P-adic numbers
And
Bruhat-Tits Tree

In the first part of these notes we give a brief introduction to p-adic numbers. In the second part we discuss some properties of the 'Bruhat-Tits' tree. It is mostly based on the first chapter of *p-adic Numbers, p-adic Analysis and Zeta-Functions by Neal Koblitz and also second and third chapters of p-adic numbers by Fernando Q.Gouvea.*

1 Field Of p-adic Numbers

In the first part of these notes, we introduce a new norm $|.|_p$ on $\mathbb{Q}$, for any prime p. We want to construct an extension of $\mathbb{Q}$, $\mathbb{Q}_p$, such that $\mathbb{Q}_p$ is complete with respect to $|.|_p$. We start with recalling some basic definitions.

If X is a nonempty set, a metric is a function $d : X \times X \to R^+$ such that
$d(x, y) = 0$ *iff* $x = y$
$d(x, y) = d(y, x)$
$d(x, y) \leq d(x, z) + d(z, y)\ \forall z \in X$
A norm on a the field F is a map denoted $\|.\|$ from F to R^+ such that:
$\|x\| = 0$ *iff* $x = 0$
$\|xy\| = \|x\|\,\|y\|$
$\|x + y\| \leq \|x\| + \|y\|$
We say that a metric d comes from a norm $\|\|$ if $d(x, y) = \|x - y\|$

1.1 Metrics On Rational Numbers

Definition 1. *Fix a prime number p. Let:*

$$ord_p : Z\setminus\{0\} \to R$$

be the function that assigns to any positive integer a, the highest power of p which divides a,i.e., the greatest m such that $a \equiv 0$ (mod p^m).

We extend ord_p to the field of rational numbers as follows: if $x = a/b \in Q^{\times}$ then

$$ord_p(x) = ord_p(a) - ord_p(b)$$

For $a = 0$, we write $ord_p(0) = \infty$. We can easily check that for any $x \in Q$, the value $ord_p(x)$ does not depend on its representation as a quotient of two integers.

The basic properties of ord_p are the following:

Lemma 1.1. *1. $ord_p(xy) = ord_p(x) + ord_p(y)$*

2. $ord_p(x+y) \geq min\left\{ord_p(x), ord_p(y)\right\}$

Definition 2. *define the map $|.|_p$ on Q as follows:*

$$|x|_p = \begin{cases} p^{-ord_p(x)} & if \ x \neq 0 \\ 0 & if \ x = 0 \end{cases}$$

Proposition 1. *$|.|_p$ is a norm on Q.*

Remark 1. *We sometimes let $|.|_\infty$ denote the usual absolute value on Q and we call it the "absolute value at infinity".*

Definition 3. *A norm is called non-Archimedean if $\|x+y\| \leq max\left(\|x\|, \|y\|\right)$ for all x, y in the field. A metric is non-Archimedean if $d(x,y) \leq max\left(d(x,z), d(z,y)\right)$ for all x, y, z; In particular, a metric is non-Archimedean if it is induced by a non-Archimedean norm,since in that case $d(x,y) = \|x-y\| = \|(x-z)+(z-y)\| \leq max(\|x-z\|, \|z-y\|) = max(d(x,z), d(z,y))$*

It follows from Lemma ... that $|.|_p$ is non-Archimedean. A norm(or a metric) which is not non-Archimedean is called Archimedean. The ordinary absolute value is an Archimedean norm on Q.

1.2 Topological Properties Of non-Archimedean Metrics

Proposition 2. *Let $\|.\|$ be a non-Archimedean absolute value on a field K. If $x, y \in K$ and $\|x\| \neq \|y\|$, then*
$\|x+y\| = max\left(\|x\|, \|y\|\right)$

Proof. If we suppose that $\|x\| > \|y\|$, then

$$\|x+y\| \leq \|x\| = max\left\{\|x\| \, \|y\|\right\}$$

We also have $x = (x+y) - y$, so

$$\|x\| \leq max\left\{\|x+y\| \, \|y\|\right\}$$

Since we know that $\|x\| > \|y\|$, this inequality can only hold if

$$max\left\{\|x+y\|, \|y\|\right\} = \|x+y\|$$

This gives the reverse inequality $\|x\| \leq \|x+y\|$.

Corollary 1.2. *In a space with a non-Archimedean metric, all triangles are isosceles.*

Proof. Let x, y, z be the vertices of our triangle, so the length of the sides of the triangle are

$$d(x,y) = \|x-y\|$$
$$d(y,z) = \|y-z\|$$
$$d(x,z) = \|x-z\|$$

We also have $(x-y)+(y-z) = (x-z)$. So if $\|x-y\| \neq \|y-z\|$, by the previous proposition $\|x-z\|$ is equal to the bigger of the two. In any case two of the sides are equal.

This really should not be too surprising if we think what this says in the case of $|.|_p$ on Q. It says that if two rational numbers are divisible by different powers of p, then their difference is divisible precisely by the lower power of p(which is what it means to be the same size as the bigger of the two).

Let $a \in K$ be an element and and r a positive real number. Consider the open ball of radius r and center a

$$B(a,r) = \{x \in K : \|x-a\| < r\}$$

and the closed ball of radius r and center a

$$\bar{B}(a,r) = \{x \in K : \|x-a\| \leq r\}$$

We have the following proposition

Proposition 3. *Let K be a field with a non-Archimedean absolute value.*

1. *If $b \in B(a,r)$, then $B(a,r) = B(b,r)$*
2. *If $b \in \bar{B}(a,r)$, then $\bar{B}(a,r) = \bar{B}(b,r)$*

3. $B(a,r)$ *is both open and closed.*

4. $\bar{B}(a,r)$ *is both open and closed for* $r \neq 0$.

5. *For* $r,s \neq 0$, *we have* $B(a,r) \cap B(b,s) \neq \phi$ *if and only if* $B(a,r) \subset B(b,s)$ *or* $B(a,r) \supset B(b,s)$.

6. *For* $r,s \neq 0$, *we have* $\bar{B}(a,r) \cap \bar{B}(b,s) \neq \phi$ *if and only if* $\bar{B}(a,r) \subset \bar{B}(b,s)$ *or* $\bar{B}(a,r) \supset \bar{B}(b,s)$.

Definition 4. *Two metrics* d_1 *and* d_2 *on a set* X *are equivalent if a sequence is Cauchy with respect to* d_1 *if and only if it is Cauchy with respect to* d_2. *Two norms are equivalent if they induce equivalent metrics.*

Remark 2. *In the definition of* $|.|_p$ *instead of* $(1/p)^{ord_p x}$ *we could have written* $\rho^{ord_p x}$ *with any* $\rho \in (0,1)$. *We would have obtained an equivalent non-Archimedean norm.*

Exercise 1.1. *Prove the remark .*

A nice property for the choice of $|.|_p = (1/p)^{ord_p x}$ *is given in the following exercise.*

Exercise 1.2. *Let* X *be a nonzero rational number. Prove that*

$$\prod_{p \leq \infty} |x|_p = 1$$

where $p \leq \infty$ *means that we take the product over all of the primes of* Q, *including the "prime at infinity".*

In the following theorem, by the trivial norm we mean the norm $\|.\|$ *on* Q *such that* $\|0\| = 0$ *and* $\|x\| = 1$ *for* $x \neq 0$

Theorem 1.3. *(Ostrowski). Every nontrivial norm* $\|.\|$ *on* Q *is equivalent to* $|.|_p$ *for some prime* p *or for* $p = \infty$.

Building Q_p

Let p *be a prime number. Define*

$$\mathcal{C} = \mathcal{C}_p(\mathbb{Q}) = \{(x_n) : (x_n) \text{ is a Cauchy sequence w.r.t } |.|_p\}$$

Proposition 4. *Defining*

$$(x_n) + (y_n) = (x_n + y_n)$$
$$(x_n)(y_n) = (x_n y_n)$$

makes $\mathcal{C}$ *a commutative ring with unity.*

Exercise 1.3. *Check that the sum and product of two Cauchy sequences, as defined above, are also Cauchy sequences.*

Exercise 1.4. *Find two non-zero Cauchy sequences (w.r.t $|.|_p$) whose product is the zero sequence.*

We should check that this ring contains $\mathbb{Q}$, since we want to construct something that extends Q. In fact all we need to do is notice that if $x \in \mathbb{Q}$, the sequence

$$x, x, x, ...$$

is certainly Cauchy. We denote this sequence by (x).

Lemma 1.4. *The map $x \to (x)$ is an inclusion of $\mathbb{Q}$ into $\mathcal{C}$*

We want that different Cauchy sequences in $\mathbb{Q}$ whose terms get close to each other have the same limit, but they are different objects in $\mathcal{C}$. So we must pass to a quotient of $\mathcal{C}$.

Definition 5. *We define $\mathcal{N} \subset \mathcal{C}$ to be the ideal*

$$\mathcal{N} = \Big\{ (x_n) : |x_n|_p \to 0 \Big\}$$

of sequences that tend to zero with respect to the absolute value $|.|_p$.

Lemma 1.5. *$\mathcal{N}$ is a maximal ideal of $\mathcal{C}$.*

Since taking a quotient by a maximal idea gives a field, we can give the next definition:

Definition 6. *We define the field of p-adic numbers to be the quotient of the ring $\mathcal{C}$ by its maximal ideal $\mathcal{N}$:*

$$\mathbb{Q}_p = \mathcal{C}/\mathcal{N}$$

In order to extend the $|.|_p$ to $\mathbb{Q}_p$ we need the next lemma:

Lemma 1.6. *Let $(x_n) \in \mathcal{C}$. If $(x_n) \notin \mathcal{N}$,then $\exists n_0$ such that for any $n, m \geq n_0$, $|x_n|_p = |x_m|_p$(The sequence is eventually stationary).*

Now we can define the p-adic norm on $\mathbb{Q}_p$.

Definition 7. *Let $x \in \mathbb{Q}_p$ and (x_n) any Cauchy sequence representing it. We define*

$$|x|_p = lim_{n\to\infty}|x_n|_p$$

.

By the lemma above, we know that this limit exists, but we still need to check that it is well-defined, i.e, if we take another sequence representing x we will get the same limit.

Exercise 1.5. *Show that $|x|_p$, as defined above, does not depend on the choice of the sequence (x_n) representing x.*

Exercise 1.6. *show that for $x \in \mathbb{Q}_p$ $|x|_p = 0$ if and only if $x = 0$.*

Exercise 1.7. *show that $|.|_p$ is non-Archimedean over $\mathbb{Q}_p$*

Remark 3. *For $x \in Q$ the definitions of $|x|_p$ in $\mathbb{Q}$ and $\mathbb{Q}_p$ are consistent. We have indeed an absolute value on $\mathbb{Q}_p$ which extends the p-adic absolute value on $\mathbb{Q}$.*

Proposition 5. *The image of $\mathbb{Q}$ under the inclusion $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$ is a dense subset of $\mathbb{Q}$.*

Proposition 6. *$\mathbb{Q}_p$ is complete w.r.t $|.|_p$.*

Remark 4. *$\mathbb{Q}_p$ is not algebraically closed and if we take the algebraic closure of $\mathbb{Q}_p$, it will not be complete. So we must complete this new field to get an algebraically closed and complete extension of $\mathbb{Q}$.*

Now that we have constructed Q_p, it would be a good idea to give a more concrete description of $\mathbb{Q}_p$.

Definition 8. *The ring of p-adic integers is*

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : \ \ |x|_p \leq 1\}$$

Proposition 7. *For any $x \in \mathbb{Z}_p$, there exists a unique Cauchy sequence a_n converging to x of the following type:*

- $a_n \in \mathbb{Z}$ *and* $0 \leq a_n \leq p^{n-1}$
- $a_n \equiv a_{n-1}$ *(mod p^{n-1})*

Let $x \in \mathbb{Z}_p$ and (a_n) be the Cauchy sequence described in the above theorem. We write them in base p. Let $a_0 = b_0$ for some $0 \leq b_0 \leq p-1$. Since $a_1 \equiv a_0$ (mod p), $a_1 = b_0 + b_1 p$ for some $0 \leq b_1 \leq p-1$. Going up the sequence, we get

$$\begin{aligned} a_0 &= b_0 & 0 &\leq b_0 \leq p-1 \\ a_1 &= b_0 + b_1 p & 0 &\leq b_1 \leq p-1 \\ a_2 &= b_0 + b_1 p + b_2 p^2 & 0 &\leq b_2 \leq p-1 \\ a_3 &= b_0 + b_1 p + b_2 p^2 + b_3 p^3 & 0 &\leq b_3 \leq p-1 \end{aligned}$$

Lemma 1.7. *Given any $x \in \mathbb{Z}_p$, the series*

$$b_0 + b_1 p + ... + b_n p^n + ...$$

obtained as above, converges to x.

Proof. The only thing that we need to notice is that a series converges to x if the sequence of its partial sums converges to x. But the partial sums of the above series are the a_n, which we already know converge to x.

So we have this equality in $\mathbb{Q}_p$

$$x = b_0 + b_1 p + ... + b_n p^n + ...$$

.

Now we need to get all of $\mathbb{Q}_p$. Let $x \in \mathbb{Q}_p$ and $|x|_p = p^m \geq 1$. Let $z = p^m x$, then $|z| = |p^m||x| = p^{-m}p^m = 1$. So we can write $x = z/p^m$ where $z \in \mathbb{Z}_p$. We get the following corollary

Corollary 1.8. *Every $x \in \mathbb{Q}_p$ can be written in the form*

$$\begin{aligned} x &= b_{-n_0}p^{-n_0} + ... + b_0 + b_1 p + +... + b_n p^n + ... \\ &= \sum_{n \geq -n_0} b_n p^n \end{aligned}$$

with $0 \leq b_n \leq p - 1$

The following theorem known as "Hensel's lemma" is one of the most important algebraic property of $\mathbb{Q}_p$. Basically, it says that often one can decide whether a polynomial has roots in $\mathbb{Z}_p$ by finding an approximate root of the polynomial.

Theorem 1.9. ***(Hensel's Lemma)*** *Let $F(x) = a_0 + a_1 x + ... + a_n x^n$ be a polynomial in $\mathbb{Z}_p[x]$. Suppose that there exits a p-adic integer $\alpha_1 \in \mathbb{Z}_p$ such that*

$$F(\alpha_1) = 0 \qquad (\textit{mod } p\mathbb{Z}_p)$$

and

$$F'(\alpha_1) \neq 0 \qquad (\textit{mod } p\mathbb{Z}_p)$$

where $F'(x)$ is the formal derivative of $F(x)$. Then there exits a p-adic integer $\alpha \in \mathbb{Z}_p$ such that $F(\alpha) = 0$ and $\alpha \equiv \alpha_1$ (mod $p\mathbb{Z}_p$)

Proof. We will construct a sequence of integers $\alpha_1, \alpha_2, ..., \alpha_n, ...$ such that, for all $n \geq 1$ we have

i) $F(\alpha_n) \equiv 0 \pmod{p^n}$

ii) $\alpha_n \equiv \alpha_{n+1} \pmod{p^n}$

Since $|\alpha_{n+1} - \alpha_n|_p \leq p^{-n}$, the sequence (α_n) is Cauchy. Its limit α satisfies $F(\alpha) = 0$(by continuity) and $\alpha \equiv \alpha_1$ (mod p)(since $\alpha_n \equiv \alpha_1$ (mod p) for all $n \geq 1$). So once we construct α_n the theorem will be proved.

Now we will construct α_2. Since $\alpha_2 \equiv \alpha_1 \pmod{p^n}$, we should have $\alpha_2 = \alpha_1 + b_1 p$ for some $b_1 \in \mathbb{Z}_p$. We have

$$\begin{aligned} F(\alpha_2) &= F(\alpha_1 + b_1 p) \\ &= F(\alpha_1) + F'(\alpha_1) b_1 p + \text{terms in } p^n \text{ with } n \geq 2 \\ &\equiv F(\alpha_1) + F'(\alpha_1) b_1 p \pmod{p^2} \end{aligned}$$

We can find α_2 if and only if we can find b_1 such that

$$F(\alpha_1) + F'(\alpha_1) b_1 p \equiv 0 \pmod{p^2}$$

We know that $F(\alpha_1) \equiv 0 \pmod{p}$, so $F(\alpha) = px$ for some x. We get

$$px + F'(\alpha_1) b_1 p \equiv 0 \pmod{p^2}$$

which gives

$$x + F'(\alpha_1) b_1 \equiv 0 \pmod{p}$$

Notice that $F'(\alpha_1)$ is not divisible by p. So it is invertible in $\mathbb{Z}_p$. Let

$$b_1 = -x(F'(\alpha_1))^{-1} \pmod{p}$$

.

In fact, we can choose such a $b_1 \in \mathbb{Z}$ such that $0 \leq b_1 \leq p - 1$. Then $\alpha_2 = \alpha_1 + b_1 p$ will satisfy the properties we want.

We can show that the same calculation works to get α_{n+1} from α_n. Hence we can construct the whole sequence.

2 Lattices Of $\mathbb{Q}_p^2$ And the Bruhat-Tits Tree

In this section we study the structures of lattices of $\mathbb{Q}_p^2$ and we explore some properties of Bruhat-Tits tree.

We start by making an observation. Let $x \in \mathbb{Z}_p$. We know that we can write $x = b_0 + b_1 p + b_2 p^2 + ...$ for some $b_i \in \mathbb{F}_p$. We define

$$\begin{aligned} \varphi : \mathbb{Z}_p &\to \mathbb{F}_p \\ x &\longmapsto b_0 \end{aligned}$$

Since

$$\begin{aligned} \varphi(x) = 0 &\Leftrightarrow b_0 = 0 \\ &\Leftrightarrow x \in p\mathbb{Z}_p \end{aligned}$$

$ker(\varphi) = p\mathbb{Z}_p$ and hence $\mathbb{Z}_p / p\mathbb{Z}_p \cong \mathbb{F}_p$.

Definition 9. *We call a subset $\mathcal{L} \subseteq \mathbb{Q}_p^2$ a lattice if $\mathcal{L}$ is a rank 2 free $\mathbb{Z}_p$-module of $\mathbb{Q}_p^2$. Equivalently $\mathcal{L}$ is a lattice of $\mathbb{Q}_p^2$ if there exists 2 independent vectors $v_1, v_2 \in \mathbb{Q}_p^2$ such that*

$$\begin{aligned} \mathcal{L} &= \mathbb{Z}_p v_1 + \mathbb{Z}_p v_2 \\ &= \{x v_1 + y v_2 | x, y \in \mathbb{Z}_p\} \end{aligned}$$

.

$\mathcal{L}_0 = \mathbb{Z}^2 = \mathbb{Z}_p(1,0) + \mathbb{Z}(0,1)$ and $\mathcal{L} = \mathbb{Z}_p(p^a, 0) + \mathbb{Z}(0, p^b)$, $a, b \in \mathbb{Z}$ are examples of lattices of $\mathbb{Q}_p^2$.

In the next discussion we fix a lattice $\mathcal{L} = \mathbb{Z}_p v_1 + \mathbb{Z}_p v_2$ and we want to characterize all lattices $\mathcal{L}'$ such that $\mathcal{L} \supseteq \mathcal{L}' \supseteq p\mathcal{L}$.

Define

$$\begin{aligned} \varphi' : \mathcal{L} &\to \mathbb{F}_p^2 \\ x v_1 + y v_2 &\longmapsto (\varphi(x), \varphi(y)) \end{aligned}$$

where φ is the map defined above.

As before we can verify that $ker(\varphi') = p\mathcal{L}$ and so we have $\mathcal{L}/p\mathcal{L} \cong \mathbb{F}_p^2$. Hence for $\mathcal{L}'$ such that $\mathcal{L} \supseteq \mathcal{L}' \supseteq p\mathcal{L}$, $\mathcal{L}'/p\mathcal{L}$ is a subspace of $\mathbb{F}_p^2$.

The extreme cases are when

$$\mathcal{L}'/p\mathcal{L} = \begin{cases} 0 & \textit{iff} \quad \mathcal{L}' = p\mathcal{L} \\ \mathbb{F}_p^2 & \textit{iff} \quad \mathcal{L}' = \mathcal{L} \end{cases}$$

The other cases are when $\mathcal{L}'/p\mathcal{L}$ is isomorphic to a one dimensional subspace of $\mathbb{F}_p^2$. There are $p+1$ such subspaces and they are generated by $(a, 1)$, $a \in \mathbb{F}_p$ and $(1, 0)$.

2.1 Bruhat-Tits Tree

We want to construct a graph whose vertices are the equivalence classes of the lattices of $\mathbb{Q}_p^2$. We define an equivalence relation on the set of lattices of $\mathbb{Q}_p$ such that

$$\mathcal{L} \sim \mathcal{L}' \Leftrightarrow \mathcal{L}' = \lambda\mathcal{L} \quad \textit{for some } \lambda \in \mathbb{Q}_p^\times$$

The Bruhat-Tits tree is the graph T, with vertices $[\mathcal{L}]$, where $[\mathcal{L}]$ is the equivalent class of some lattice $\mathcal{L}$ of $\mathbb{Q}_p^2$. There is an edge between two vertices v_1 and v_2 of T if and only if

$$\begin{aligned} \exists\mathcal{L} \quad & \textit{s.t } v_1 = [\mathcal{L}] \\ \exists\mathcal{L}' \quad & \textit{s.t } v_2 = [\mathcal{L}'] \end{aligned}$$

and

$$\mathcal{L} \supset \ \mathcal{L}' \subset p\mathcal{L}$$

We notice that since $\mathcal{L} \supset \ \mathcal{L}' \supset \ p\mathcal{L}$ implies $\mathcal{L}' \supset \ p\mathcal{L} \supset \ p\mathcal{L}'$, T is actually an undirected graph.

Now consider the group

$$GL_2(\mathbb{Q}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Q}_p, ad - bc \neq 0 \right\}$$

.

Define the following action of $Gl_2(\mathbb{Q}_p)$ on the set of vertices of T: for any $M \in \mathbb{Q}_p$ let

$$f_M : \quad v = [\mathcal{L}] \longmapsto [M\mathcal{L}]$$

where

$$\begin{aligned} M\mathcal{L} &= \{Ml : l \in \mathcal{L}\} \\ &= span\, \langle Mv_1, Mv_2 \rangle \end{aligned}$$

for $\mathcal{L} = span_{\mathbb{Z}_p} \langle v_1, v_2 \rangle$.
Since $M\lambda.\mathcal{L} = \lambda.M\mathcal{L}$

$$\mathcal{L} \sim \mathcal{L}' \Leftrightarrow M\mathcal{L} \sim M\mathcal{L}'.$$

Hence the action f_M *is well defined.*

Now we consider one specific vertex of T, $[\mathcal{L}_0]$ *the equivalence class of lattice* $\mathcal{L}_0 = \mathbb{Z}_p(1,0) + \mathbb{Z}_p(0,1)$ *of* T. *By the discussion that we had before, we already know all* $\mathcal{L}'$ *such that* $\mathcal{L}_0 \supset \mathcal{L}' \supset p\mathcal{L}_0$. *So the neighbors of* $\mathcal{L}_0$ *are*

$$\begin{cases} p\mathcal{L}_0 + \mathbb{Z}_p(a,1) & a \in \mathbb{F}_p \\ p\mathcal{L}_0 + \mathbb{Z}_p(1,0) \end{cases}$$

The only thing that we need to verify is that these formulations actually give us lattices of $\mathbb{Q}_p^2$. *For* $a \in \mathbb{F}_p$,

$$\begin{aligned} p\mathcal{L}_0 + \mathbb{Z}_p(a,1) &= span_{\mathbb{Z}_p} \langle (p,0), (0,p), (a,1) \rangle \\ &= span_{\mathbb{Z}_p} \langle (p,0), (a,1) \rangle \end{aligned}$$

since $(0,p) = p(a,1) - a(p,0)$. *In the case* $p\mathcal{L}_0 + \mathbb{Z}_p(1,0)$ *we have*

$$p\mathcal{L}_0 + \mathbb{Z}_p(1,0) = span_{\mathbb{Z}_p} \langle (1,0), (0,p) \rangle$$

$Gl_2(\mathbb{Q}_p)$ *also acts on edges of* T, *since if* $\mathcal{L} \supset \mathcal{L}' \supset p\mathcal{L}$, *then* $M\mathcal{L} \supset M\mathcal{L}' \supset Mp\mathcal{L} = pM\mathcal{L}$. *Hence if* $[\mathcal{L}]$ *and* $[\mathcal{L}']$ *are neighbors,* $[M\mathcal{L}]$ *and* $[M\mathcal{L}']$ *are also neighbors.*

Claim 1. $Gl_2(\mathbb{Q}_p)$ *acts transitively on the vertices of* T.

To see that let $\mathcal{L} = \mathbb{Z}_p v_1 + \mathbb{Z}_p v_2$ *be an arbitrary lattice of* $\mathbb{Q}_p^2$. *Since* v_1, v_2 *are independent over* $\mathbb{Q}_p$, $M = (v_1|v_2) \in Gl_2(\mathbb{Q}_p)$ *and we have* $M\mathcal{L}_0 = \mathcal{L}$. *Since* M *is invertible, we can also write* $\mathcal{L}_0 = M^{-1}\mathcal{L}$. *Now take any two lattices* $\mathcal{L}_1$, $\mathcal{L}_2$. $\exists M_1, M_2 \in Gl_2(\mathbb{Q}_p)$ *such that* $M_1\mathcal{L}_0 = \mathcal{L}_1$ *and* $M_2^{-1}\mathcal{L}_2 = \mathcal{L}_0$. *Hence* $M_1 M_2^{-1}\mathcal{L}_2 = \mathcal{L}_1$

$$\mathcal{L}_2 \overset{f_{M_2^{-1}}}{\to} \mathcal{L}_0 \overset{f_{M_1}}{\to} \mathcal{L}_1$$

We have seen that $\mathcal{L}_0$ *is of degree* $p+1$. *Since* $Gl_2(\mathbb{Q}_p)$ *acts transitively on vertices and also acts on edges, all other vertices in* T *are of degree* $p+1$*(*T *is* $(p+1)$ *regular).*

We need the following theorem in order to verify the next fact about the set of vertices of T.

Theorem 2.1. *Let a group G act transitively on a set S. Then there is a bijection between S and the quotient group $G/Stab_G(s_0)$ for any $s_0 \in S$, given by*

$$f : G/Stab_G(s_0) \to S$$
$$gStab_G(s_0) \longmapsto g.s_0$$

It is not hard to see that $stab_{Gl_2(\mathbb{Q}_p)}(\mathcal{L}_0) = Gl_2(\mathbb{Z}_p)$. By the previous theorem

$$\{\textit{vertices of } T\} \overset{bijection}{\leftrightarrow} Gl_2(\mathbb{Q}_p)/Gl_2(\mathbb{Z}_p)$$

Since $GL_2(\mathbb{Z}_p)$ fixes $\mathcal{L}_0$, it acts on the edges incident to $\mathcal{L}_0$. This action is also transitive: let $\mathcal{L} = span_{\mathbb{Z}_p} \langle (p,0),(a,1)\rangle$ and $\mathcal{L}' = span_{\mathbb{Z}_p} \langle (p,0),(b,1)\rangle$ be two arbitrary neighbors of $\mathcal{L}_0$. Then we can verify that $\begin{pmatrix} 1 & b-a \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{Z}_p)$ takes $\mathcal{L}$ to $\mathcal{L}'$.

Using this and the fact that $GL_2(\mathbb{Q}_p)$ acts transitively on the vertices of T, we can also show that $GL_2(\mathbb{Q}_p)$ acts transitively on the edges of T. In fact, we only need to show that we can take any edge in T to any edge incident to $\mathcal{L}_0$.

Now we try to find the stabilizer of the edge $e : [\mathcal{L}_0] - [\mathcal{L}_0']$ where $\mathcal{L}_0' = span_{\mathbb{Z}_p} \langle (1,0),(0,p)\rangle$. Let

$$B = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_p) : c \in p\mathbb{Z}_p \right\}$$

.

Since $B \subset GL_2(\mathbb{Z}_p)$, it fixes $\mathcal{L}_0$. It is not hard to see that B also fixes $\mathcal{L}_0'$. Hence

$$B \subset \textit{Stab}(e)$$

There is another possibility for $M \in$ stab(e). M can switch $\mathcal{L}_0$ and $\mathcal{L}_0'$. All such M belongs to $\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} B$, since $\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$ switches $\mathcal{L}_0$ and $\mathcal{L}_0'$:

$$\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \mathcal{L}_0 = \mathcal{L}_0'$$
$$\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \mathcal{L}_0' = p\mathcal{L}_0 \sim \mathcal{L}_0$$

Hence stab(e)=$B \cup \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} B = B'$. By using the theorem 2.1 again we get

$$\{\textit{edges of } T\} \overset{bijection}{\leftrightarrow} GL_2(\mathbb{Q}_p)/B'.$$