Lecture 2
Pertrov-Galerkin methods
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§1 Applications to the Poisson problem

In this section we will apply some of the previous analysis regarding bilinear forms to the Poisson problem. But first, note the following. Recall the definition of α in (1.5). If A is invertible, A being the linear operator characterized by $a(\cdot,\cdot)$,

$$\alpha = \inf_{x\in X}\left(\frac{1}{\|x\|}\sup_{y\in Y}\frac{\langle Ax,y\rangle}{\|y\|}\right) = \inf_{x\in X}\frac{\|Ax\|_{Y^*}}{\|x\|},$$

so by a "change of variables", said definition is equivalent to

$$\alpha = \inf_{y^*\in Y^*}\frac{\|y^*\|_{Y^*}}{\|A^{-1}y^*\|_X} > 0, \tag{2.1}$$

which implies that $\alpha^{-1} = \|A^{-1}\|$. Similarly, $\beta^{-1} = \|(A^*)^{-1}\|$.

Example 2 (Dirichlet). Let $X = Y = H^1_0(\Omega)$ with $\Omega \subset \mathbb{R}^n$ is bounded. Define

$$a(u,v) = \int_\Omega \nabla u\cdot\nabla v + \kappa\int_\Omega uv. \tag{2.2}$$

Let $f \in L^2(\Omega)$. Then $Au = f$ corresponds to

$$\begin{cases} -\Delta u + \kappa u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Indeed, recalling the definition $A : H^1_0(\Omega) \to H^1_0(\Omega)$, we have $(*) : a(u,v) = \langle f, v\rangle$ for all $u,v \in H^1_0(\Omega)$. Let $v \in C^1_c(\Omega)$ and assume that $u \in C^2(\Omega)\cap C^1(\overline{\Omega})$. We have

$$a(u,v) = \int_\Omega \nabla u\cdot\nabla v + \kappa\int_\Omega uv = -\int_\Omega v\Delta u + \kappa\int_\Omega uv = \int_\Omega fv \quad \forall v \in C^1_c(\Omega),$$

by partial integration on $\int_\Omega \nabla u\cdot\nabla v$ (i.e. Green's first identity) and noting that v is compactly supported in Ω. This is extended to $v \in H^1_0(\Omega)$ via a density argument which makes the statement $(*)$ equivalent to $-\Delta u + \kappa u = f$ on Ω. Moreover, $u \in H^1_0(\Omega)$ and by assumption $u \in C^1(\overline{\Omega})$, we conclude that $u|\partial\Omega = 0$.

We wish to verify when $a(\cdot,\cdot)$ is coercive. If $\kappa > 0$,

$$a(u,u) \geq \|\nabla u\|_{L^2}^2 + \kappa\|u\|_{L^2}^2 \geq \min\{1,\kappa\}\|u\|_{H_0^1}^2.$$

If $\kappa \leq 0$, we have for $u \in H_0^1(\Omega)$ the Friedrich inequality $\|u\|_{L^2} \leq C_F\|\nabla u\|_{L^2}$ which makes, for any given $\epsilon > 0$,

$$a(u,u) \geq \epsilon\|\nabla u\|_{L^2}^2 + (1-\epsilon)\|\nabla u\|_{L^2}^2 + \kappa\|u\|_{L^2}^2 \geq \epsilon\|\nabla u\|_{L^2}^2 + \left(\frac{1-\epsilon}{C_F^2} + \kappa\right)\|u\|_{L^2}^2.$$

It follows that $a(\cdot,\cdot)$ is coercive if $|\kappa| < C_F^{-2}$. Moreover, A is invertible if κ is not an eigenvalue of the Laplacian. We conclude this example by mentioning that the solution u is smooth in Ω. Moreover, if $\partial\Omega$ is sufficiently smooth, then u is smooth up to the boundary. $\diamond$

Example 3 (Neumann). Let $X = Y = H^1(\Omega)$ and let $a(\cdot,\cdot)$ be as above. $Au = f$ corresponds to

$$\begin{cases} -\Delta u + \kappa u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

To see this, we first assume that $\partial\Omega \in C^1$, $u \in C^2(\overline{\Omega})$ and $v \in C^1(\overline{\Omega})$. By similar reasoning in the previous example we may write

$$a(u,v) = -\int_\Omega v\Delta u + \int_{\partial\Omega} v\partial_\nu u + \kappa\int_\Omega uv = \int_\Omega fv \quad \forall v \in C^1(\overline{\Omega}). \tag{2.5}$$

Noting that $C^1(\overline{\Omega}) \supset C_c^1(\Omega)$, the integral $\int_{\partial\Omega}$ vanishes and we obtain the PDE $-\Delta u + \kappa u = f$ in Ω when v is extended to $H^1(\Omega)$. Finally, for any $v \in C^1(\overline{\Omega})$ we have $\int_{\partial\Omega} v\partial_\nu u = 0$ which makes $\partial_\nu u = 0$ on $\partial\Omega$.

The bilinear form $a(\cdot,\cdot)$ is coercive if $\kappa > 0$. If $\partial\Omega \in C^1$, then the spectrum is discrete. $\diamond$

Remark. If $X = Y = H^1(\mathbb{T}^n)$ where $\mathbb{T}^n = \mathbb{R}^n/(2\pi\mathbb{Z})^n$, then we have the same set up s the Neumann problem.

§2 Petrov-Galerkin methods

Let X, Y be Banach spaces, let $a : X \times Y \to \mathbb{R}$ be a bounded bilinear form and let $A : X \to Y^*$ be an invertible bounded linear map defined by $\langle Ax, y\rangle = a(x,y)$ for all $x \in X$ and all $y \in Y$. Let $b \in Y^*$ and let $x_0 \in X$ be such that $Ax_0 = b$. We formulate the Petrov-Galerkin method (PG) in which we seek an approximation to x_0 from a given subspace of X. Take $\widehat{X} \subset X$ and $\widehat{Y} \subset Y$ to be closed linear subspaces and suppose that $\widehat{x} \in \widehat{X}$ satisfies

$$a(\widehat{x}, y) = b(y) \quad \forall y \in \widehat{Y}. \tag{2.6}$$

From the assumption $Ax_0 = b$ we have $a(x_0, y) = b(y)$ for all $y \in Y$ which makes $a(x_0 - \widehat{x}, y) = 0$ for all $y \in \widehat{Y}$. Intuitively speaking, if $\widehat{X} = \widehat{Y}$, $\widehat{x}$ serves as "best approximation" of x_0 in $\widehat{X}$.

Suppose now that the inf-sup condition holds in the subspaces. Then

$$\widehat{\alpha} = \inf_{x \in \widehat{X}} \sup_{y \in \widehat{Y}} \frac{a(x, y)}{\|x\| \|y\|} > 0.$$

Then for all $x \in \widehat{X}$,

$$\begin{aligned} \|x_0 - \widehat{x}\| &\le \|x_0 - x\| + \|x - \widehat{x}\| \\ &\le \|x_0 - x\| + \widehat{\alpha}^{-1} \sup_{y \in \widehat{Y}} \frac{a(x - \widehat{x}, y)}{\|y\|} \\ &= \|x_0 - x\| + \widehat{\alpha}^{-1} \sup_{y \in \widehat{Y}} \frac{a(x - x_0, y)}{\|y\|} \\ &\le (1 + \widehat{\alpha}^{-1} \|a\|) \|x_0 - x\|, \end{aligned}$$

where the second inequality holds because for any $x \in \widehat{X}$

$$\widehat{\alpha} \le \sup_{y \in Y^*} \frac{a(x - \widehat{x}, y)}{\|x - \widehat{x}\| \|y\|} \iff \|x - \widehat{x}\| \le \widehat{\alpha}^{-1} \sup_{y \in Y^*} \frac{a(x - \widehat{x}, y)}{\|y\|},$$

whereas the equality holds because

$$a(x - \widehat{x}, y) = a(x - x_0 + x_0 - \widehat{x}, y) = a(x - x_0, y).$$

We have the following lemma:

Lemma 2.1 (Cea's lemma)**.** *Under the hypotheses made above,*

$$\|x_0 - \widehat{x}\| \le (1 + \widehat{\alpha} \|a\|) \inf_{x \in \widehat{X}} \|x_0 - x\|. \tag{2.7}$$

2.1 Operator point of view

Let V be a linear space, $\widehat{V} \subset V$ be a linear subspace. Let $J_V : \widehat{V} \to V$ denote an extension operator. We have

$$\begin{tikzcd} \widehat{Y} \arrow[r, "J_Y"] \arrow[rr, bend right, dashed, "b \circ J_Y"'] & Y \arrow[r, "b"] & \mathbb{R} \end{tikzcd}$$

Notice that the composition $b \circ J_Y : \widehat{Y} \to \mathbb{R}$ agrees with b on $\widehat{Y}$ and is equal to $J_Y^* b$. Now let $J_Y^* : \widehat{Y} \to \widehat{Y}^*$ and define $\widehat{A} : \widehat{X} \to \widehat{Y}^*$ by

$$\langle \widehat{A}x, y \rangle = a(x, y) \quad \forall x \in \widehat{X},\ \forall y \in \widehat{Y}. \tag{2.8}$$

From

$$\begin{tikzcd} X \arrow[r, "A"] & Y^* \arrow[d, "J_Y^*"] \\ \widehat{X} \arrow[u, "J_X"] \arrow[r, "\widehat{A}"] & \widehat{Y}^* \end{tikzcd}$$

we see that $\widehat{A} = J_Y^* A J_X$ and the statement $\widehat{A}\widehat{x} = J_Y^* b$ is equivalent to (2.6) i.e. the Petrov-Galerkin method; have $\widehat{x} = \widehat{A}^{-1} J_Y^* A x_0$ as the solution obtained by (PG). Moreover, if $x \in \widehat{X}$,

$$\widehat{A}x = J_Y^* A J_X x = J_Y^* A x \iff x = \widehat{A}^{-1} J_Y^* A x.$$

The last relation together with the previous expression of $\widehat{x}$ in terms of x_0, we have for any $x \in \widehat{X}$

$$\|x_0 - \widehat{x}\| = \|x_0 - x + \widehat{A}^{-1} J_Y^* A x - \widehat{x}\| \leq \|x_0 - x\| + \|\widehat{A}^{-1} J_Y^* A x\| \|x - x_0\|,$$

which again implies the conclusion of Cea's lemma;

$$\|x_0 - \widehat{x}\| \leq (1 + \|\widehat{A}^{-1}\| \|A\|) \inf_{x \in \widehat{X}} \|x_0 - x\|. \tag{2.9}$$

2.2 Application to Hilbert spaces

Suppose that $\widehat{X} = \operatorname{span}\{\phi_j\}_{j \in J}$ and $\widehat{Y} = \operatorname{span}\{\psi_k\}_{k \in K}$ with $|J|, |K| < \infty$, we may write $u \in \widehat{X}$ and $v \in \widehat{Y}$ as $u = \sum_{j \in J} u_j \phi_j$ and $v = \sum_{k \in K} v_k \psi_k$ so that

$$a(u, v) = \sum_{j,k} u_j v_k a(\phi_j, \psi_k) \quad \text{and} \quad b(v) = \sum_{k \in K} v_k b(\psi_k).$$

In particular, the (PG) method suggests approximate solution $u \in \widehat{X}$ satisfying $a(u, v) = b(v)$ for all $v \in \widehat{Y}$ can now be viewed as a seeking solution u whose coefficients satisfy $\sum_{j \in J} u_j a(\phi_j, \psi_k) = b(\psi_k)$ for all $k \in K$. Therefore, by defining

$$\overline{A}_{jk} = a(\phi_j, \psi_k) \quad \text{and} \quad \overline{b}_k = b(\psi_k) \quad \text{for} \quad j \in J,\ k \in K, \tag{2.10}$$

the Petrov-Galerkin method is equivalent to solving the linear system $\overline{A}\overline{u} = \overline{b}$.

Remark. In $\overline{b}_k$, we typically have $\int_\Omega f \psi_k$. In practice, this integral is usually computed by means of Gaussian quadratures.

Example 4. Consider $-u'' + u = f$ on the torus $\mathbb{T} = (0, 2\pi)$. We have

$$a(u, v) = \int_0^{2\pi} u'v' + uv,$$

with $\widehat{X} = \widehat{Y} = \mathrm{span}\{1, \sin x, \cos x, ..., \sin Nx, \cos Nx\}$. Due to orthogonality, the *stiffness* matrix $\overline{A}$ is diagonal therefore (PG) solution $\overline{u}$ is given by

$$\overline{u}_k = \frac{1}{(k^2+1)\pi} \int_0^{2\pi} f\psi_k,$$

which is essentially the Nth truncation of the Fourier series of the true solution u.

$\diamond$