# Spherically Symmetric Collapse of Stars

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**Abstract**

In this project, we will study the gravitational collapse of a class of spherically symmetric stars both analytically and numerically.

**Introduction**

Spherically symmetric collapse has been studied by many people from different aspects. Oppenheimer and Snyder [11] described the collapse of a spherically symmetric and homogeneous dust ball(the density and pressures given by ($\rho = \rho(t), p = 0$) whose singularity is neither locally or globally naked. This solution leads to a generally accepted result that the final fate of gravitational collapse is a black hole. However, in case of that the density is inhomogeneous, the collapse of spherically symmetric dust, the outcome is generically either a black hole or a naked singularity, depending on the initial data of the collapse. The work of Oppenheimer and Snyder are also considered to be the beginning of modern theory of black holes.

For some high-density matter, the assumption of pressureless matter is obviously not appropriate. In the aspect of when pressures are important, the gravitational collapse of perfect fluids, has been studied both analytically and numerically. Under the assumption of self similarity, Ori and Piran [13] investigated the collapse of a perfect fluid numerically and then Harada [4] constructed the null coordinate to detect naked singularities. Joshi [8] have already concluded the results that both black holes and naked singularity could be the final fate of gravitational collapse in a more general setting. We will give a glimpse at these results about spherically symmetric collapse so far.

**1. Oppenheimer–Snyder spherical dust collapse**

Oppenheimer and Snyder(1939) [11] described the collapse of a spherical star with uniform density and zero pressure to a Schwarzschild black hole. This solution is often considered to be one of the most useful analytic solutions of the Einstein equations. Although the collapse scenario of this solution is highly idealized, we can still learn some general features of gravitational collapse and black hole formation can be learned, the analytic solution would also be helpful to treat more complicated scenarios numerically [2].

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A black hole is a region of spacetime that cannot communicate with the outside Universe. While Spacetime singularities form the interior black holes, Einstein's equations describe the outside Universe but they fail in the interior black holes due to the singularity. The Schwarzschild metric is a special case for stationary black hole solution to Einstein's equations, which uniquely specified by only one parameter: the mass $M$.

Consider a 4-dimensional spacetime manifold $M$ with local coordinates $(t, r, \theta, \phi)$. The Schwarzschild solution for a vacuum spherical spacetime is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{1}$$

*Remark* 1. (1). The Schwarzschild solution holds in the vacuum region of any spherical spacetime even a spacetiem containing matter. The vacuum exterior solution of collapsing stars must given by the Schwarzschild solution by Birkhoff's theorem.
(2). We have two singularities, $r = 0$ is a physical spacetime singularity, $r = 2M$ is a coordinate singularity, which can be removed by coordinate transformation. Moreover, the black hole event horizon is located at $r = 2M$, known as the Schwarzschild radius. In fact, the Krerschmann scalar:

$$R^{abcd}R_{abcd} = \frac{48m^2}{r^6},$$

which blows up at the origin, showing that existence of the physical spacetime singularity. One alternative coordinate choice that removes the coordinate singularity at $r = 2M$ is *Kruskal-Szekeres* coordinate system.
(3). One useful Schwarzschild metric in numerical computations is the Schwarzschild metric in isotropic radial coordinates:

$$ds^2 = -\left(\frac{1 - M/2\tilde{r}}{1 + M/2\tilde{r}}\right) dt^2 + \left(1 + \frac{M}{2\tilde{r}}\right)^4 \left[d\tilde{r}^2 + \tilde{r}^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \tag{2}$$

where

$$\begin{cases} r = \tilde{r}(1 + M/2\tilde{r}) \\ \tilde{r} = \frac{1}{2}\left[r - M \pm (r - 2M)^{1/2}\right] \end{cases}.$$

Note that the isotropic coordinate $\tilde{r}$ only describes region of Schwarzschild geometry with $r \geq 2M$. The black hole event horizon is located at $\tilde{r} = M/2$ in these coordinates.

According to the Birkhoff's theorem, the Schwarzschild solution is the most general description outside a nonrotating, spherically symmetric star. The metric describing stellar interior in the time-independent form is given by:

$$ds^2 = -e^{2\Phi} dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{3}$$

where $\Phi$ and $m$ are function of $r$. Suppose the stellar matter is described as a perfect fluid, we want to relate them with density $\rho(r)$ and pressure $p(r)$. The stress energy tensor is given by

$$T^{\mu v} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}, \tag{4}$$

where $u$ is the 4-velocity of the fluid elements. We choose the co-moving coordinates and take $u$ to be pointing in the timelike direction. We also normalize $u$ by setting $u_\mu u_\nu = -1$ s.t. $u = (e^\Phi, 0, 0, 0)$. Therefore,

$$T^{\mu\nu} = \text{diag}\left(\rho e^{-2\Phi}, p\left(1 - \frac{2m(r)}{r}\right), p/r^2, p\left(r^2\sin^2\theta\right)^{-1}\right)$$

By Einstein equations:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{5}$$

note that $R_{\mu\nu}$ is the Ricci tensor, $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar. We start by computing all the nonzero Christoffel symbols $\Gamma^\alpha_{\mu\nu}$, denote $\partial_r m(r) = m'$:

$$\begin{aligned}
&\Gamma^t_{tr} = \Phi', \Gamma^r_{tt} = \Phi' e^{2\Phi}\left(1 - \frac{2m}{r}\right), \Gamma^r_{rr} = \frac{rm' - m}{r^2 - 2rm} \\
&\Gamma^\theta_{r\theta} = \Gamma^\phi_{r\phi} = \frac{1}{r}, \Gamma^r_{\theta\theta} = 2m - r, \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta;
\end{aligned}$$

then the nonzero components of Ricci tensor are

$$\begin{aligned}
R_{tt} &= e^{2\Phi}\left[\left(\Phi'' + (\Phi')^2\right)\left(1 - \frac{2m}{r}\right) + \Phi'\left(\frac{2r - 3m - rm'}{r^2}\right)\right], \\
R_{rr} &= \left(1 - \frac{2m}{r}\right)^{-1}\left[\frac{(rm' - m)(2 + r\Phi')}{r^3}\right) - \left(\Phi'' + (\Phi')^2\right], \\
R_{\theta\theta} &= \left(\sin^2\theta\right)^{-1} R_{\phi\phi} = (2m - r)\Phi' + m' + \frac{m}{r};
\end{aligned}$$

Therefore, we have the Ricci scalar:

$$R = 2\left[\frac{2m'}{r^2} + \Phi'(3m - 2r + rm') - \left(1 - \frac{2m}{r}\right)\left(\Phi'' + (\Phi')^2\right)\right]. \tag{6}$$

Substituting (6) and $T_{tt} = \rho e^{2\Phi}, T_{rr} = p\left(1 - \frac{2m(r)}{r}\right)^{-1}$ into (5) which gives

$$G_{tt} = \frac{2m' e^{2\Phi}}{r^2} = 8\pi\rho e^{2\Phi} \implies m' = 4\pi r^2 \rho \tag{7}$$

$$G_{rr} = \frac{2}{r}\left(\Phi' - \frac{m}{1 - 2m/r}\right) = \frac{8\pi p}{1 - 2m/r} \implies \Phi' = \frac{m + 4\pi r^3 p}{r(r - 2m)}. \tag{8}$$

An easy way to formulate a differential equation for the pressure is using conservation of energy thus the divergence of the stress-energy tensor vanishes, which implies

$$\begin{aligned}
0 = \nabla_\nu T^{r\nu} &= \frac{\partial T^{r\nu}}{\partial x^\nu} + T^{\sigma\nu}\Gamma^r_{\sigma\nu} + T^{r\sigma}\Gamma^\nu_{\sigma\nu} \\
&= \frac{\partial T^{rr}}{\partial r} + T^{tt}\Gamma^r_{tt} + T^{rr}\Gamma^r_{rr} + T^{\theta\theta}\Gamma^r_{\theta\theta} + T^{\phi\phi}\Gamma^r_{\phi\phi} + T^{rr}\Gamma^\nu_{r\nu} \\
&= \left(1 - \frac{2m}{r}\right)\left[p' + (p + \rho)\Phi'\right] \implies p' = -(\rho + p)\Phi'.
\end{aligned} \tag{9}$$

The Oppenheimer-Volkhoff(OV) equations [12] are the following coupled first-order ODEs

$$\frac{dm}{dr} = 4\pi r^2 \rho \tag{10}$$

$$\frac{d\Phi}{dr} = \frac{m + 4\pi r^3 p}{r(r-2m)} \tag{11}$$

$$\frac{dp}{dr} = -(\rho + p)\frac{d\Phi}{dr}, \tag{12}$$

which describes the stellar structure for spherically symmetric stars.

*Remark* 2. The total mass of the star is $M = \int_0^R 4\pi r^2 \rho dr$ where $R$ is the stellar radius ($p = \rho = 0$). For uniform density($\rho = \rho_0$), incompressible stars, $m = \frac{4}{3}\pi\rho_0 r^3$, we integrate the OV equation exactly:

$$p = \rho_0 \frac{\sqrt{1-2M/R} - \sqrt{1-2Mr^2/R^3}}{\sqrt{1-2Mr^2/R^3} - 3\sqrt{1-2M/R}} \tag{13}$$

then the central pressure is

$$p_c = \frac{3M}{4\pi R^3}\frac{1-\sqrt{1-2M/R}}{3\sqrt{1-2M/R}-1}. \tag{14}$$

Taking $p_c \to \infty$ then $M/R = 4/9$, the infinite pressure are necessary to support a star with a radius greater than $9/8$ of the Schwarzschild radius $R = 2M$ this means that a spherically-symmetric, uniform density, perfect-fluid cannot have radius smaller than the Schwarzschild radius, the final fate can only be a black hole.

As we introduced before, OS solution described the collapse of of uniform-density pressureless dust into a black hole. By Birkhoff's theorem, the metric outside the dust is a Schwarzschild solution while inside is a Friedmann-Robertson-Walker(FRW) solution. We denote the inside and outside regions of the collapsing dust by $\Sigma_-$ and $\Sigma_+$ respectively. Therefore, we have

$$ds^2_{\Sigma_+} = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{15}$$

$$ds^2_{\Sigma_-} = -d\tau^2 + a^2(\tau)\left(d\chi^2 + \sin^2\chi d\Omega^2\right). \tag{16}$$

By Einstein field equation,

$$\ddot{a}^2 + 1 = \frac{8\pi}{3}\rho a^2 \tag{17}$$

then energy-momentum conservation in the absence of pressure imply that

$$\rho a^3 = \text{constant} = \frac{3a_{\max}}{8\pi}. \tag{18}$$

Therefore, we have a parametric form of solution for (17) and (18):

$$a(\eta) = \frac{1}{2}a_{\max}(1+\cos\eta), \tau(\eta) = \frac{1}{2}a_{\max}(1+\sin\eta), \tag{19}$$

which describe that the collapse starts at $\eta = 0, a = a_{\max}$ and it ends at $\eta = \pi, a = 0$. The hypersurface *Sigma* coincides with the surface of the collapsing star, which is located at $\chi = \chi_0$ in our comoving coordinates. From outside the dust, *Sigma* can be described by the parametric equations $r = R(\tau), t = T(\tau)$. Now we have the same induced metrics on both sides of the hypersurface

$$ds_{\Sigma}^2 = -\left(F\dot{T}^2 - F^{-1}\dot{R}^2\right)d\tau^2 + R^2(\tau)d\Omega^2(F = 1 - 2m/R), \tag{20}$$

$$ds_{\Sigma}^2 = -d\tau^2 + a^2(\tau)\left(d\chi_0^2 + \sin^2\chi d\Omega^2\right), \tag{21}$$

which implies

$$R(\tau) = a(\tau)\sin\chi_0 \tag{22}$$

$$F\dot{T}^2 - F^{-1}\dot{R}^2 = 1 \implies FdotT = \sqrt{\dot{R}^2 + F} = \beta(R, \dot{R}). \tag{23}$$

In [16], the author shows that

$$m = \frac{4\pi}{3}\rho R^3. \tag{24}$$

This provides an intuitive interpretation of Oppenheimer-Snyder problem since the density of the dust ball becomes infinite as $R(\tau) \to 0$, indicating the formation of a black hole as the ball collapses to zero size.

## 2. Spherically symmetric collapse of more general forms of matter

In the previous section we present the classic OS solution of spherically symmetric collapse of a homogeneous dust ball. The singularity theorem [6] states that the singularity in the generic gravitational collapse of a massive star. Hence, Penrose [15] raised a cosmic censorship conjecture which says that all singularities are hidden in black holes or no singularity cannot be seen by an observer. However, many counterexamples have been found. In OS solution, a singularity is neither locally or globally naked thus any observer cannot see the singularity which satisfied the strong version of cosmic censorship conjecture, and the description that the final fate of gravitational collapse is a black hole has been generally accepted. Once the density and velocity of stars is inhomogeneous, OS solution does not hold.

Tolman-Bondi solution proved that in this case from very generic initial data the singularity can be either locally or globally naked [3]. Formation of a naked singularity in this solution leads to the blow up of the density.

*2.1. Naked singularities in self-similar spherical collapse*

Ori and Piran[13] investigated the spherical collapse of a perfect fluid under the assumption of self-similarity with an adiabatic equation of state

$$p = (\gamma - 1)\rho, \tag{25}$$

when $\gamma - 1 \ll 1$, the naked singularities might appear. This solution increases significantly the range of matter fields that should be ruled out thus the cosmic censorship conjecture will hold.

From [14], a spherical space-time is self-similar if there exists a radial area coordinate $r$ and an orthogonal time coordinate $t$ such that for $c > 0$

$$g_{tt}(t,r) = g_{tt}(ct,cr), g_{rr}(t,r) = g_{rr}(ct,cr). \tag{26}$$

*2.1.1. Self-similar Schwarzschild coordinates*

We define a "Schwarzschild-like" coordinates $(r,t)$ as

$$ds^2 = -e^{\nu}dt^2 + e^{\lambda}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{27}$$

The collapse is described by the total energy density $\rho = d(x)/t^2$, the velocity $u^r(x)$ and the metric function $g_{rr}(x) = e^{\lambda(x)}, g_{tt}(x) = -e^{\nu(x)}$. $\rho$ and $p$ are related by adiabatic equation of state, while $u^t$ and $u^r$ are related by

$$-e^{\nu}(u^t)^2 + e^{\lambda}(u^r)^2 = -1 \tag{28}$$

We define $x \equiv r/|t|$ as self-similar variable and our goal is looking for a solution of $u^r, \lambda, \nu$. The spherical self-similar relativistic collapse equations are

$$T^t_t : 8\pi\gamma u_t u^r = \frac{e^{-\lambda} - 1}{x} + 8\pi d\left(1 + \gamma u^r u_r\right)x, \tag{29}$$

$$T^r_r 8\pi d(1 + \gamma u_t u^t) = -e^{-\lambda}\left(\frac{1}{x^2} + \frac{v'}{x}\right) + \frac{1}{x^2}, \tag{30}$$

$$T^t_r : 8\pi\gamma d u_t u^r = e^{-\lambda}\lambda', \tag{31}$$

$$u^{\alpha}T^{\alpha}_{\alpha;\beta} : \left[u^r d' + u^t\left(2d + xd'\right)\right] + e^{-\frac{\lambda+\nu}{2}}\left[\left(e^{\frac{\lambda+\nu}{2}}u^r\right)' + x\left(e^{\frac{\lambda+\nu}{2}}u^t\right)'\right] = 0. \tag{32}$$

Note that $()' = \partial_x()$. Regularity at origin needs $u^r(0) = \lambda(0) = 0$, we set $\nu(0) = 0$.

*Remark* 3. (1). The central density $\rho(0) = d_0/t^2$, diverges at $t = 0$ if $d_0 \neq 0$. This singularity is a basic feature of the solution but does not reflect any singularity in in the solution of equations above.
(2). The solution is characterized by two parameters $\gamma$ and $d_0$ and we want to integrate the collapse equations numerically from $x = 0$ to $x = \infty$. We only consider $\gamma < 1.015$ here, if not the solutions contain trapped surfaces and a black hole form before $t = 0$.
(3). The self-similar solution has a sonic point $x_s$, where the soluion is generally discontinuous. One of a discrete regular solution is a called the the general-relativistic equivalent of the Penston's Newtonian solution, which is one of the simplest naked singularity.

Figure 1 shows a numerical solution of the spherical self-similar relativistic collapse equations with $\gamma = 1.01$ and $d_0 = 0.1144$. To understand this example, it is sufficient to consider the solution near the origin and at infinity. It can be seen that the solution describes an almost homogeneous, $d \approx d_0 - d_2 x^2$, uniform, $u^r \approx -2x/3\gamma$ and almost Newtonian

$$2m/r = 1 - g_{rr}^{-1} \approx 8\pi x^2 d_0/3 \ll 1$$

collapse. The solution goes over asymptotically to an isothermal

$$d \approx d_\infty x^{-2}, 2m/r \approx (2m/r)_\infty$$

constant velocity $u^r = -u_\infty$ flow. Like the shell-focusing singularities[10], this singularity has a Newtonian character $2m/r < 1$ for all $x$. The system remains almost Newtonian and a black hole does not appear before the singularity is formed.

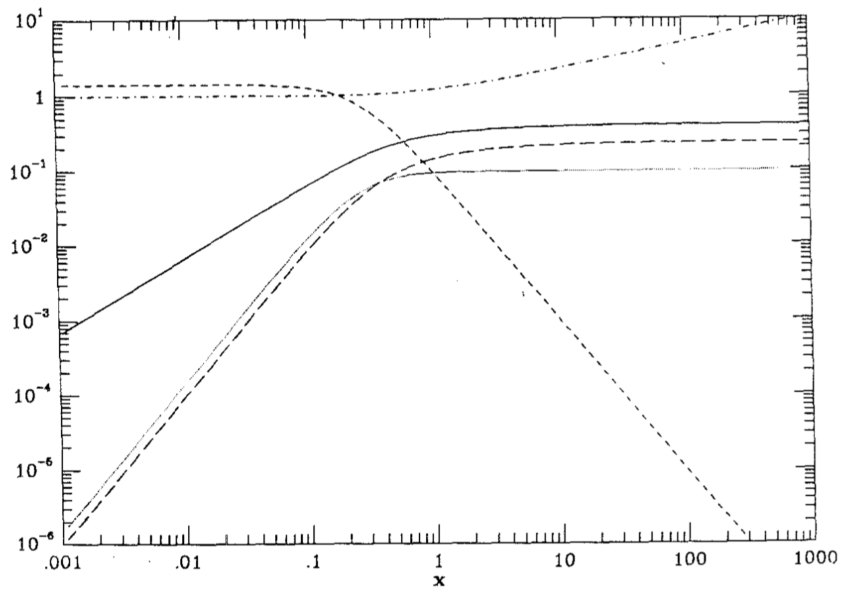


Figure 1: Collapse expressed in self-similar Schwarzschild coordinates for $t < 0$ [13]: $|u^r$(solid line), $4\pi x^2 d = 4\pi p r^2$(dotted line), $4\pi d$(short-dashed line), $2m/r$(long-dashed line), $|g_{tt}|$(dash-dotted line)

*2.1.2. Self-similar comoving coordinates*

Ori and Piran[13] also showed an example for self-similar collapse in comoving coordinates whose singularity and its causal structure seem like the shell-focsing singularities found in the Tolman-Bondi solutions[3]. Now, we present the self-similar comoving coordinates. The comoving coordinates is defined by the metric

$$ds^2 = -e^{\Psi} dT^2 + e^{\Lambda} dR^2 + r^2 d\Omega^2, u^\mu = u^T \delta^\mu_T. \tag{33}$$

The normalization condition $u^\mu u_\mu = -1$ leads that $u^T = e^{-\Psi/2}$. We define a comoving similarity variable $y = R/T$ and we want to seek a solution where $g_{TT}(y) = e{-}\Psi/2$,

$g_{RR}(y) = e^{\Lambda}$, $D(y) = \rho T^2$ and $\tilde{r}(y) = r/T$. The comoving self-similar collapse equation are

$$y\tilde{r}'' - \frac{\gamma-1}{\gamma}(\tilde{r} - y\tilde{r}')\frac{D'}{D} + y\tilde{r}'\left(\gamma\frac{D'}{D} + \frac{2\tilde{r}'}{\tilde{r}} + \frac{2(\gamma-1)}{y}\right) = 0 \tag{34}$$

$$e^{-\Psi}\left(2y^2\tilde{r}\tilde{r}'' + (\tilde{r} - y\tilde{r}')^2 - \frac{2(\gamma-1)y\tilde{r}}{\gamma}(\tilde{r} - y\tilde{r}')\frac{D'}{D}\right)$$
$$+ e^{-\Lambda}\left((\tilde{r}')^2 - \frac{2(\gamma-1)\tilde{r}\tilde{r}'}{\gamma}\frac{D'}{D}\right) = -8\pi(\gamma-1)D\tilde{r}^2 - 1 \tag{35}$$

$$e^{-\Psi} = (4\pi D)^{\frac{2(\gamma-1)}{\gamma}} \tag{36}$$

$$e^{-\Lambda} = (4\pi D)^{2\gamma}\tilde{r}^4 y^{4(\gamma-1)}. \tag{37}$$

More detailed analysis and numerical solution about this type of self-similar collapse, see [13],[14].

*2.2. Final fate of the spherically symmetric collapse of a perfect fluid*

Harada contructed the null coordinates to solve the difficulties in detection of naked singularities [4]. Furthermore, using an outgoing null coordinate ("observer time coordinate"), the coordinates never cross an event horizon and thus the singularity is globally naked as a trivial result. The procedure to obtain a numerical solution is

1. Preparing initial data on a spacelike hypersurface $t = 0$.
2. Solving Misner-Sharp equations [9] from initial data $t = 0$ and storing data on the first null ray which emanates from the center at $t = 0$.
3. When the ray reaches the stellar surface, solving Hernandez-Misner equations [7] with the stored data on the first null ray as initial data $u = 0$.

Complete numerical schemes and difference equations can be found in [1] and [17]. This result is also free from the assumption of self-similarity.

## 3. General conclusion of spherically symmetric collapse

In the following, we will review a general conclusion of spherically symmetric collapse. Joshi[8] showed that for an arbitrary regular distribution of a general matter field at the initial state, there always exists an evolution from this initial data which would collapse into either a black hole or a naked singularity, depending on the allowed choice of free functions available in the solution. This method generates new families of black hole solutions from spherically symmetric collpse without the assumption of cosmic censorship hypothesis.

We consider a general type I matter fields[5], which include most of the physically important forms of matter such as dust, perfect fluids and so on, nearly all observed forms of matter and equations of state are included in this general class. For our purpose, we consider a gravitational collapse of a matter cloud that evolves from a regular initial data defined on an initial spacelike surface. The matter field in a general coordinate system is given by

$$T^{ab} = \psi_1 E_1^a E_1^b + \psi_2 E_2^a E_2^b + \psi_3 E_2^3 E_2^3 + \psi_4 E_4^a E_4^b,$$

where $(E_1, E_2, E_3, E_4)$ is an orthonormal basis. Note that $E_1, E_2, E_3$ are spacelike eigenvectors w.r.t. eigenvalue $\psi_1, \psi_2, \psi_3$ respectively; $E_4$ is timelike eigenvector w.r.t. eigenvalue $\psi_4$. The metric for such spherically symmetric matter distribution is

$$ds^2 = -e^{2\nu}dt^2 + e^{2\lambda}dr^2 + R^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{38}$$

where $\mu, \lambda, R$ are functions of $t$ and $r$ and the stress-energy tensor is

$$T_a^b = \mathrm{diag}\left(-\rho, p_r, p_\theta, p_\theta\right).$$

Here, density $\rho$, radial pressure $p_r$, tangential stresses $p_\theta$ are the eigenvalues of $T_a^b$. We also assume that the matter fields satisfied the weak energy condition: the energy density measured by any observer must be nonnegtive, for any timelike vector $u^a$ we must have

$$T_{ab}u^a u^b \geq 0 \implies \rho \geq 0, \rho + p_r \geq 0, \rho + p_\theta \geq 0. \tag{39}$$

Since the dynamical evolution of the initial data at any time from when collapse starts, we have the following functions of $r$

$$\begin{cases} \nu(t_i, r) = \nu_0(r) \\ \lambda(t_i, r) = \lambda_0(r) \\ R(t_i, r) = R_0(r) \\ \rho(t_i, r) = \rho_0(r) \\ p_r(t_i, r) = p_{r_0}(r) \\ p_\theta(t_i, r) = p_{\theta_0}(r) \end{cases}, \tag{40}$$

$t = t_i$ is the initial time. The dynamical evolution of these initial data is determined by Einstein equation, thus we have the following equations from the given metric:

$$T_t^t = -\rho = -\frac{F'}{k_0 R^2 R'}, T_r^r = p_r = -\frac{F'}{k_0 R^2 \dot{R}}, \tag{41}$$

$$\nu'(\rho + p_r) = 2(p_\theta - p_r)\frac{R'}{R} - p_r', \tag{42}$$

$$-2\dot{R}' + R'\frac{\dot{G}}{G} + \dot{R}\frac{H'}{H} = 0, \tag{43}$$

$$G - h = 1 - \frac{F}{R}. \tag{44}$$

Note that in this equations $\dot{()} = \partial_r(), ()' = \partial_t(),$

$$G(t, r) = e^{-2\lambda}(R')^2, H(t, r) = e^{-2\nu}(\dot{R})^2 \tag{45}$$

$F(t,r) \geq 0$ iS treated as mass function for the cloud. To preserve the regularity at $t_i$, we must let the mass function vanishes at the centre of cloud thus $F(t_i, 0) = 0$. Moreover, the initial data are not all independent, they satisfied

$$\nu_0'(\rho_0 + p_{r_0}) = 2(p_{\theta_0} - p_{r_0})\frac{R_0'}{R_0} - p_{r_0}' \implies \nu_0(r) = \int \left(\frac{2(p_{\theta_0} - p_{r_0})R_0'}{R_0(\rho_0 + p_{r_0})} - \frac{p_{r_0}'}{(\rho_0 + p_{r_0})}\right) dr. \tag{46}$$

We have five equations and seven unknowns, this scenario gives us freedom of choice of two functions. Therefore, we select two free functions and then the given initial data and weak energy condition will determine the specific evolution for the initial data. We need to ensure the regularity of the initial data which means that the curvatures and the initial density and pressures are all finite. To ensure that the curvatures are finite, the Kretchmann scalar

$$\begin{aligned} K &= R^{abcd}R_{abcd} \\ &\frac{4}{3}\left(\frac{F}{R^3} + p_r + p_\theta - \rho\right)^2 - \frac{1}{3}\left(p_r + 2p_\theta - \rho\right)^2 + 2\left(\rho^2 + p_r^2 + 2p_\theta^2\right) \end{aligned}$$

must be bounded. A singularity will appear on the initial surface if either $\rho$ or one of the pressures are unbounded at any point on the initial surface, or $(F/R)^3 \to \infty$ at any point. We can see if the density and pressures are finite and bounded,

$$F(t_i, r) = \int \left(\rho_0(R_0')R_0^2\right) dR_0$$

the spacetime is singularity-free initially in the sense that the Kretchmann scalar, density and pressures are finite. Furthermore, there is a coordinate freedom left in the choice of the scaling of the coordinate $r$ which can reduce the number of independent initial data to four.

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