Range closedness and the inf-sup conditions

04/09/2013

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§1 Necessity and sufficiency for invertibility

In this section we derive a sufficient and necessary condition for existence of an inverse of a bounded linear map on Banach spaces. The following result, known as *Banach's bounded inverse theorem*, is an immediate consequence of the open mapping theorem:

> *Let X, Y be Banach spaces and let $A : X \to Y$ be a bounded linear map. Suppose that A is invertible, then $A^{-1} : Y \to X$ is also bounded.*

Suppose that A is as in the context of the previous result. Then

$$\|x\| = \|A^{-1}Ax\| \leq c\|Ax\| \quad \forall x \in X. \tag{1.1}$$

Definition 1.1. We say that A is *bounded below* if $\|x\| \leq c\|Ax\|$ for all $x \in X$ for some $c > 0$.

Remark. Note that if A is as such, then $\mathrm{Ker}(A) = \{0\}$, i.e., A is injective.

It turns out that (1.1) is not sufficient to guarantee invertibility of A.

Lemma 1.2. *Let X, Y be Banach spaces and let $A : X \to Y$ be bounded and linear. A is bounded below if and only if A is injective and the range of A is closed.*

Proof. Suppose that A is bounded below and that $x_n \in X$ with $Ax_n \to y \in Y$. Then $\|x_n - x_m\| \leq c\|Ax_n - Ax_m\| \to 0$ as $n, m \to \infty$; the sequence $(x_n)_{n\geq 1}$ is Cauchy. By completeness $x_n \to x \in X$ and so $Ax_n \to Ax$ which makes $y = Ax$. Therefore y belongs to the range of A.

Conversely, suppose that $A : X \to \mathrm{Ran}(A)$ is invertible. Since $\mathrm{Ran}(A)$ is a Banach space, we can use the bounded inverse theorem

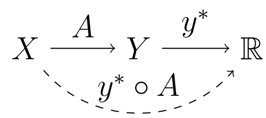
$$\|x\| = \|A^{-1}Ax\| \leq c\|Ax\|_{\mathrm{Ran}(A)} = c\|Ax\|_Y.$$

□

Remark. Injectivity here can be substituted by injectivity of $\tilde{A} : X/\mathrm{Ker}(A) \to Y$ to derive a condition similar to (1.1) that is necessary and sufficient for a general (i.e., not necessarily injective) operator to have a closed range.

In view of the preceding lemma, we need a bit more than boundedness below of A to get invertibility.

First, we need a set of preliminary results. Define $A^* : Y^* \to X^*$ by $\langle Ax, y^* \rangle = \langle x, A^*y^* \rangle$ for all $x \in X$ and for all $y^* \in Y^*$. In other words,



which makes $A^*y^* = y^* \circ A$. We have the following results:

- $\text{Ker}(A^*) = \text{Ran}(A)^\perp$ because if $y^* \in \text{Ker}A^*$, then $A^*y^* = 0$ and $\langle Ax, y^* \rangle = \langle x, A^*y^* \rangle = 0$ for all $x \in X$; in other words $y \in \text{Ran}(A)^\perp$. On the other hand, $y \in \text{Ran}(A)^\perp$ implies $\langle x, A^*y^* \rangle = 0$ for all $x \in X$ which means $y^* \in \text{Ker}(A^*)$.
- $\text{Ker}A = \text{Ran}(A^*)^\perp$. This holds by a similar argument made above.
- $\text{Ran}(A)$ is closed if and only if $\text{Ran}(A) = \text{Ker}(A^*)^\perp$. To see this, let $M \subset X$ and define its *annihilator* by

$$M^\perp = \{x^* \in X^* : \langle x, x^* \rangle = 0 \ \forall x \in M\}. \tag{1.2}$$

We also define

$$M^{\perp\perp} = \{x \in X : \langle x, x^* \rangle = 0 \ \forall x^* \in M^\perp\}. \tag{1.3}$$

Clearly $M \subset M^{\perp\perp}$ since for all $x \in M$ and for all $x^* \in M^\perp$, $\langle x, x^* \rangle = 0$. Now suppose that M is a closed linear space and let $x \notin M$. A consequence of the Hahn-Banach theorem gives an $x^* \in X^*$ such that $x^*(x) \neq 0$ and $x^*|_M = 0$. In other words, $x^* \in M^\perp$ which by definition makes $x \notin M^{\perp\perp}$. It follows that $M = M^{\perp\perp}$ if $M \subset X$ is a closed linear space. Returning to the claim above, $\text{Ran}(A)$ is linear and closed so take M to be $\text{Ran}(A)$ and use the first point made above.

- $\text{Ran}(A^*)$ is closed if and only if $\text{Ran}(A^*) = \text{Ker}(A)^\perp$.

Using the previous results we have the following theorem:

Theorem 1.3. *Let X, Y be Banach spaces and let $A : X \to Y$ be bounded and linear. A is invertible if and only if A and A^* are bounded below.*

Proof. Assume that A and A^* are bounded below. We have already proved in Lemma 1.2 that A is injective and that $\text{Ran}(A)$ is closed. By the third (bullet point) result above, $\text{Ran}(A) = \text{Ker}(A^*)^\perp$ but since A^* is also injective, it follows that $\text{Ran}(A) = Y$.

Conversely, suppose that A is invertible, then A is bounded below by (1.1). For below boundedness of A^*, it suffices to show that A^* is invertible and note that $\mathrm{Ker}(A^*) = \mathrm{Ran}(A)^\perp = \{0\}$. Suppose that $x^* \in X^*$ and define $y^* \in Y^*$ by $\langle Ax, y^* \rangle = \langle x, x^* \rangle$ for all $x \in X$; i.e. $\langle y, y^* \rangle = \langle A^{-1}y, x^* \rangle$ for all $y \in Y$. This makes $\langle x, A^*y^* \rangle = \langle x, x^* \rangle$ for all $x \in X$, equivalently, $\langle y, y^* \rangle = \langle y, (A^{-1})^* x^* \rangle$ for all $y \in Y$; there exists $(A^*)^{-1}$. □

§2 Bilinear forms

In this section we derive the *inf-sup conditions*.

Let X, Y be Banach spaces and let $a : X \times Y \to \mathbb{R}$ be a bounded bilinear form.

$$\|a\| = \sup_{x\in X} \sup_{y \in Y} \frac{a(x,y)}{\|x\| \|y\|} < \infty. \tag{1.4}$$

Then we define $A : X \to Y^*$ by $\langle Ax, y \rangle = a(x,y)$ for all $x \in X$ and for all $y \in Y$.

$$\|Ax\| = \sup_{y\in Y} \frac{\langle Ax, y \rangle}{\|y\|} = \sup_{y \in Y} \frac{a(x,y)}{\|y\|} \leq \|a\| \|x\|.$$

The *adjoint* $A^* : Y^{**} \to X^*$, assuming that Y is reflexive,

$$\langle x, A^*y \rangle = \langle Ax, y \rangle = a(x,y).$$

Observe that A is bounded below if and only if for some $c > 0$

$$\|x\| \leq c\|Ax\| = c \sup_{y\in Y} \frac{\langle Ax, y \rangle}{\|y\|} = c \sup \frac{a(x,y)}{\|y\|},$$

if and only if

$$\alpha := \inf_{x\in X} \sup_{y \in Y} \frac{a(x,y)}{\|x\| \|y\|} > 0. \tag{1.5}$$

Again, A^* is bounded below if and only if for some $c > 0$

$$\|y\| \leq c\|A^*y\| = c \sup_{x\in X} \frac{\langle x, A^*y \rangle}{\|x\|} = c \sup_{x\in X} \frac{a(x,y)}{\|x\|},$$

if and only if

$$\beta := \inf_{y\in Y} \sup_{x \in X} \frac{a(x,y)}{\|x\| \|y\|} > 0. \tag{1.6}$$

These are called the *inf-sup conditions*.

Example 1. Let $X = Y$ be reflexive with $a(x,x) \geq c\|x\|^2$ for $c > 0$.

$$\sup_{y \in X} \frac{a(x,y)}{\|y\|} \geq \frac{a(x,x)}{\|x\|} \geq c\|x\|.$$

Divide by $\|x\|$ and take an infimum over X makes

$$\inf_{x \in X} \sup_{y \in X} \frac{a(x,y)}{\|x\|\|y\|} \geq c.$$

This is known as the Lax-Milgram lemma. ◇