Alon-Boppana Lower Bound and Ramanujan Graphs

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(This lecture is based on [HLW06])

Recall that if G is an undirected graph with n vertices, then its adjacency matrix has n real eigenvalues $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. Then, in order to construct a family of d-regular expanders, by the Alon-Milman theorem, we need to bound the spectral gap $(d - \lambda_2)$ from below. Hence it is important to understand the behavior of λ_2.

1 Main statement and definition

Theorem 1.1 (Alon-Boppana)**.** *There exists a constant c such that for every connected finite regular graph G,*

$$\lambda_2(G) \geq 2\sqrt{d-1}\left(1 - \frac{c}{\Delta^2}\right)$$

where $\Delta = \mathrm{diam}(G)$ and $d = \deg(v)$ for every vertex v.

Corollary 1.2. *Let $(G_m)_{m=1}^{\infty}$ be a family of connected, d-regular, finite graphs with $|V(G_m)| \to \infty$ as $m \to \infty$. Then,*

$$\liminf_{m\to\infty} \lambda_2(G_m) \geq 2\sqrt{d-1}$$

In view of this corollary, we define Ramanujan graphs as graphs that are optimal in this sense:

Definition 1.3. An (n, d)-graph G (n vertices and d-regular) is called **Ramanujan** if

$$\lambda(G) \leq 2\sqrt{d-1}$$

where $\lambda(G) = \max\limits_{|\lambda_i| \neq d} |\lambda|$.

2 The infinite tree T_d and its spectrum

Throughout this section $T = T_d$, $V = V(T)$ and $N(v)$ denotes the set of neighbors of a vertex $v \in V$. We can define

$$A_T : l_2(V) \to l_2(V)$$

just like in the finite case, that is,

$$(A_T f)(v) = \sum_{w \in N(v)} f(w).$$

We view A_T in $\mathcal{B}(l_2(V))$, the Banach algebra of bounded linear operators on $l_2(V)$.

Definition 2.1. We say a function $f : V \to \mathbb{C}$ is **spherical around vertex** v if $f(u)$ depends only on the distance between u and v ($\mathrm{dist}(u, v)$).

For any function $f : V \to \mathbb{C}$, we can define its **spherical symmetrization around** v to be a function $\tilde{f}$ that is spherical around v and such that $\sum\limits_{\mathrm{dist}(u,v)=i} \tilde{f}(u) = \sum\limits_{\mathrm{dist}(u,v)=i} f(u)$ for every $i \geq 0$.

Definition 2.2. The **spectrum of** A_T is

$$\sigma(A_T) := \{\lambda : \lambda I - A_T \text{ is not invertible}\}$$

(For basic properties of the spectrum, see [Rud91])

Theorem 2.3 (Cartier)**.** $\sigma(A_T) = [-2\sqrt{d-1}, 2\sqrt{d-1}]$

Proof. (sketch)

We start by fixing a vertex $v \in V$ and consider it to be the 'root' of our tree.

It can show that in our case,

$$\lambda \in \sigma(A_T) \Longleftrightarrow \delta_v \notin \mathrm{img}(\lambda I - A_T)$$

where δ_v is the characteristic function of v ($\delta_v(v) = 1$ and $\delta_v(u) = 0$ for all the other $u \in V$).

So, it is enough to show that

$$\delta_v = (\lambda I - A_T) \cdot f \tag{1}$$

has a solution (in $l_2(V)$) if $|\lambda| < 2\sqrt{d-1}$ and does not have a solution if $|\lambda| > 2\sqrt{d-1}$ (see theorems 12.26 and 10.13 in [Rud91]).

Claim 1. We may assume f in equation (1) is spherical around v (more precisely, if (1) has a solution for some f, then it also has a solution for some $\tilde{f}$ spherical around v).

In fact, if f satisfies (1), then it is easy to show its spherical symmetrization around v also satisfies (1).

If f is spherical around v, then it is determined by a sequence $f_0, f_1, f_2, \ldots$ such that $f(u) = f_i$ for every u satisfying $\mathrm{dist}(u, v) = i$. Using this notation it is not hard to see that a spherical function f satisfies (1) if and only if:

$$\begin{aligned} &\lambda f_0 = d f_1 + 1 \\ &\lambda f_i = f_{i-1} + (d-1) f_{i+1} \quad \text{for } i \geq 1 \end{aligned} \tag{2}$$

Using linear algebra we can show the solutions $\{f_i\}$ of (2) are of the form $f_i = \alpha \rho_+^i + \beta \rho_-^i$, where $\rho_\pm = \frac{\lambda \pm \sqrt{\lambda^2 - 4(d-1)}}{2(d-1)}$.

Now, if $|\lambda| < 2\sqrt{d-1}$ then $|\rho_\pm| = \frac{1}{\sqrt{d-1}}$. Hence, $|f_i| = \Theta((d-1)^{-\frac{i}{2}})$ [according to the authors, this is an easy computation; the upper bound is easy to check but I could not verify the lower bound]. Since there are $\Theta((d-1)^i)$ vertices at distance i, this means such an f would not be in l_2 (in fact, $||f||_2^2 \geq C \sum_{i=0}^{\infty} (d-1)^i ((d-1)^{-\frac{i}{2}})^2 = C \sum_{i=0}^{\infty} 1 = \infty$). Hence, if $|\lambda| < 2\sqrt{d-1}$, then $\lambda \notin \sigma(A_T)$.

If $\lambda > 2\sqrt{d-1}$, then $r := |\rho_-| < \frac{1}{\sqrt{d-1}}$. In this case, we choose $\alpha = 0$, giving $f = \beta\rho_-^i$. Then, $||f||_2^2 \leq C \sum_{i=0}^{\infty} (d-1)^i (|\beta r^i|)^2 = C|\beta|^2 \sum_{i=0}^{\infty} ((d-1)r^2)^i$. Since $r < \frac{1}{\sqrt{d-1}}$, we obtain $(d-1)r^2 < 1$ and, thus, $||f||_2^2 < \infty$. Hence, $f \in l_2$. It is clear that it satisfies (2) for all $i \geq 1$. We just have to check it satisfies $\lambda f_0 = df_1 + 1$, i.e., $\lambda\beta = d\beta\rho_- + 1$. One can check that this holds for some choice of β. So, if $\lambda > 2\sqrt{d-1}$, then $\lambda \in \sigma(A_T)$.

A similar argument shows that if $\lambda < -2\sqrt{d-1}$, then $\lambda \in \sigma(A_T)$. $\square$

3 A proof of the Alon-Boppana lower bound

We proceed now to the proof of theorem 1.1. In this section, G is a graph as in theorem 1.1 and $A = A_G$. It is not hard to see that $\lambda_2(G) = \max\limits_{f \perp \mathbf{1}} \frac{f^T A f}{||f||^2}$ (where $\mathbf{1}$ denotes the constant function which maps everything to 1). So, we will define a convenient f that will give us the required lower bound for $\lambda_2(G)$.

Strategy of the proof: Consider $\Delta = \mathrm{diam}(G)$ and $s, t \in V(G)$ such that $\mathrm{dist}(s,t) = \Delta$. Roughly speaking, we will define f such that its values for vertices 'near' s are positive, its values for vertices 'near' t are negative and the remaining ones are mapped to zero. More specifically, we let $k = \lfloor \frac{\Delta}{2} \rfloor - 1$ and consider $T_{d,k}$, the d-'regular' tree of height k (see figure 1). We construct an eigenvector g for $A_{T_{d,k}}$ (the adjacency matrix of $T_{d,k}$) whose eigenvalue satisfies $\mu \geq 2\sqrt{d-1}(1 - \frac{c}{\Delta^2})$. By defining the values of f according to the values of g in a certain way (and normalizing its positive and negative values such that $< f, \mathbf{1} >= \sum f(x) = 0$), we can show that $\frac{f^T A f}{||f||^2} \geq \mu$, giving us the lower bound we wanted.

We want to construct an eigenvector g for $A_{T_{d,k}}$ (with eigenvalue μ). If we assume g is spherical around v (the root of $T_{d,k}$), we get the following equations for g:

$$\begin{aligned} \mu g_0 &= d g_1 \\ \mu g_i &= g_{i-1} + (d-1) g_{i+1}, \text{ for } i = 1, ..., k \\ g_{k+1} &= 0 \end{aligned} \tag{3}$$

(to simplify notation we assume there is a $(k+1)$-th level and the value of g at this level is zero)

Claim 3.1. *There is a $\mu > 1 - \frac{c}{\Delta^2}$ (with $c \approx 2\pi^2$) such that there is a real solution g of (3) that is non-negative and non-increasing.*

Proof. Define $h : \{0, ..., k+1\} \to \mathbb{R}$ by $h(i) := (d-1)^{-\frac{i}{2}} \sin((k+1-i)\theta)$.

It is easy to see that $h_{k+1} = 0$. Let us check that h satisfies (3) regardless of the value of θ:

$$\begin{aligned} h_{i-1} + (d-1)h_{i+1} &= (d-1)^{-\frac{i-1}{2}} \cdot [\sin((k+2-i)\theta) + \sin((k-i)\theta)] \\ &= \sqrt{d-1}(d-1)^{-\frac{i}{2}} \cdot 2\sin((k+1)\theta)\cos(\theta) = \mu h_i \end{aligned}$$

The condition for $i = 0$ reads

$$(2d-2) \cdot \cos(\theta) \cdot \sin((k+1)\theta) = d \cdot \sin(k\theta)$$

The smallest positive root of this equation is in $(0, \frac{\pi}{k+1})$ because the difference of the two terms of this equation change sign between 0 and $\frac{\pi}{k+1}$. So, $\theta \in (0, \frac{\pi}{k+1})$. Hence, $\theta_0 < \frac{\pi}{k+1} \approx \frac{2\pi}{\Delta}$, since $k = \lfloor \frac{\Delta}{2} \rfloor - 1$. By the Taylor expansion of cos, $\cos(\theta_0) > 1 - \frac{c}{\Delta^2}$ (so $c \approx 2\pi^2$).

Moreover, since $\theta \in (0, \frac{\pi}{k+1})$, h is non-negative and non-decreasing. $\square$

Let s and t be two vertices that realize the distance Δ. We define the sets of points 'near' s, 'near' t and the rest of them:

$$\begin{aligned} S_i &:= \{v : \mathrm{dist}(s, v) = i\} && \text{for } i = 0, ..., k \\ T_i &:= \{v : \mathrm{dist}(t, v) = i\} && \text{for } j = 0, ..., k \\ Q &:= V(G) \setminus \bigcup_{0 \le i \le k} (S_i \cup T_i) \end{aligned}$$

Notice that the sets S_i and T_j are disjoint (for any i, j). We are now ready to define $f : V(G) \to \mathbb{R}$:

$$f(v) = \begin{cases} c_1 g_i & \text{if } v \in S_i \\ -c_2 g_i & \text{if } v \in T_i \\ 0 & \text{otherwise} \end{cases}$$

where c_1 and c_2 are positive constants that will be determined later.

Claim 3.2. *With this definition we have*

$$(Af)_v \ge \mu f_v \quad \textit{for } v \in \bigcup_i S_i \qquad \textit{and} \qquad (Af)_v \le \mu f_v \quad \textit{for } v \in \bigcup_i T_i$$

Proof. Let $v \in T_i$ for some $i > 0$. Then, of its neighbors, $p \geq 1$ belong to T_{i-1}, q belong to T_i and $(d-p-q)$ belong to T_{i+1}. Thus,

$$(Af)_v = -(p \cdot c_2 g_{i-1} + q \cdot c_2 g_i + (d-p-q) \cdot c_2 g_{i+1})$$

Now, by (3) and claim 3.1,

$$\begin{aligned}
(Af)_v &= -c_2 \cdot (p g_{i-1} + q g_i + (d-p-q) g_{i+1}) \\
&= -c_2 \cdot (g_{i-1} + (p-1) g_{i-1} + q g_i + (d-p-q) g_{i+1}) \\
&\leq -c_2 \cdot (g_{i-1} + (p-1) g_{i+1} + q g_{i+1} + (d-p-q) g_{i+1}) \\
&= -c_2 \cdot (g_{i-1} + (d-1) g_{i+1}) \\
&= -c_2 \cdot (A_{T_{d,k}} g)_i = -c_2 \mu g_i = \mu f_v.
\end{aligned}$$

A similar argument works for $v \in S_i$. ∎

As a consequence of claims 3.1 and 3.2, we obtain the following

Theorem 3.3 (Alon-Boppana)**.**

$$\lambda_2(G) \geq 2\sqrt{d-1}\left(1 - \frac{c}{\Delta^2}\right)$$

Proof. By claim 3.2,

$$\begin{aligned}
f^T A f &= \sum_{v \in V(G)} f_v (Af)_v \\
&= \sum_{v \in \cup S_i} f_v (Af)_v + \sum_{v \in \cup T_i} f_v (Af)_v + \sum_{v \in Q} f_v (Af)_v \\
&\geq \sum_{v \in \cup S_i} f_v \mu f_v + \sum_{v \in \cup T_i} f_v \mu f_v = \mu f^T f = \mu f^T f = \mu \left|\left|f\right|\right|^2
\end{aligned}$$

Finally, by choosing suitable c_1 and c_2, we get

$$\sum_{v \in \cup S_i} f_v = -\sum_{v \in \cup T_i} f_v$$

and, thus, $f \perp \mathbf{1}$.

Therefore, by claim 3.1,

$$\lambda_2(A) \geq \frac{f^T A f}{\left|\left|f\right|\right|^2} \geq \mu \geq 2\sqrt{d-1}\left(1 - \frac{c}{\Delta^2}\right)$$

∎

4 Further Remarks

Conjecture 4.1. *For every integer $d \geq 3$ there exists arbritrarily larde d-regular Ramanujan graphs.*

Theorem 4.2 (Lubotzky-Phillips-Sarnak [LPS88], Margulis [Mar88], Morgenstern [Mor94])**.** *For every prime p and every positive integer k there exist infinitely many d-regular Ramanujan graphs with $d = p^k + 1$.*

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