Lecture 5
Polynomial approximation in Sobolev spaces

§1 Piecewise polynomial spaces

We define the spaces of piecewise polynomials with respect to a partition P

$$S^{m,r}(P) = \{u \in C^r(\Omega) : u|_\tau \in S(\tau)\}. \tag{5.1}$$

where $S(\tau) = \mathbb{P}_{m-1}$ or $S(\tau) = \mathbb{P}_{m-1} \times \cdots \times \mathbb{P}_{m-1}$. Here, r denotes the overall regularity of the basis elements. For example:

- $r = -1$ corresponds to *no* condition on regularity; discontinuous piecewise polynomials.
- $r = 0$ corresponds to the $\mathscr{C}^0$ Lagrange finite elements.
- $r = 1$ corresponds to $\mathscr{C}^1$ finite elements.
- $r = m - 2$ corresponds to splines.

Generally, for (PG) approximations,

$$\|u - u_P\|_{W^{k,p}(\Omega)} \leq C \inf_{v \in S^{m,r}(P)} \|u - v\|_{W^{k,p}(\Omega)}. \tag{5.2}$$

We start with only one element.

§2 Polynomial approximation in Sobolev spaces

Let $\Omega \subset \mathbb{R}^n$ be a (bounded) domain such a triangle, square or tetraherdron. We recall the multi-index notation: Let $\alpha \in \mathbb{N}_0$, $\alpha = (\alpha_1, ..., \alpha_n)$. We have the following:

- $\partial^\alpha u = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} u$,
- $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$,
- $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$,
- $\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!$.

Recall the Sobolev norm:

$$\|u\|_{W^{k,p}(\Omega)} = \max_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^p(\Omega)},$$

and the semi-norm:

$$|u|_{W^{k,p}(\Omega)} = \max_{|\alpha| = k} \|\partial^\alpha u\|_{L^p(\Omega)},$$

where

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : \|u\|_{W^{k,p}(\Omega)} < \infty\} \qquad (1 \le p \le \infty).$$

Remark. We may adopt the short hand notation $\|\cdot\|_{k,p}$ and $|\cdot|_{k,p}$.

We have an important density theorem.

Theorem 5.1 (Meyers-Serrin)**.** *The set* $\{u \in C^\infty : \|u\|_{k,p} < \infty\}$ *is dense in* $W^{k,p}(\Omega)$*.*

Recall the Taylor theorem for a function $f \in C^m[0,t]$,

$$f(t) = \sum_{k=0}^{m-1} \frac{f^{(k)}(s)}{k!} t^k + \int_0^t \frac{f^{(m)}(s)}{(m-1)!}(t-s)^{m-1}\, ds.$$

In the multi-dimensional setting, for $x, y \in \mathbb{R}^n$, define $f(s) = u(y + s(x-y))$ and obtain

$$f'(s) = [(x_1 - y_1)\partial_i + \cdots + (x_n - y_n)\partial_n]u(y + s(x-y)).$$

Successively,

$$f^{(k)}(s) = \sum_{|\alpha|=k} \frac{k!}{\alpha!}(x-y)^\alpha \partial^\alpha u(y + s(x-y)),$$

evaluated at $s = 1$ we obtain the expression

$$u(x) = \sum_{|\alpha|<m} \frac{\partial^\alpha u(y)}{\alpha!}(x-y)^\alpha + \int_0^1 \sum_{|\alpha|=m} \frac{m}{\alpha!}(x-y)^\alpha (1-s)^{m-1} \partial^\alpha u(y + s(x-y))\, ds. \tag{5.3}$$

We denote the first term in the expression above by $T_y^m u \in \mathbb{P}_{m-1}$ and write

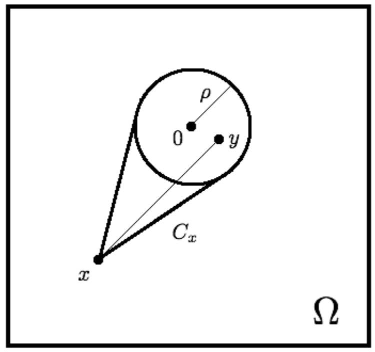
$$E = \sum_{|\alpha|=m} \frac{m}{\alpha!}(x-y)^\alpha \int_0^1 s^{m-1} \partial^\alpha u(x + s(y-x))\, ds. \tag{5.4}$$

Remark. Note that we replaced $1-s$ by s.

This was all under the assumption that $u \in C^m(\Omega)$, however, such smoothness cannot always be assumed in Sobolev spaces. We consider the *averaged Taylor polynomial*: Let $\mathcal{B} = B(0,\rho)$, let $\phi_0 \in C_c^\infty(\mathcal{B}_1)$ be positive with $\int \phi_0 = 1$. Write $\phi(x) = \rho^{-n}\phi_0(x/\rho)$ so that $\phi \in C_c^\infty(\mathcal{B})$ and that $\int \phi = 1$ as well. Now define

$$Q_m u(x) = \int_{\mathcal{B}} T_y^m u(x)\phi(y)\, dy. \tag{5.5}$$

That is, to approximate $u(x)$, we construct a Taylor polynomial, based at y and averaged over $y \in \mathcal{B}$. We have



$$\begin{aligned}
Q_m u(x) &= \sum_{|\alpha|<m} \frac{1}{\alpha!} \int_{\mathcal{B}_1} \partial^\alpha u(y)(x-y)^\alpha \phi(y)\,dy \\
&= \sum_{|\alpha|,|\beta|<m} a_{\alpha,\beta} \int_{\mathcal{B}_1} \partial^\alpha u(y) y^{\alpha-\beta} \phi(y)\,dy \\
&= \sum_{|\alpha|,|\beta|<m} (-1)^{|\alpha|} a_{\alpha,\beta} x^\beta \int_{\mathcal{B}_1} u(y) \partial^\alpha \left(y^{\alpha-\beta}\phi(y)\right)\,dy.
\end{aligned}$$

Let $\psi_{\alpha,\beta} = \partial^\alpha \left(y^{\alpha-\beta}\phi(y)\right)$.

Definition 5.2. Ω is called *star-shaped* with respect to $\mathcal{B}$ if for all x belonging to Ω, there exists a point $y \in \mathcal{B}$ such that the straight segment $[x, y] \subset \Omega$.

We have the following lemma:

Lemma 5.3. *Suppose now that Ω is star-shaped with respect to $\mathcal{B}$. Then the polynomial $Q_m u \in \mathbb{P}_{m-1}$ with degree less than or equal to $m-1$ and*

$$\|Q_m u\|_{W^{k,p}(\Omega)} \le C\|u\|_{L^1(\mathcal{B})}. \tag{5.6}$$

In other words, $Q_m : L^1(\mathcal{B}) \to W^{k,p}(\Omega)$ is bounded.

Proof. Notice that $|Q_m u|_{W^{m,p}} = 0$ and

$$\|Q_m u\|_{W^{k,\infty}(\Omega)} \le C \sum_{|\alpha|,|\beta|<m} \|x^\beta\|_{W^{k,|\alpha|}(\Omega)} \|u\|_{L^1(\mathcal{B})} \|\psi_{\alpha,\beta}\|_{L^\infty(\mathcal{B})}.$$

□

Now we direct our focus to the error term $u - Q_m u$.

$$R_m u(x) = u(x) - Q_m(x) = \int_{\mathcal{B}} \int_0^1 \sum_{|\alpha|=m} \frac{m}{\alpha!}(x-y)^\alpha s^{m-1} \partial^\alpha u(x+s(y-x))\phi(y)\,ds dy,$$

take $z = x + s(y-x)$, have $\frac{z-x}{s} = y - x$ which makes $dz = s^n\, dy$ and

$$\mathcal{A} = \left\{ (z,s) = 0 < s \le 1 : \left| x + \frac{z-x}{s} \right| < \rho \right\}. \tag{5.7}$$

Here, $(x-y)^\alpha = s^{-|\alpha|}(x-z)^\alpha$, so we have

$$\begin{aligned} R_m u(x) &= \sum_{|\alpha|=m} \frac{m}{\alpha!} \int_{\mathcal{A}} (x-z)^\alpha s^{-n-1} \partial^\alpha u(z) \phi\left(x + \frac{z-x}{s} \right) dsdz \\ &= \sum_{|\alpha|=m} \frac{m}{\alpha!} \int_{C_x} (x-z)^\alpha \partial^\alpha u(z) K(z,x)\, dz, \end{aligned}$$

where we wrote

$$K(z,x) = \int_0^1 \mathbf{1}_{\mathcal{A}}(z,s) s^{-n-1} \phi\left(x + \frac{z-x}{s} \right) ds.$$

For $(z,s) \in \mathcal{A}$, $\frac{|z-x|}{s} < |x| + \rho$ implies $s > \frac{|z-x|}{|x|+\rho} = t$, which makes

$$|K(z,x)| \le \left(\int_t^1 s^{-n-1}\, ds \right) \|\phi\|_{L^\infty(\mathcal{B})} = \frac{t^{-n}-1}{n} \|\varphi\|_{L^\infty(\mathcal{B})} \le C\rho^{-n} \left(\frac{|x|+\rho}{|z-x|} \right)^n,$$

since $\|\rho\|_{L^\infty} \le c\rho^{-n}$. We have

$$|K(z,x)| \le C(n,\phi) \left(1 + \frac{|x|}{\rho} \right) |z-x|^{-n},$$

and $\frac{|x|}{\rho} \le \frac{\operatorname{diam}\Omega}{\rho} = \gamma$, where γ is referred to as *Chunkiness paramater*. We have

$$|R_m u(x)| \le C(m,n,\gamma) \int_{C_x} |x-z|^{m-n} g_m(z)\, dz, \tag{5.8}$$

where $g_m(z) = \max_{|\alpha|-m} |\partial^\alpha u(z)|$. Define now

$$I_{m,h}(x) = \mathbf{1}_{\{|x|<h\}} |x|^{m-n}, \quad h = \operatorname{diam}\Omega, \tag{5.9}$$

where $|x|^{m-n}$ is said to be the *Riesz potential.* We claim that

$$|R_m u(x)| \le C(I_{m,h} * g_m)(x), \tag{5.10}$$

Indeed, from (5.8)

$$\begin{aligned} |R_m u(x)| &\le C \int_{C_x} |x-z|^{m-n} \mathbf{1}_{\{|x-z| \le h\}} g_m(z)\, dz \\ &\le C \int_{\mathbb{R}^n} I_{m,h}(x-z) g_m(z)\, dz, \end{aligned}$$

where g_m is extended by 0 outside of Ω. Recall now by Holder that $\|u * v\|_\infty \leq \|u\|_p \|v\|_q$,

$$\|R_m u\|_{L^\infty} \leq C\|I_{m,h}\|_{L^q} \|g_m\|_{L^p} \leq C\|I_{m,h}\|_{L^q} |u|_{W^{m,p}(\Omega)},$$

but since $\|I_{m,h}\|_{L^\infty} \leq h^{m-n}$ for $m \geq n$,

$$\int\limits_{|x|<h} |x|^{q(m-n)} \, dx \leq C \int_0^h r^{q(m-n)} r^{n-1} \, dr \leq C h^{q(m-n)+n},$$

if we require that $q(m-n)+n > -1$ or $, m > n(1-\frac{1}{q}) = \frac{n}{p}$,

$$\|I_{m,h}\|_{L^q} \leq C h^{m-n/p}, \tag{5.11}$$

we may conclude that

$$\|R_m u\|_{L^\infty} \leq C h^{m-n/p} |u|_{W^{m,p}(\Omega)}, \quad m > \frac{n}{p}, \tag{5.12}$$

and if $p = 1$, then all the above holds even for $m = n$.