Lecture 6
Integration error for Lagrange finite elements

§1 Bramble-Hilbert lemma

We will first prove the *Sobolev lemma.*

Theorem 6.1 (*Sobolev lemma*)**.** *Let Ω be a finite union of star-shaped domains. Then $W^{m,p}(\Omega) \hookrightarrow C(\Omega)$ if $m > \frac{n}{p}$, $m = 1$ is allowed if $p = 1$.*

Proof. Let $u \in W^{m,p}(\Omega)$. Then there exists a sequence $\{u_k\} \subset C^\infty(\Omega)$ such that $u_k \to u$ in $W^{m,p}$.

$$\begin{aligned}\|u_j - u_k\|_\infty &\le \|Q_m(u_j - u_k)\|_\infty + \|R_m(u_j - u_k)\|_\infty \\ &\le C\|u_j - u_k\|_{L^1} + C|u_j - u_k|_{W^{m,p}} \to 0 \text{ as } k, j \to \infty.\end{aligned}$$

It therefore follows that $\{u_j\}$ is Cauchy in L^∞ and there exists $w \in C(\overline{\Omega})$ such that $u_k \to w$ in L^∞; but $u_j \to u$ in L^1 so $u = w$ almost everywhere. □

We claim that $\partial^\beta Q_m u = Q_{m-|\beta|}\partial^\beta u$. To see this look at derivatives of

$$T_y^m u(x) = \sum_{|\alpha|<m} \frac{(x-y)^\alpha}{\alpha!}\partial^\alpha u(y).$$

We have $\partial_x^\beta (x-y)^\alpha = \frac{\alpha!}{(\alpha-\beta)!}(x-y)^{\alpha-\beta}$,

$$\partial^\beta T_y^m u(x) = \sum_{|\alpha|<m,\, \alpha\ge\beta} \frac{(x-y)^{\alpha-\beta}}{(\alpha-\beta)!}\partial^{\alpha-\beta}\partial^\beta u(y) = \sum_{|\gamma|<m-|\beta|} \frac{(x-y)^\gamma}{\gamma!}\partial^\gamma\partial^\beta u(y).$$

We now prove *Young's inequality* using duality.

$$\begin{aligned}\langle u * v, \varphi\rangle &= \iint u(x-y)v(y)\varphi(x)\,dydx \\ &= \iint \tilde{u}(y-x)\varphi(x)v(y)\,dxdy \\ &= \langle \tilde{u} * \varphi, v\rangle\,,\end{aligned}$$

provided that $\langle|\tilde{u} * \varphi, |v|\rangle < \infty$. Note that $\tilde{u} = u(-x)$.

$$\langle|\tilde{u} * \varphi|, |v|\rangle \le \|\tilde{u} * \varphi\|_{L^\infty}\|v\|_{L^1} \le \|u\|_{L^p}\|\varphi\|_{L^q}\|v\|_{L^\infty},$$

which makes $|\langle u * v, \varphi\rangle| \le \|u\|_p\|\varphi\|_q\|v\|_\infty$. By duality

$$\|u * v\|_{L^p} \le \sup_{\varphi\in L^q} \frac{\langle u * v, \varphi\rangle}{\|\varphi\|_{L^q}} \le \|u\|_{L^p}\|v\|_{L^1}. \tag{6.1}$$

Remark. Young's inequality says for $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, $\|u * v\|_r \leq \|u\|_p \|v\|_q$.

Theorem 6.2 (*Bramble-Hilbert lemma*)**.** *Let* $u \in W^{m,p}(\Omega)$ *with* $0 \leq k \leq m$. *Then*

$$\inf_{v \in \mathbb{P}_{m-1}} \|u - v\|_{W^{k,p}(\Omega)} \leq C|h|^{m-k}|u|_{W^{m,p}(\Omega)}. \tag{6.2}$$

Proof. Let $u \in C^\infty(\Omega)$. Estimate (5.11) gives

$$\|R_m u\|_{L^p} \leq C\|I_{m,h}\|_{L^1}|u|_{W^{m,p}} \leq Ch^m |u|_{W^{m,p}},$$

and

$$\partial^\beta R_m u = \partial^\beta(u - Q_m u) = \partial^\beta u - Q_{m-|\beta|}\partial^\beta u = R_{m-|\beta|}\partial^\beta u.$$

We conclude for smooth u

$$|R_m u|_{W^{k,p}} \leq \max_{|\beta|=k} \|R_{m-k}\partial^\beta u\|_{L^p} \leq Ch^{m-k}|u|_{W^{m,p}}.$$

Let now $\{u_k\} \subset C^\infty(\Omega)$ be a sequence with $u_n \to u$ in $W^{m,p}$. By Lemma 5.3

$$\|Q_m u_j - Q_m u_\nu\|_{W^{k,p}} \leq C\|u_j - u_\nu\|_{L^1} \to 0,$$

so define $Q_m u = \lim_{j\to\infty} Q_m u_j$,

$$|u - Q_m u|_{W^{k,p}} \leq |u - u_j|_{W^{k,p}} + |u_j - Q_m u_j|_{k,p} + |Q_m u_j - Q_m u|_{W^{k,p}}.$$

By assumption, $|u - u_j|_{k,p}, |Q_m u_j - Q_m u|_{k,p} \to 0$ and by the above argument for smooth functions,

$$|u - Q_m u|_{W^{k,p}} \leq C' h^{m-k}|u_j|_{W^{m,p}} + o(1) \leq Ch^{m-k}|u|_{W^{m,p}} + o(1),$$

as $j \to \infty$. □

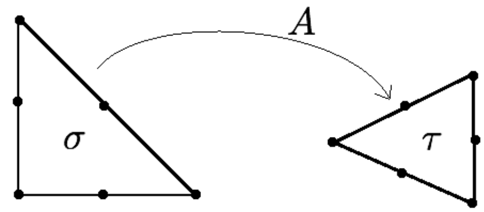
§2 Interpolation error for the Lagrange finite element

Let $\mathbb{P}_{d-1}$ be the shape function space defined on a finite element domain τ, τ being an n-simplex.

The standard simplex in $\mathbb{R}^n$ is given

$$\sigma = \left\{ x \in \mathbb{R}^{n-1} : \sum_{i=1}^{n-1} x_i = 1, x_i \geq 0 \right\}. \tag{6.3}$$

where its vertices are (conveniently) taken to be spanned by $\{0, e_1, ..., e_n\}$, e_i being the standard unit vector in $\mathbb{R}^n$. The dimension of $\mathbb{P}_{d-1} = \#\{\overline{\sigma} \cap \mathbb{Z}^n\}$. There exists an affine bijection $A : \sigma \to \tau$. Take



$$\mathcal{N} = \{\delta_z : z \in A(\overline{\sigma} \cap \mathbb{Z}^n)\},$$

evidently, $\#\mathcal{N} = \dim \mathbb{P}_{d-1}$. Let V_τ denote $A(\overline{\sigma} \cap \mathbb{Z}^n)$, the set of points which characterizes the shape functions; these determine the degrees of freedom. Recalling Definition 4.4, we have an equivalent definition for the local interpolation for a function u,

$$I_\tau u = \sum_{z \in V_\tau} u(z)\phi_z,$$

defined for $u \in W^{m,p}$ with $m > n/p$. We wish to determine an estimate on $\|u - I_\tau u\|_{W^{k,p}(\tau)}$:

$$\|u - I_\tau u\|_{k,p} \le \|u - Q_d u\|_{k,p} + \|Q_d u - I_\tau u\|_{k,p}.$$

Noting that $\|u - Q_d u\|_{k,p} \le Ch^{d-k}|u|_{d,p}$ by the Bamble-Hilbert lemma, we now study the behaviour of the second term. We first estimate $\|I_\tau\|_{W^{k,p}(\tau)}$:

$$\|I_\tau v\|_{k,p} \le \left(\sum_{z \in V_\tau} \|\phi_z\|_{k,p} \right) \|v\|_\infty.$$

Take $h = \operatorname{diam} \tau$, and consider a function $\widehat{\phi}$ whose domain τ is modified so that the modification is of size 1; i.e. $\phi_z(x) = \widehat{\phi}(x/h)$. From this, we conclude that $D^k \phi_z \sim h^{-k}$ which makes $\int_\tau |D^k \phi_z|^p \sim h^{n-kp}$ hence $\|\phi_z\|_{k,p} \le Ch^{n/p-k}$. It therefore follows that

$$\|I_\tau v\|_{k,p} \le Ch^{-k+\frac{n}{p}} \|v\|_\infty, \quad v \in W^{m,p}(\tau).$$

Noting that $I_\tau Q_d u = Q_d u$, we conclude that

$$\|Q_d u - I_\tau u\|_{k,p} = \|I_\tau(u - Q_d u)\|_{k,p} \le Ch^{-k+\frac{n}{p}} \|u - Q_d u\|_\infty.$$

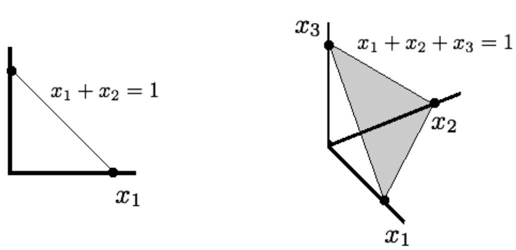


Figure 3: An example of a standard simplex in two and three dimensions.

We wish to bound the the term $\|u - Q_d u\|_\infty$, present in the last expression, we recall that $\|R_m u\|_\infty \le Ch^{m-\frac{n}{p}}|u|_{m,p}$, $0 \le k \le d \le m$ with d corresponding to the order of the finite element space and $m > n/p$ is chosen so as to ensure $u \in C$. Here, we do not wish to restrict ourself to finite element spaces of dimensions satisfying such restrictions as $d > n/p$ so as to ensure necessary regularity. For this, we consider the following argument.

$$\|u - Q_d u\|_\infty \le \|u - Q_m u\|_\infty + \|Q_m u - Q_d u\|_\infty \le Ch^{m-\frac{n}{p}}|u|_{m,p} + \|Q_m u - Q_d u\|_\infty,$$

but

$$Q_m u(x) - Q_d u(x) = \sum_{d \le |\alpha| < m} \int_{\mathcal{B}} \frac{\partial^\alpha u(y)}{\alpha!}(x-y)^\alpha \phi(y)\, dy,$$

therefore

$$\|Q_m u - Q_d u\|_\infty \le C \sum_{j=d}^{m-1} h^j |u|_{j,p} \|\phi\|_{L^q} \le C \sum_{j=d}^{m-1} h^{j+\frac{n}{q}-n} |u|_{j,p}.$$

We have

$$\begin{aligned}\|u - I_\tau u\|_{W^{k,p}(\tau)} &\le C \sum_{j=d}^{m} h^{j-\frac{n}{p}-k+\frac{n}{p}} |u|_{j,p} \\ &= C\left(h^{d-k}|u|_{d,p} + \cdots + h^{m-k}|u|_{m,p}\right) \le Ch^{d-k}\|u\|_{W^{m,p}(\tau)}.\end{aligned}$$

Remark. The previous implies that the accuracy/rate of convergence depends on the order of polynomial used.

Since

$$\|u - I_P u\|^p_{W^{k,p}(\Omega)} = \sum_{\tau \in P} \|u - I_\tau u\|^p_{W^{k,p}(\tau)}, \tag{6.4}$$

we conclude

$$\inf_{v \in S^d(P)} \|u - v\|_{W^{k,p}(\Omega)} \le Ch^{d-k}\|u\|_{W^{m,p}(\Omega)}, \tag{6.5}$$

where $h = \max_{\tau \in P} \operatorname{diam} \tau$.