Graphs With Large Girth And Large Chromatic Number

In the first part of these notes we use a probabilistic method to show the existence of graphs with large girth and large chromatic number. In the second part we give an explicit example of such graphs. It is mostly based on the third chapter of *Some Applications Of Modular Forms* by Peter Sarnak and also the third and forth chapters of *Elementary Number Theory, Group Theory, And Ramanujan Graphs* by Giuliana Davidoff, Peter Sarnak and Alain Valette.

1 Existence Of Graphs With Large Girth And Large Chromatic Number

An interesting combinatorial problem is to construct graphs with large chromatic number and large girth. Since adding edges increases the chromatic number and decrease the girth, it is not obvious at all that such graphs exist.

Let start with reminding some basic definitions and lemmas.

Definition 1. *Let $X = (V, E)$ be a finite graph without loop. A proper coloring of X is a coloring of vertices of X such that two adjacent vertices get different colors.*

Definition 2. *Denote by A, the adjacency matrix of $X = (V, E)$. We call a set $S \subset V$ a stable set if $A_{x,y} = 0$ for all $x, y \in S$.*

Definition 3. *The chromatic number $\chi(X)$ is the minimal number n, such that there exists a proper coloring of X using n colors. In other words, $\chi(X)$ is the minimal number n of classes in a partition $V = V_1 \cup .... \cup V_n$, such that each V_i is a stable set.*

Definition 4. *The independence number $i(x)$ of a graph is the maximal cardinality of a stable set of X.*

Definition 5. *The girth of a graph denoted by $g(x)$ is the length of a shortest cycle contained in the graph.*

Lemma 1.1. *Let X be a finite graph without loop, on n vertices. Then $n \leq i(X)\chi(X)$.*

Proof. *Let*

$$V_i = \{v \in V : v \text{ is colored by color } i\} \quad \forall 1 \le i \le \chi(X)$$

Since V is the disjoint union of V_i, we have $|V| = \sum_{i=1}^{\chi(X)} |V_i|$. We also have for all i, $|V_i| \le i(X)$ since each V_i is a stable set. Hence,

$$n = |V| \le i(X)\chi(X).$$

Our goal is to show that graphs with large girth and large chromatic number exists. In order to do that, we use a method known as the probabilistic method. The method consists of examining the graphs of a certain shape which do not have the desired property and to show that there are not so many of them. The method does not give an explicit example of such graphs.

Let k and c be large numbers. We will show that there exits graph X such that $g(X) \ge k$ and $\chi(X) \ge c$. We let n goes to infinity in the following discussion. We denote by $\chi_{n,m}$ the set of all graphs with n labeled vertices and m edges. fix $0 < \epsilon < 1/k$ and set $m = \lfloor n^{1+\epsilon} \rfloor$.

We have $|\chi_{n,m}| = \binom{\binom{n}{2}}{m}$, because in order to construct a graph on n vertices with m edges, we need to choose m edges out of $\binom{n}{2}$ existing edges.

For large n by lemma 1.1, small independence number leads to a large chromatic number. So, we are interested in those graphs in $\chi_{n,m}$ with small independence number. To measure how small is the independence number of a graph, we would require that it has to meet the complete graph k_p on any subset of the vertices with p elements,for some integer p, in a large number, let say at least n edges. Let fix $0 \le \eta \le \epsilon/2$ and set $p = \lfloor n^{1-\eta} \rfloor$. Now we count the number of "bad" X, the ones which meet a given complete graph K_p in less than n edges.

If $0 \le l \le n$, then the number of X which meet a given K_p in exactly l edges is

$$\binom{\binom{p}{2}}{l}\binom{\binom{n}{2} - \binom{p}{2}}{m-l}$$

Since we need to choose exactly l edges out of $\binom{p}{2}$ edges in the given K_p and the remaining $m - l$ edges out of edges not in the K_p.

Let $N(K_p, n, m)$ be the number of $X \in \chi_{n,m}$ such that X meets the given K_p in at most n edges. Then $N(K_p, n, m)$ is equal to

$$N(K_p, n, m) = \sum_{l=0}^{n} \binom{\binom{p}{2}}{l}\binom{\binom{n}{2} - \binom{p}{2}}{m-l}$$

For $n \le N/2$ and $l \le n$, we have

$$\binom{N}{l} \leq \binom{N}{n}$$

.

Sine $\binom{p}{2} \approx \frac{n^{2(1-\eta)}}{2}$ *then, for* n *large and* $l \leq n$*, we have*

$$\binom{\binom{p}{2}}{l} \leq \binom{\binom{p}{2}}{n}$$

and

$$\binom{\binom{n}{2} - \binom{p}{2}}{m-l} \leq \binom{\binom{n}{2} - \binom{p}{2}}{m}$$

Thus,

$$\begin{aligned}
N(K_p, n, m) &\leq (n+1)\binom{\binom{p}{2}}{n}\binom{\binom{n}{2} - \binom{p}{2}}{m} \\
&\leq p^{2n}\binom{\binom{n}{2} - \binom{p}{2}}{m} \\
&= p^{2n}/m! \left[\binom{n}{2} - \binom{p}{2}\right]\left[\binom{n}{2} - \binom{p}{2} - 1\right] ... \left[\binom{n}{2} - \binom{p}{2} - m + 1\right],
\end{aligned}$$

Where, the second inequality comes from the fact that

$$\begin{aligned}
(n+1)\binom{\binom{p}{2}}{n} &= (n+1)/n! \left[\binom{p}{2}\right]\left[\binom{p}{2} - 1\right] ... \left[\binom{p}{2} - n + 1\right] \\
&\leq \binom{p}{2}^n \leq (p^2)^n.
\end{aligned}$$

Now, using the fact that for $l \leq m$

$$\binom{n}{2} - \binom{p}{2} - l \leq \left(\binom{n}{2} - l\right)\left(1 - \frac{\binom{p}{2}}{\binom{n}{2}}\right)$$

,

we get

$$\begin{aligned}
N(K_p, n, m) &\leq \frac{p^{2n}}{m!}\left[\binom{n}{2}\right]\left[\binom{n}{2} - 1\right] ... \left[\binom{n}{2} - m + 1\right]\left(1 - \frac{\binom{p}{2}}{\binom{n}{2}}\right)^m \\
&= p^{2n}\binom{\binom{n}{2}}{m}\left(1 - \frac{\binom{p}{2}}{\binom{n}{2}}\right)^m \\
&\leq p^{2n}\binom{\binom{n}{2}}{m}\left(1 - \left(\frac{p-1}{n-1}\right)^2\right)^m
\end{aligned}$$

We know that for $0 \leq x \leq 1$, we have $(1-x)^m \leq e^{-mx}$. thus,

$$N(K_p, n, m) \leq p^{2n} \binom{\binom{n}{2}}{m} e^{-m\left(\frac{p-1}{n-1}\right)^2}$$
$$= p^{2n} e^{-m\left(\frac{p-1}{n-1}\right)^2} \left|\chi_{n,m}\right|.$$

Let $N(n, m)$ be the number of graph $X \in \chi_{n,m}$ such that X meets some K_p in less than n edges. Then, we have

$$N(n, m) \leq \binom{n}{p} N(K_p, n, m)$$

since there are $\binom{n}{p}$ such K_p.

We have $\binom{n}{p} \leq n^p$. We also have $n^p \leq p^n$, since $p = \lfloor n^{1-\eta} \rfloor$. Hence, we get

$$N(n, m) \leq p^{3n} e^{-m\left(\frac{p-1}{n-1}\right)^2} \left|\chi_{n,m}\right|$$

Since $m = \lfloor n^{1+\epsilon} \rfloor$, $p = \lfloor n^{1-\eta} \rfloor$ and $0 \leq \eta \leq \frac{\epsilon}{2}$, we get

$$p^{3n} e^{-m\left(\frac{p-1}{n-1}\right)^2} \approx n^{(1-\eta)3n} e^{-n^{1+\epsilon-2\eta}} \to 0$$

as n goes to infinity. Hence,

$$lim_{n \to \infty} \frac{N(n, m)}{\left|\chi_{n,m}\right|} = 0$$

,

which gives us

$$N(n, m) = o\left(\left|\chi_{n,m}\right|\right)$$

This guarantees that the proportion of the graphs with small independence number in $\chi_{n,m}$ goes to 1 as n goes to infinity.

Next, let look at the girth. Nothing ensures that the graphs mentioned previously, with small independence number, have large girth. We should manage to make disappear small circuits form X without increasing its independence number.

Let k be a large integer. Define

$$F(X) = \left|\{c : \text{ c is a circuit of X with length } l \ \leq k\}\right|$$

In other words, $F(X)$ is the number of small circuits in the graph X. Denote by $A(n, k)$ the average of F over $\chi_{n,m}$:

$$A(n,k) = \frac{1}{|\chi_{n,m}|} \sum_{X \in \chi_{n,m}} F(X)$$

We can also calculate $A(n,k)$ *in another way: let* $c:\ x_1x_2...x_lx_1$ *be a circuit of length* $3 \le l \le k$. *Such a circuit appears in* $\binom{\binom{n}{2}-l}{m-l}$ *graphs* $X \in \chi_{n,m}$. *Hence, it contributes* $\binom{\binom{n}{2}-l}{m-l}$ *to the sum. There are* $n(n-1)...(n-l+1)$ *such circuits of length* l. *Hence, we have*

$$\begin{aligned}
A(n,k) &= \frac{1}{|\chi_{n,m}|} \sum_{l=3}^{k} n(n-1)...(n-l+1) \binom{\binom{n}{2}-l}{m-l} \\
&\le \sum_{l=3}^{k} n^l \frac{\binom{\binom{n}{2}-l}{m-l}}{\binom{\binom{n}{2}}{m}} \qquad \textit{(since } |\chi_{n,m}| = \binom{\binom{n}{2}}{m}\textit{)} \\
&= \sum_{l=3}^{k} n^l \frac{m(m-1)...(m-l+1)}{\binom{n}{2}\left(\binom{n}{2}-1\right)...\left(\binom{n}{2}-l+1\right)} \\
&\le \sum_{l=3}^{k} \frac{n^l m^l}{\binom{n}{2}\left(\binom{n}{2}-1\right)...\left(\binom{n}{2}-l+1\right)} \\
&= \sum_{l=3}^{k} \frac{n^l m^l}{\binom{n}{2}^l} \left[1 + \left(\frac{\binom{n}{2}^l}{\binom{n}{2}\left(\binom{n}{2}-1\right)...\left(\binom{n}{2}-l+1\right)} - 1\right)\right]
\end{aligned}$$

We have

$$\left(\frac{\binom{n}{2}^l}{\binom{n}{2}\left(\binom{n}{2}-1\right)...\left(\binom{n}{2}-l+1\right)} - 1\right) \to \left(\frac{n^{2l}}{n^{2l}} - 1\right) \to 0$$

as $n \to \infty$. *Hence,*

$$\begin{aligned}
A(n,k) &\le \sum_{l=3}^{k} \frac{n^l m^l}{\binom{n}{2}^l} (1+o(1)) \\
&= \sum_{l=3}^{k} \left(\frac{2m}{n-1}\right)^l (1+o(1)) \\
&\le k \left(\frac{2m}{n-1}\right)^k (1+o(1)) \\
&\to \frac{2^k \left(n^{1+\epsilon}\right)^k}{n^k} (1+o(1)) = o(n)
\end{aligned}$$

Since $m = \lfloor n^{1+\epsilon} \rfloor$ and $\epsilon < \frac{1}{k}$.

We conclude that

$$\frac{1}{|\chi_{m,n}|} \sum_{X \in \chi_{m,n}: F(X) \geq \frac{n}{k}} \frac{n}{k} \leq \frac{1}{|\chi_{m,n}|} \sum_{X \in \chi_{m,n}} F(X)$$
$$= A(n,k) = o(n)$$

as $n \to \infty$. Thus,

$$\frac{\frac{n}{k} \left| \left\{ X \in \chi_{m,n} : F(X) \geq \frac{n}{k} \right\} \right|}{|\chi_{m,n}|} = o(n)$$

.

So,

$$\frac{\left| \left\{ X \in \chi_{m,n} : F(X) \geq \frac{n}{k} \right\} \right|}{|\chi_{m,n}|} = o(1)$$

.

as $n \to \infty$.

Now, we reach the last step. Consider a graph $X \in \chi_{m,n}$ such that X has the 2 following properties:

1. *X meets every K_p in at least n edges*
2. *$F(X) < \frac{n}{k}$*

From what we saw previously, the proportion of $X \in \chi_{m,n}$ which satisfy these 2 properties goes to 1 as $n \to \infty$. So, we can choose such X. Let X' be the graph obtained from X by deleting all edges which lie on a circuit of length at most k. Then, obviously $g(X') > k$. Also, by (2), there are at most $\frac{n}{k}$ such circuits in X. So we delete less than $\frac{n}{k}k = n$ edges for going from X to X'. Hence X' meets any K_p in at least one edges and we have $i(X) \leq p$. By lemma 1.1, $\chi'(X') \geq \frac{n}{p}$, which is of order n^{η} and so is greater than c when n is large enough. So, X' has the required properties.

2 Explicit Example of Expander Graphs With Large Girth And Large Chromatic Number

In the previous section we discussed the existence of graphs with large girth and large chromatic number. In this section we will give an explicit example of expander graphs with large girth and large chromatic number(for definition of expander graphs we refer to the third lecture of the course). We will not give proofs *for the most of the theorems in this section. The reader can look for the proofs in the both references of the notes. We start with giving some definitions.*

Definition 6. *Let $GL_2(q)$ be the group of 2-by-2 matrices with coefficients in $\mathbb{F}_q$ and $SL_2(q)$ the subgroup of $GL_2(q)$ of matrices with determinant 1. We denote by $PGL_2(q)$ the quotient group*

$$PGL_2(q) = GL_2(q)/\left\{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in \mathbb{F}_q^*\right\}$$

and by $PSL_2(q)$ the quotient group

$$PSL_2(q) = SL_2(q)/\left\{\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} : \epsilon = \pm 1\right\}$$

Proposition 1.

- *(a)* $|GL_2(q)| = q(q-1)(q^2-1)$
- *(b)* $|SL_2(q)| = |PGL_2(q)| = q(q^2-1)$
- *(c)* $|PSL_2(q)| = \begin{cases} q(q^2-1) & \textit{if } q \textit{ is even} \\ q(q^2-1)/2 & \textit{if } q \textit{ is odd} \end{cases}$

Proof.*(a). For constructing a 2-by-2 matrices in $GL_2(q)$, we choose the first column, a nonzero vector in $\mathbb{F}_q^2$: we have (q^2-1) such vectors. Then we need to choose the second column: another vector in $\mathbb{F}_q^2$, independent from the first one; there are (q^2-q) of them.*

(b). Since $PGL_2(q)$ is the quotient of $GL_2(q)$ by a subgroup of cardinality q, we get the result for the cardinality of $PGL_2(q)$. For the cardinality of $SL_2(q)$, we basically repeat the argument used for $|GL_2(q)|$, just for the second column we only have q choices, since we want the determinant to be 1.

(c). It follows by the same kind of arguments.

We will need the following theorem:

Theorem 2.1. *Let $r_{2s}(n)$ be the number of representations of n as a sum of $2s$ squares. Then*

$$r_4(n) = 8 \sum_{d|n, 4\nmid d} d.$$

Now let p, q be two different primes such that

$$p \equiv 1, q \equiv 1 (mod 4)$$

Consider the reduction modulo q:

$$\tau_q : \mathbb{Z} \rightarrow \mathbb{F}_q$$

and let i be an integer such that $i^2 \equiv -1$ (mod q)(Since $q \equiv 1(mod4)$, we know that such i exists).

From theorem 2.1 we know that there are $8(p+1)$ solutions $\alpha = (a_0, a_1, a_2, a_3)$ to

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 = p.$$

There are exactly $p+1$ solutions α with $a_0 > 0$ and odd and a_j even for $j = 1, 2, 3$. Let $\overline{\alpha}$ be the component-wise image of α under τ_q:

$$\overline{\alpha} = (\tau_q(a_0), \tau_q(a_1), \tau_q(a_2), \tau_q(a_3))$$

We associate to $\overline{\alpha}$ the matrix $\overline{\alpha}'$ in $PGL_2(q)$:

$$\overline{\alpha}' = \begin{pmatrix} \tau_q(a_0) + i\tau_q(a_1) & \tau_q(a_2) + i\tau_q(a_3) \\ -\tau_q(a_2) + i\tau_q(a_3) & \tau_q(a_0) - i\tau_q(a_1) \end{pmatrix}$$

Let $S \subseteq PGL_2(q)$ be the set of matrices that we get in this way.

Theorem 2.2. *If q is large enough with respect to p(for example $q > 2\sqrt{p}$), then $|S| = p+1$*

Recall that we define the Legendre symbol (m/p) to be

$$(m/p) = \begin{cases} 0 & \text{if } p \mid m \\ 1 & \text{if } p \nmid m \text{ and m is a square modulo } p \\ -1 & \text{if } p \nmid m \text{ and m is not a square modulo } p \end{cases}$$

For any $\overline{\alpha}' \in PSL_2(q)$, defined as above,

$$\begin{aligned} det(\overline{\alpha}') &= \tau_q(a_0^2 - (ia_1)^2 - (ia_3)^2 + a_2^2) \\ &= \tau_q(a_0^2 + a_1^2 + a_3^2 + a_2^2) \\ &= \tau_q(p) \end{aligned}$$

Now suppose that $(p/q) = 1$, then there exists $d \in \mathbb{F}_q$ such that $p = d^2$. Now consider a matrix $M \in S$. So, $det(M) = p = d^2$. Let $M' = \begin{pmatrix} \frac{1}{d} & 0 \\ 0 & \frac{1}{d} \end{pmatrix} M$. Then,

$$det(M') = \frac{1}{d^2} det(M) = \frac{1}{p} p = 1.$$

Thus, $M' \in PSL_2(q)$. Since, M and M' represent the same equivalent class in $PGL_2(q)$, we conclude that $M \in PSL_2(q)$. Hence, when $(p/q) = 1$, $S \subseteq PSL_2(q)$.

Define $X^{p,q}$ to be the Cayley graph of

- *$PGL_2(q)$ relative to the S if $(p/q) = -1$*
- *$PSL_2(q)$ relative to the S if $(p/q) = 1$*

(For the definition of Cayley graph we refer to the seventh lecture of this course). In fact, $X^{p,q}$ is the connected component of the identity in the Cayley graph of $PGL_2(q)$ relative to the S. So $X^{p,q}$ is a $(p+1)$-regular graph on $n = q(q^2-1)$ or $n = q(q^2-1)/2$ depending on the sign of (p/q).

The following theorem gives us the main result:

Theorem 2.3. *1. Case(1)$(p/q) = -1$*

   (a) $X^{p,q}$ is a bipartite Ramanujan graph

   (b) $girth(X^{p,q}) \geq 4log_p q - log_p 4$

   (c) $diam(X^{p,q}) \leq 2logn + 2log_p 2 + 1$

2. Case(2)$(p/q) = 1$

   (a) $X^{p,q}$ is a Ramanujan graph

   (b) $girth(X^{p,q}) \geq 2log_p q$

   (c) $diam(X^{p,q}) \leq 2log_p n + 2log_p 2 + 1$

   (d) $i(X^{p,q}) \leq \frac{2\sqrt{p}}{p+1} n$

   (e) $\chi(X^{p,q}) \geq \frac{p+1}{2\sqrt{p}}$

(For the definition of Ramanujan graphs we refer to the third lecture of the course). Hence, when $(p/q) = 1$, $X^{p,q}$ is an explicit example of Ramanujan graphs(a special case of expanders) with large girth and large chromatic number.