Lecture 7
Direct and inverse estimates and Mesh refinement

§1 Direct (Jackson) estimates

Let Ω be star-shaped with respect to a ball, $h = \operatorname{diam}\Omega$ and let $u \in W^{m,p}(\Omega)$. We have shown

$$\|u - Q_m u\|_{L^\infty(\Omega)} \le Ch^{m-\frac{n}{p}}|u|_{W^{m,p}(\Omega)}, \quad (m > n, p),$$

and

$$\|u - Q_m u\|_{W^{k,p}(\Omega)} \le Ch^{m-k}|u|_{W^{m,p}(\Omega)}, \quad (m \ge k \ge 0),$$

where $Q_m u \in \mathbb{P}_{m-1}$ is the averaged Taylor polynomial and $C = C(n, m, \gamma)$, γ being the chunkiness parameter. For the Lagrange finite elements of order d, on a simplex τ,

$$I_\tau v = \sum_{z \in \mathcal{N}_\tau} v(z)\phi_z \quad v \in C(\overline{\tau}),$$

We have obtained the estimate

$$\|u - I_\tau u\|_{W^{k,p}(\tau)} \le C\left(h^{d-k}|u|_{d,p} + \cdots + h^{m-k}|u|_{m,p}\right) \le Ch^{d-k}\|u\|_{W^{m,p}(\tau)},$$

where typically $k \le d \le m$.

Remark. We would expect $W^{d,p}$ instead of $W^{m,p}$ in the right hand-side of the last estimate above, however, we need $m > n/p$ in order for $I_\tau v$ to be well defined. Recall that the argument depended on $\|\phi_z\|_{W^{k,p}(\tau)} \le Ch^{-k+n/p}$.

For a conforming partition P, we defined $I_P u \in S_P$ by by $(I_P u)|_\tau = I_\tau u$ for $\tau \in P$, where $S_P = S^d(P)$. In other words,

$$I_P u = \sum_{z \in \mathcal{N}_P} u(z)\phi_z,$$

now with ϕ_z the globally defined nodal basis function. Assume that $S_P \subset W^{k,p}$, i.e $k = 0, 1$ (we cannot achieve high k since regularity is restricted along interfaces for piecewise polynomials). Then

$$\|u - I_P u\|_W \le C\left(h^{d-k}|u|_{d,p} + \cdots + h^{m-k}|u|_{m,p}\right), \tag{7.1}$$

where $h = \max_{\tau \in P} h_\tau$, $h_\tau = \operatorname{diam}\tau$ and $C = C(n, m, \gamma)$; $(\gamma = \max_{\tau \in P} \gamma_\tau)$. Notice that we have to choose γ as such so as to ensure that constant C remains bounded; geometrically, this requires that the collection P has no long and thin triangles.

Estimate (7.1) implies for $u \in W^{m,p}$,

$$\inf_{v \in S_P} \|u - v\|_{W^{k,p}(\Omega)} \leq Ch^{d-k} \|u\|_{W^{m,p}(\Omega)}, \quad (m > n/p).$$

Remark. This kind of estimate is called a *direct estimate* or *Jackson-type estimate*.

§2 Mesh refinement

For good accuracy, we want h to be small and γ not be too large. A *refinement rule* is a procedure that transforms a partition P into a new partition P' with smaller h.

Example 5 (*Red refinement, loops, RSS*). Notice that $h \mapsto h/2$ and γ remains constant. ◇

Example 6 (*Newest vertex bisection, standard subdivision*). Notice that $h \mapsto \lambda h$, with $\lambda < 1$, and γ remains bounded (shape is regular). In order to avoid hanging nodes, initial vertices must be chosen carefully in order to ensure conformity. ◇

Remark. Note that the first example is necessarily a global procedure whereas the second can perform local refinement.

A model for (uniform) refinement algorithm runs as follows:

§3 Inverse (Bernstein) estimates

Because S_P is finite-dimensional, For $v \in S_P$, we have $\|v\|_{k,p} \leq C\|v\|_{m,p}$. We want to keep track of the dependence of constant C on h. Recall that $\tau = A(\sigma)$, where σ is the standard simplex and $A : \sigma \to \tau$ is an affine bijection.

Let $\widehat{v} = v \circ A$ and consider the following technical lemma.

Lemma 7.1. *Let A be a matrix of constant coefficients, let $v : \mathbb{R}^n \to \mathbb{R}$ and let $x \in \mathbb{R}^n$. Then*

$$\nabla(v \circ A) = A^T(\nabla v) \circ A.$$

Proof. We are interested in the derivatives of $v \circ A$. We may write

$$\partial_j(v \circ A)(x) = \partial_j v(Ax) = \sum_{k=1}^{n} \frac{\partial v}{\partial y_k} \frac{\partial y_k}{\partial x_j}, \quad y = Ax.$$

For $k = 1, ..., n$,

$$\frac{\partial y_k}{\partial x_j} = \partial_j (Ax)_k = \sum_{i=1}^{n} A_{ki} \partial_j x_i = A_{kj},$$

meanwhile $\frac{\partial v}{\partial y_k} = \partial_k v \circ A$. Summing up,

$$\partial_j (v \circ A) = \sum_{k=1}^{n} (\partial_k v \circ A) A_{kj} = \left(A^T \nabla v \circ A \right)_j ,$$

corresponding to the jth entry of said gradient vector. $\square$

In order to obtain estimates on $|\nabla v|$ and $|\nabla \widehat{v}|$, (notice that $\nabla v \circ A \in \mathbb{R}^n$),

$$|\nabla \widehat{v}| \leq \|A\| |\nabla v| \quad \text{and} \quad |\nabla v| \leq \|A^{-1}\| |\nabla \widehat{v}|.$$

To bound the norms of A and its inverse, look at the behaviour of A. $A : \sigma \to \tau$ is affine, meaning, A sends edges to edges, so the maximum scaling A performs here is determined by the vertex of σ whose image under A is the longest vertex belonging to τ. In precise,

$$\|A\| \leq \frac{\operatorname{diam} \tau}{C} = \quad \text{and} \quad \|A^{-1}\| \leq \frac{C}{\gamma_\tau \cdot \operatorname{diam} \tau}.$$

Remark. Here, $C = 1$ since the vertices of a standard simplex is always 1.

Hence we have $\|A\| \leq c h_\tau$ and $\|A^{-1}\| \leq C(\gamma_\tau) h_\tau^{-1}$. We have $|D^k v| \leq C h_\tau^{-k} |D^k \widehat{v}|$ and $|D^k \widehat{v}| \leq C h_\tau^k |D^k v|$. Note also that

$$\int_\tau v = |\det A| \int_\sigma \widehat{v}, \quad |\det A| = c \operatorname{Vol} \tau \sim h_\tau^n.$$

Remark. The constant c corresponds to the number of standard simplices required to form an n-dimensional cube; i.e. $c = n!$.

Let $p, q \geq 1$ and let $k \geq m$. For $v \in \mathbb{P}_{d-1}$, evidently

$$|v|_{W^{k,p}}^p \leq \|D^k v\|_{L^p}^p \leq C h_\tau^n h_\tau^{-kp} \|D^k v\|_{L^p}^p(\sigma),$$

because $|D^k v| \leq C h_\tau^{-k} |D^k \widehat{v}|$. Moreover,

$$\|D^k \widehat{v}\|_{L^p(\sigma)} \leq C(p, q, \sigma) \|D^k \widehat{v}\|_{L^q(\sigma)},$$

since $\|\cdot\|_{L^p} \simeq \|\cdot\|_{L^q}$ with $p, q \geq 1$ over $\mathbb{P}_{d-1}$ (recall $\dim \mathbb{P}_{d-1} < \infty$). Similarly, $\|\cdot\|_{W^{k-m,q}} \simeq \|\cdot\|_{L^q}$, so we may write $\|D^{k-m} D^m \widehat{v}\|_{L^q(\sigma)} \leq C(q, k-m, \sigma) \|D^m \widehat{v}\|_{L^q(\sigma)}$. Summing up,

$$|v|_{W^{k,p}(\tau)}^p \leq C h_\tau^{n-kp} \|D^m \widehat{v}\|_{L^q(\sigma)}^p \leq C h^{n-kp} \left(h_\tau^{-n+mq} \int_\tau |D^m v|^q \right)^{p/q},$$

where the last inequality follows from $|D^k\widehat{v}| \leq Ch_\tau^k|D^kv|$. The constants depend only on γ, σ, p, q, m and k. We conclude that

$$|v|_{W^{k,p}(\tau)} \leq Ch^{\frac{n}{p}-\frac{n}{q}+m-k}|v|_{W^{m,q}(\tau)}, \tag{7.2}$$

for some positive constant C depending on γ.

We wish to make estimate (7.2) global. Let $v \in L^\infty(\Omega)$ such that $v|_\tau \in \mathbb{P}_{d-1}$; we do not presume global continuity. For $p = \infty$ with $k \geq m - \frac{n}{q}$,

$$|v|_{W^{k,\infty}(\Omega)} \leq \max_{\tau\in P} Ch_\tau^{m-\frac{n}{q}-k}|v|_{W^{m,q}(\tau)} \leq Ch_{min}^{m-\frac{n}{q}-k}\left(\sum_{\tau\in P}|v|^q_{W^{m,q}(\tau)}\right)^{1/q},$$

so if $p \geq q$ with $k = m$, from (7.2) we obtain

$$\left(\sum_{\tau\in P}|v|^p_{W^{k,p}(\tau)}\right)^{1/p} \leq Ch_{min}^{\frac{n}{p}-\frac{n}{q}}\left(\sum_{\tau\in P}|v|^p_{W^{k,q}(\tau)}\right)^{1/p} \leq Ch_{min}^{\frac{n}{p}-\frac{n}{q}}\left(\sum_{\tau\in P}|v|^q_{W^{k,q}(\tau)}\right)^{1/q}.$$

In particular, for $k = m = 0$, we obtain the estimate

$$\|v\|_{L^p(\Omega)} \leq Ch_{min}^{\frac{n}{p}-\frac{n}{q}}\|v\|_{L^q(\Omega)}, \quad (1 \leq q \leq p \leq \infty). \tag{7.3}$$

Again, for the case $p < q$ with $k \geq m$, there exists a real positive number r satisfying $1/p = 1/r + 1/q$ and

$$\begin{aligned}\left(\sum_{\tau\in P}|v|^p_{W^{k,p}(\tau)}\right)^{1/p} &\leq Ch_{min}^{m-k}\left(\sum_{\tau\in P}h_\tau^{\left(\frac{n}{p}-\frac{n}{q}\right)p}|v|^p_{W^{m,q}(\tau)}\right)^{1/p}\\ &\leq Ch_{min}^{m-k}\left(\sum_{\tau\in P}h_\tau^{\left(\frac{n}{p}-\frac{n}{q}\right)r}\right)^{1/r}\left(\sum_{\tau\in P}|v|^q_{W^{k,q}(\tau)}\right)^{1/q},\end{aligned}$$

where the second inequality is true due to Holder. Notice that the quantity

$$\sum_{\tau\in P}h_\tau^{\left(\frac{n}{p}-\frac{n}{q}\right)r} = \sum_{\tau\in P}h_\tau^n \leq c\,\mathrm{Vol}\,\Omega.$$

Remark. If $S_P \subset W^{k,p}(\Omega) \cap W^{m,q}(\Omega)$ then,

$$\left(\sum_{\tau\in P}|v|^p_{W^{k,p}(\tau)}\right)^{1/p} \equiv |v|_{W^{k,p}(\Omega)}, \quad v \in S_P.$$