdot R^G_+$ of R. By Hilbert's basis theorem

$\exists\ f_1,\dots,f_N \in R^G_+$ s.t. $R\cdot R^G_+ = \langle f_1,\dots,f_N\rangle_R$. WLOG, f_i homogeneous (of pos. degree).

<u>Claim:</u> $R^G = \mathbb{C}[f_1,\dots,f_N]$.

Let $\varphi \in R^G$ to show $\varphi \in \mathbb{C}[f_1,\dots,f_N]$ we may assume φ is homog. and we argue by induction on its degree. Degree 0 being obvious.

$$\varphi = \sum_i a_i f_i\ , \quad a_i \in R\ , \quad \text{wlog } a_i \text{ homog. and } \deg(a_i f_i) = \deg(\varphi).$$

$$\varphi = \varphi^\natural = \sum a_i^\natural f_i. \qquad \text{Wlog } a_i^\natural \text{ homog. } \deg(a_i^\natural f_i) = \deg(\varphi).$$

As $\deg a_i^\natural < \deg\varphi$, $a_i^\natural \in \mathbb{C}[f_1,\dots,f_N]$. $\checkmark$

Example: SL_2 acts on symmetric bilinear forms $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$ by

$$g * q = g q {}^t g$$

The orbit are represented by $\left\{ \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} : t \in \mathbb{C}^\times \right\}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

The function $\det(q)$ is the only invariant. Namely

$$R^G = \mathbb{C}[\det(q)] \cong \mathbb{C}[x]$$

$\mathbb{C}^3 /\!/ SL_2 \cong \mathbb{A}^1$, but the orbits are not in bijection with the points of $\mathbb{A}^1$.

Remark: Same holds for the action of SL_N on symmetric bilinear forms in N variables.

Theorem (Chevalley-Iwahori-Nagata): The set orbits always surjects onto V^G (for any action of a reductive group on an affine algebraic variety). It is bijective iff every orbit is Zariski-closed.

Example: The orbit of $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \left\{ \begin{pmatrix} a & b \\ b & d \end{pmatrix} : ad - b^2 = 0,\ \{a,b,d\} \neq \{0\} \right\}$ is not closed. $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ belongs to the closure.

References:

Algebraic groups: books by A. Borel, T. Springer, J.E. Humphreys, Goodman & Wallach.

Galois cohomology: Serre - local fields.

Representation theory: Humphreys (Lie algebras), Fulton - Harris

Invariant theory: Mumford GIT, paper by Dieudonné - Carrell, Goodman - Wallach