Exercises (Algebraic Geometry 2001-02)

1. (Hartshorne Ex. 1.9, p. 8) Let $\mathfrak{a}$ be an ideal of $k[x_1, \ldots, x_n]$ which can be generated by r elements. Prove that every irreducible component of $\mathcal{Z}(\mathfrak{a})$ has dimension $\geq n - r$.

2. (Hartshorne Ex. 2.14, p.13) *The Segre Embedding.* Let $\psi : \mathbb{P}^r \times \mathbb{P}^s$ be the map defined by

$$((a_0 : \cdots : a_r), (b_0 : \cdots : b_s)) \mapsto (\cdots : a_i b_j : \ldots)$$

where the coordinates $a_i b_j$ are given in lexicographic order and $N = (r+1)(s+1) - 1$. Prove first that ψ is well defined and injective. Show that the image of ψ is a subvariety of $\mathbb{P}^N$.

3. *Grassmannian Varieties.* Assume that k is a field of characteristic different from 2. Prove that a vector w in $\bigwedge^2 V$ is of the form $v_1 \wedge v_2$ for some vectors $v_1, v_2 \in V$ if and only if $w \wedge w = 0$. Conclude that $G(2, V)$ is a variety cut by $\binom{n}{4}$ quadratic relations.

Compute these relations explicitly for $n = 4, 5$ (you may use Maple). Do you get a radical ideal? (You may use Macaulay2).

4. *Regular functions and quasi-affine not affine variety.*
   (1) Let Y be a variety and Z an open subvariety of Y. Prove that $K(Z) \cong K(Y)$.
   (2) Prove that the ring of regular functions on $\mathbb{A}^2 \setminus Z(x)$ is isomorphic to $k[x, y][x^{-1}]$ (the localization of $k[x, y]$ in the multiplicative set $\{1, x, x^2, \ldots\}$).
   (3) Prove that the ring of regular functions on $\mathbb{A}^2 - \{0\}$ is isomorphic to $k[x, y]$.
   (4) Conclude that $\mathbb{A}^2 - \{0\}$ is not affine (Hint: Theorem 3.11 in the notes).

5. *Properties of products.*
   (1) Prove that if X and Y are irreducible (affine) varieties so is $X \times Y$.
   (2) Prove that $\dim(X \times Y) = \dim(X) + \dim(Y)$. (I will accept partial solutions here.)

6. *The information contained in the local ring.* (Hartshorne, Ex. 4.7, p. 31) Let X and Y be two varieties. Suppose that there are points $P \in X$ and $Q \in Y$ such that the local rings $\mathcal{O}_{X,P}$ and $\mathcal{O}_{Y,Q}$ are isomorphic as k-algebras. Then show that there are open sets $P \in U \subset X$ and $Q \in V \subset Y$ and an isomorphism of U to V that sends P to Q.

7. *Singular points and tangent cones.* Find the singular points of the surface $xy^2 = z^2$ in $\mathbb{A}^3$ assuming k is of characteristic different from 2. Calculate the tangent cone to the

surface at every singular point. Draw the surface with Maple (or anything similar) and see that your answer makes sense!

Prove this surface a rational surface (i.e., birational to $\mathbb{P}^2$) and find isomorphic open sets.

8. *A quotient singularity.* Let the group $\mathbb{Z}/3\mathbb{Z}$ act as automorphisms on $\mathbb{A}^2$ by letting 1 act as

$$(x, y) \mapsto (\zeta x, \zeta y),$$

where $\zeta \in k$ is a primitive third root of unity (assume char$(k) \neq 3$).

- Prove that the ring of invariants for this action is $B := k[x^3, x^2y, xy^2, y^3]$.
- Find a variety $Y \subset \mathbb{A}^4$ with $A(Y) \cong B$.
- Show that there is a canonical morphism $\pi : \mathbb{A}^2 \longrightarrow Y$, whose fibers are the orbits for the action. Show, moreover, that given an affine variety Z and a morphism $f : \mathbb{A}^2 \longrightarrow Z$, which is invariant under the action of $\mathbb{Z}/3\mathbb{Z}$, there is a unique morphism $f' : Y \longrightarrow Z$ such that the following diagram commutes:

$$\begin{tikzcd} \mathbb{A}^2 \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Z \\ & Y \arrow[ur, "f'"'] & \end{tikzcd}$$

  We say that Y is the *quotient* of $\mathbb{A}^2$ by (or 'for') the group action of $\mathbb{Z}/3\mathbb{Z}$.
- Find the singular points of Y and for each such point the tangent space and tangent cone. (Expect a very simple answer).
- Let $0 \in Y$ be $\pi(0)$. (If you chose Y in a natural fashion it will be the zero point of $\mathbb{A}^4$). Calculate the blow-up $\widetilde{Y}$ of $Y \subset \mathbb{A}^4$ at 0. Calculate the special fiber and find if $\widetilde{Y}$ is singular or not.

9. *Normal and Singular Varieties.* Let k be a field of characteristic not 2. Let $f(x) \in k[x_1, \ldots, x_n]$ be a square free polynomial, i.e., there does not exist a non-constant polynomial $g(x) \in k[x_1, \ldots, x_n]$ such that $g^2 | f$. Prove that the closed set Z given by

$$Z(y^2 - f(x)) \subseteq \mathbb{A}^{n+1}_{x_1,\ldots,x_n,y}$$

is normal. Conclude, in particular, that the cone $y^2 = x_1^2 + x_2^2$ is a singular and normal variety.

10. *Extending maps defined on an open set.* Do Hartshorne, Exercise 6.3.

11. *Automorphisms of* $\mathbb{P}^1$. Do Hartshorne, Exercise 6.6.