Lecture 4
Characterization of finite element methods

There are two main characteristics of finite element methods (FEM):

- Petrov-Galerkin approach.
- Element-by-element computation.
- Local nodal basis leading to sparse stiffness matrices.

Remark. Contrary to working basis-by-basis.

§1 Finite element spaces

Definition 4.1. Let $\Omega \subseteq \mathbb{R}^2$ be bounded and polygonal. Then a *triangulation* (partition) of Ω is a collection $\mathcal{P} = \{\tau\}$ of (open) triangles such that

$$\overline{\Omega} = \bigcup_{\tau \in \mathcal{P}} \overline{\tau}; \quad \tau \cap \sigma = \emptyset \quad \text{for} \quad \tau, \sigma \in \mathcal{P}.$$

$\mathcal{P}$ is called *conforming* if it does not contain *hanging nodes*.

Remark. There are generalizations regarding extensions to higher dimensions, other shapes and curved boundaries.

Here are a two examples of finite elements:

- **Lagrange C^0-finite element** space of order d:

$$S^d_{\mathcal{P}} = \{u \in C(\Omega) : u|_\tau \in \mathbb{P}_{d-1} \ \forall \tau \in \mathcal{P}\}, \tag{4.1}$$

  and

$$S^0_{\mathcal{P}} = \{u \in L^\infty(\Omega) : u|_\tau \in \mathbb{R} \ \forall \tau \in \mathcal{P}\}. \tag{4.2}$$

- **C^1-finite elements**:

$$A^d_{\mathcal{P}} = \{u \in C^1(\Omega) : u|_\tau \in \mathbb{P}_{d-1} \ \forall \tau \in \mathcal{P}\}, \tag{4.3}$$

Remark. There are finite element spaces of vectors (or tensors) fields.

§2 Finite elements

Definition 4.2 (Ciaret)**.** A finite element is a triple $(\tau, S, \mathcal{N})$ where

- $\tau \subset \mathbb{R}^n$ is a domain with piecewise smooth boundary (element domain).
- S is a finite-dimensional space of functions on τ (shape functions).
- $\mathcal{N} = \{N_1, ..., N_k\}$ is a basis for S^* (nodal variables).

Example for $v \in S$,

$$\begin{matrix} N_1(v) = v(x_1) \\ N_2(v) = v(x_2) \end{matrix} \quad \text{and maybe} \quad \begin{matrix} N_3(v) = v'(x_1) \\ N_4(v) = v'(x_2). \end{matrix}$$

Definition 4.3. If $\{\phi_k\} \subset S$ is a basis of S with $N_i(\phi_k) = \delta_{ik}$ then we call $\{\phi_k\}$ the *nodal basis* of S.

Remark. $\mathcal{N}(v) = \begin{bmatrix} N_1(v) \\ \vdots \\ N_n(v) \end{bmatrix}$, $\mathcal{N} : S \to \mathbb{R}^{\dim S}$ is invertible. In particular $\phi_1 = \mathcal{N}^{-1}((1, 0, ..., 0))$.

Definition 4.4. The *local interpolation* for a function v is given by

$$I_\tau v = \sum_i N_i(v)\phi_i \in S. \tag{4.4}$$

Lemma 4.5. *We have the following:*

- $I_\tau : V \to S$ *is linear,* V *is some function space on which* $\{N_i\}$ *are defined.*
- $N_i(I_\tau v) = N_i(v)$ *for all* i*.*
- $I_\tau(p) = p$ *for all* $p \in S$ *(this means* I_τ *is a projection).*

Global interpolant: $I_\mathcal{P} v$ is defined by $(I_\mathcal{P} v)|_\tau = I_\tau v$ for all $\tau \in \mathcal{P}$.