Schoen-Yau's proof of positive mass theorem

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Abstract

This is a project paper for Schoen and Yau's classic proof of the positive mass theorem for dimension 3.

1 Introduction

Let $(M, \bar{g})$ be a space-time, the positive mass theorem states that if M has non-negative local mass density then the total mass of M as viewed from spatial infinity(the ADM mass) must be positive unless M is the flat Minkowski space-time. Mathematically, the theorem can be described as follows: Suppose $N \hookrightarrow M$ is an oriented three dimensional space-like hypersurface in M with induced metric g and second fundamental form h_{ij}. Then the local mass density μ and current density J^i can be expressed as

$$\mu = \frac{1}{2}[R - \sum_{i,j} h^{ij} h_{ij} + (\sum_i h_i^i)^2]$$

$$J^i = \sum_j \nabla_j [h^{ij} - (\sum_k h_k^k) g^{ij}]$$

where R is the scalar curvature of N. We assume that μ and J^i satisfies the dominant energy condition

$$\mu \geq (\sum_i J^i J_i)^{\frac{1}{2}} \tag{1.1}$$

Furthermore, we assume that g is asymptotically flat, i.e. there exists a compact subset K of N so that $N \backslash K$ consists of finitely number of components N_1, N_2, $\ldots N_r$ with each N_k diffimorphic to R^3 minus a ball(we call each N_k the end of N). We also assume that g is of C^5 and be asymptotically flat in the sense that each boundary component of N has positive mean curvature with respect to the outward normal, and each N_k admits a coordinate system (x^1, x^2, x^3) in which g has the expansion $g = g_{ij} dx^i dx^j$ with g_{ij} satisfying

$$\begin{aligned} g_{ij} &= (1 + \frac{M_k}{2r})\delta_{ij} + h_{ij}, \quad |h_{ij}| \leq \frac{k_1}{1+r^2}, \\ |\partial h_{ij}| &\leq \frac{k_2}{1+r^3}, \quad |\partial^2 h_{ij}| \leq \frac{k_3}{1+r^4}. \end{aligned} \tag{1.2}$$

where $r = (\sum_{i=1}^{3}(x^i)^2)^{\frac{1}{2}}$ and ∂ is the Euclidean gradient

$$M_k = \lim_{r\to\infty}\frac{1}{16\pi}\int_{S_r}\sum_{i,j}(\partial_j g_{ij} - \partial_i g_{jj})d\sigma_i$$

is the total mass of N_k. Here we only consider the case when $\sum_i h_i^i = 0$ in this project survey, the general case can be reduced to this one (see [7]). First we note that, in the case $\sum_i h_i^i = 0$, the dominant energy condition (1.1) implies that $R \geq 0$. So, we can state the theorem as

Theorem 1.1. *Let g be a asymptotically flat metric on an oriented 3-manifold N. If $R \geq 0$ on N, then the total mass of each end is nonnegative. Furthermore, if in addition h_{ij} satisfies*

$$|\partial^3 h_{ij}| + |\partial^4 h_{ij}| + |\partial^5 h_{ij}| \leq \frac{k_4}{1+r^5} \tag{1.3}$$

for some positive constant k_4 on an end N_k and the total mass of N_k is zero, then g is flat, N is isomorphic to $\mathbb{R}^3$ with stand metric.

2 Ideals of Proof

The proof of Theorem 1.1 can be divided into two parts. First, they prove the nonnegativity of the total mass by contradiction by using the minimal surface theory. Then they prove the uniqueness of N with an evolution of the metric g. First, suppose some N_k has negative mass $M_k < 0$, then the proof of nonnegativity of M_k can be divided into three steps:

- step 1: replace the initial metric g with a conformally equivalent metric $\widetilde{g}$ which is still asymptotically flat and satisfies $\widetilde{R} \geq 0$ on N, $\widetilde{R} > 0$ outside a compact subset of N_k, and having negative total mass N_k.
- step 2: construct a complete area minimizing surface M properly embedded in N so that $S \cap (N \setminus N_k)$ is compact and $S \cap N_k$ lies between two parallel Euclidean 2-planes in the 3-space defined by x^1, x^2, x^3.
- step 3: use a result in [2] and the asymptotically flat property to show that the minimal surface constructed in step 2 does not exist to get a contradiction.

For the uniqueness of N, suppose $M_k = 0$ for some k, the proof also consists of three steps:

- step 1: use the assumption $M_k = 0$ and $R \geq 0$ to show that N is scalar flat.

- step 2: define a family of metric $g(t)$ on N by $g(t) = \sum_{i,j=1}^{3}(g_{ij}+tS_{ij})dx^i dx^j$, where S_{ij} is the Ricci curvature of g. Use the condition that $M_k = 0$ and $R \geq 0$ to find a asymptotically flat metric $\phi^4(t)g(t)$ with mass $M(t) = -\frac{1}{32\pi}\int_N R(t)\phi(t)\sqrt{g(t)}dx$.

- step 3: show that $M'(0) = \frac{1}{32\pi}\int_N \|Ric\|^2 dx = 0$ to prove that $Ric \equiv 0$. Since dim$N$=3, this implies that g is the Euclidean metric.

3 None-negativity of M_k

- **Step1** Suppose that x^1, x^2, x^3 are asymptotically flat coordinates describing N_k on $\mathbb{R}^3 \setminus B_{\sigma_0}(0)$,where $B_{\sigma_0}(0) = \{x \in \mathbb{R}^3 || x| = (\sum_{i=1}^{3}(x^i)^2)^{\frac{1}{2}} < \sigma_0\}$. Let Δ be the Laplacian operator on N_k. We use the assumption $M_k < 0$ to construct the conformal metric $\widetilde{g}$ as needed. First, let us calculate the asymptotically expansion of $\Delta\frac{1}{r}$ on $\mathbb{R}^3 \setminus B_{\sigma_0}(0)$. By (1.2), we know

$$\sqrt{g} = det^{\frac{1}{2}}(g_{ij}) = (1 + \frac{6M_k}{r} + \mathcal{O}(r^{-2}))^{\frac{1}{2}} = 1 + \frac{3M_k}{r} + \mathcal{O}(r^{-2})$$

$$g^{ij} = (1 - \frac{M_k}{2r})^4\delta_{ij} + \mathcal{O}(r^{-2}) = 1 - \frac{2M_k}{r}\delta_{ij} + \mathcal{O}(r^{-2})$$

So we have

$$\begin{aligned}
\Delta\frac{1}{r} &= \frac{1}{\sqrt{g}}\sum_{i,j=1}^{3}\frac{\partial}{\partial x^i}(\sqrt{g}g^{ij}\frac{\partial(\frac{1}{r})}{\partial x^j}) \\
&= (1 - \frac{3M_k}{r} + \mathcal{O}(r^{-2}))\sum_{i,j=1}^{3}\frac{\partial}{\partial x^i}[(1 + \frac{3M_k}{r} + \mathcal{O}(r^{-2}))((1 - \frac{2M_k}{r})\delta_{ij} + \mathcal{O}(r^{-2}))(-\frac{x^j}{r^3})] \\
&= \sum_{i=1}^{3}\frac{\partial}{\partial x^i}((-\frac{2M_k}{r} + \frac{3M_k}{r})(-\frac{x^j}{r^3})) + \mathcal{O}(r^{-5}) \quad (\text{since}\sum_{i=1}^{3}\frac{\partial}{\partial x^i}(\frac{x^i}{r^3}) = 0) \\
&= \frac{M_k}{r^4} + \mathcal{O}(r^{-5})
\end{aligned} \tag{3.1}$$

Since $M_k < 0$, so there exists a number $\sigma > \sigma_0$ such that $\Delta\frac{1}{r} < 0$ for $r > \sigma$. Using this fact, we can construct a metric whose scalar curvature is strictly positive outside a compact set on N_k. In fact, let $t_0 = -\frac{M_k}{8\sigma_0}$

and let $\zeta(t)$ be a C^5 function which satisfies

$$\zeta(t) = \begin{cases} t & t \leq t_0 \\ \frac{3t_0}{2} & t > 2t_0 \end{cases}$$
$$\zeta'(t) \geq 0, \quad \zeta''(t) \leq 0, \quad \text{for} \quad t \in (0, +\infty) \tag{3.2}$$

Define a C^5 function $\varphi(x) : N \to \mathbb{R}$ such that

$$\varphi = \begin{cases} 1 + \frac{3t_0}{2} & \text{on} \quad N \setminus N_k \\ 1 + \zeta(-\frac{M_k}{4r}) & \text{on} \quad \mathbb{R}^3 \setminus B_{\sigma_0}(0) = N_k. \end{cases}$$

Due to (3.1) and (3.2), we have

$$\begin{aligned} \Delta\varphi \leq 0 \qquad & \text{on} \quad N, \\ \Delta\varphi < 0 \qquad & \text{for} \quad r > 2\sigma. \end{aligned} \tag{3.3}$$

Then the function φ will give us the conformal metric needed. In fact, define $\widetilde{g} = \varphi^4 g$. Then

- $\widetilde{g}$ is asymptotically flat on N since it is a multiple on $N \setminus N_k$ since it is a constant multiple of g on them and it is asymptotically flat on N_k since
$$\begin{aligned} \widetilde{g}_{ij} &= (1 - \frac{M_k}{4r})^4 (1 + \frac{M_k}{2r})^4 \delta_{ij} + \mathcal{O}(r^{-2}) \\ &= (1 + \frac{M_k}{4r})^4 \delta_{ij} + \mathcal{O}(r^{-2}) \end{aligned}$$
on N_k when r is large
- N_k has new mass $\widetilde{M_k} = \frac{M_k}{2} < 0$;
- $\widetilde{R} \geq 0$ on N and $\widetilde{R} > 0$ for $r > 2\sigma$ on N_k (recall $\widetilde{R} = \varphi^{-5}[-8\Delta\varphi + R\varphi]$, see page 156 of [5]).

So we can always assume $R \geq 0$ on N and $R > 0$ outside a compact set of N_k by replacing R by $\widetilde{R}$.

- **Step2** We construct a complete minimal surface S properly embedded in N so that $S \cap (N \setminus N_k)$ is compact and $S \cap N_k$ lies between two parallel Euclidean 2-planes in the 3-space defined by x^1, x^2, x^3.

  First note that for $\sigma > 2\sigma_0$, let C_σ be the circle of radius σ lying in the $x^1 - x^2$ plane with center origin. From the theory of minimal surface (see chapter 4-6 in [1]), there is a smooth imbedded oriented area minimal surface S_σ among all competing surfaces regardless of topological type having boundary curve C_σ. Our plan is to prove some compactness result for the set $A = \{S_\sigma | \sigma > 2\sigma_0\}$ and extract a sequence $\sigma_i \to \infty$ such that S_{σ_i} converges to the required surface S.

– Claim1: there exists a compact set $K_0 \subset N$ so that

$$S_\sigma \cap (N \setminus N_k) \subset K_0 \qquad \text{for every} \quad \sigma > 2\sigma_0. \tag{3.4}$$

To prove this claim, we need first the lemma(see [8])

Lemma 3.1. *Let E be a convex set bounded by bounded convex surface H, suppose an P is an interior point of a connected minimal surface $S \subset E$ and is contained in H, then all of S is a subset of H.*

We use this lemma to prove claim1: Let $N_{k'}$ be any other end with an asymptotically flat coordinate system y^1, y^2, y^3 associating $N_{k'}$ with $\mathbb{R}^3 \setminus B_{\tau_0}$. In this coordinate system, the metric g has the form $g = g'_{ij} dy^i dy^j$ with g'_{ij} satisfying (1.2) . We calculate the Laplacian of $|y|^2$. By (1.2)

$$\begin{aligned}\Delta|y|^2 &= {g'}^{ij} \nabla_{ij} |y|^2 \\ &= (\delta_{ij} + \mathcal{O}(|y|^{-1}))(\frac{\partial^2 |y|^2}{\partial y^i y^j} - (\nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j})(|y|^2)) \\ &= (\delta_{ij} + \mathcal{O}(|y|^{-1}))(2\delta_{ij} - (\Gamma^k_{ij} \frac{\partial |y|^2}{\partial y^k})) \\ &= 6 + \mathcal{O}(|y|^{-1})\end{aligned}$$

So there exists some $\tau_1 > \tau_0$ such that $|y|^2$ is convex for $|y| > \tau_1$. Now fix a $\sigma_1 > 2\sigma_0$ and choose $\tau > \tau_1$ such that $S_{\sigma_1} \cap N_{k'} \subset B_\tau(0)$. Since $\partial S_\sigma = C_\sigma$ is contained in N_k which does not intersect $N_{k'}$, we know from Lemma 3.1 that $S_\sigma \cap N_{k'}$ will never contact $\partial B_\tau(0)$, thus is whole contained in $B_\tau(0)$ for any $\sigma > 2\sigma_0$. Since k' is arbitrary, we have established claim1.

– Claim 2: the height of S_σ is bounded, i.e. there exists a number $h > 0$ such that

$$S_\sigma \cap N_k \subset E_h \qquad \text{for every} \quad \sigma > 2\sigma_0 \tag{3.5}$$

where $E_h = \{x \in \mathbb{R}^3 || x^3| < h\}$.
Proof of claim2: We calculate the asymptotic behavior of Laplacian on $S_\sigma \cap N_k$ of x^3 and use maximum principle to estimate x^3. Suppose x^3 attains its maximum $\bar{h}$ on $S_\sigma \cap N_k$ at $x_0 \in S_\sigma$. If $\bar{h} \le \sigma_0$, then we are done. Now suppose $\bar{h} > \sigma_0$. The tangent space of S_σ at x_0 is spanned by $\frac{\partial}{\partial x^1}(x_0), \frac{\partial}{\partial x^2}(x_0)$. Extend $\frac{\partial}{\partial x^1}(x_0), \frac{\partial}{\partial x^2}(x_0)$ to be a frame field $\{v_1, v_2\}$ of TS_σ near x_0. Let $\bar{g}$ and $\bar{\nabla}$ be the induced metric and connection on S_σ respectively. Since

$$\begin{aligned}&\bar{\nabla}_{v_i} v_j = (\nabla_{v_i} v_j)^\top = \nabla_{v_i} v_j - g(\nabla_{v_i} v_j, \nu)\nu \\ &\bar{\nabla}_{ij} x^3 = v_i v_j (x^3) - (\bar{\nabla}_{v_i} v_j)(x^3) \\ &= v_i v_j (x^3) - (\nabla_{v_i} v_j)(x^3) + (g(\nabla_{v_i} v_j, \nu)\nu)(x^3) \\ &= \nabla_{ij} x^3 + (g(\nabla_{v_i} v_j, \nu)\nu)(x^3) \quad (\text{at } x_0) \\ &= \nabla_{ij} x^3 + (b_{ij}\nu)(x^3) \quad (\text{at } x_0)\end{aligned}$$

where ν is the unit normal vector field of S_σ, b_{ij} is the second fundamental form of S_σ. From (1.2), we know

$$\begin{aligned}\nabla_{ij}x^3 &= \frac{\partial x^3}{\partial x^i x^j} - (\nabla_{x_i}x_j)(x^3)\\ &= -\Gamma_{ij}^3\\ &= \frac{M_k x^j}{r^3}\delta_{i3} + \frac{M_k x^i}{r^3}\delta_{j3} - \frac{M_k x^3}{r^3}\delta_{ij} + \mathcal{O}(\frac{1}{r^3})\end{aligned} \tag{3.6}$$

Since S_σ is a minimal surface, we have

$$\bar{g}^{ij}b_{ij} = 0 \tag{3.7}$$

Thus, we get

$$\bar{\Delta}x^3 = \bar{g}^{ij}\bar{\nabla}_{ij}x^3 = -\frac{2M_k\bar{h}}{r^3} + \mathcal{O}(\frac{1}{r^3}) \tag{3.8}$$

Since $M_k < 0$, we have $\bar{\Delta}x^3 > 0$ at x_0 for $\bar{h}$ sufficiently large which contradicts the fact that x_0 is a maximum point of x^3. Similar argument gives an estimate of the minimum of x^3 on $S_\sigma \cap N_k$.

From (3.4) and (3.5) and regularity theory for minimal surface, we can find a sequence $\sigma_i \to \infty$ so that $S_{\sigma_i} \to S$, an imbedded C^2 surface, uniformly in C^2 norm on compact subset of N. Moreover, we know $S \cap (N \setminus N_k) \subset K_0$ is compact and $S \cap N_k \subset E_h$ which is the region between two parallel plane in $\mathbb{R}^3$.

- **Step3** Use asymptotically flat property and the condition on scalar curvature to derive a contradiction. We estimate $\int_S K$, where K is the Gauss curvature of S.

Let e_1, e_2, e_3 be a local orthonormal vector field on N. Let K_{ij} the sectional curvature of the section spanned by e_i, e_j. Choose e_1, e_2, e_3 so that e_1, e_2 is tangent to S and $e_3 = \nu$ normal to S. Let b be the second fundamental form of S, i.e. $b_{ij} = g(\nabla_{e_i}\nu, e_j)$. Then the condition that S is a minimal surface is

$$\mathrm{Trace}(b) = b_{11} + b_{22} = 0 \tag{3.9}$$

The seond variation inequality for S to be area minimizing is

$$\int_S f[\Delta f + (Ric(\nu) + \|A\|^2)f] \leq 0 \tag{3.10}$$

for any C^2 function f with compact support on S, where $\|A\|^2 = \sum_{i,j=1}^{2} b_{ij}^2$ is the length of the second fundamental form of S. Integration by parts, we get

$$\int_S (Ric(\nu) + \|A\|^2)f^2 \leq \int_S \|\nabla f\|^2 \tag{3.11}$$

for any C^2 function f with compact support on S (by approximation argument, (3.11) holds for any Lipschitz function f with compact support on S).From Gaussian equation and (3.9), we know

$$\begin{aligned} K &= K_{12} + b_{11}b_{22} - b_{12}^2 \\ &= R - K_{13} - K_{23} - \frac{1}{2}\|A\|^2 \\ &= R - Ric(\nu) - \frac{1}{2}\|A\|^2 \end{aligned} \tag{3.12}$$

Substitute this into (3.10), we get

$$\int_S (R - K + \frac{1}{2}\|A\|^2) f^2 \leq \int_S \|\nabla f\|^2 \tag{3.13}$$

Now, choose a suitable cutoff function for f to get our needed estimates. For $\sigma > \sigma_0$ define exhaustion sets $S_{(\sigma)} = [S \cap (N \setminus N_k)] \cup [S \cap B_\sigma(0)]$ and cutoff functions φ

$$\varphi = \begin{cases} 1 & \text{on} \quad S_{(\sigma)} \\ \frac{\log \frac{\sigma^2}{r}}{\log \sigma} & \text{on} \quad S_{(\sigma^2)} \setminus S_{(\sigma)} \\ 0 & \text{outside} \quad S_{(\sigma^2)}. \end{cases}$$

Let g be a Lipschitz funtion on S satisfying $|g| \leq 1$ and $g = 1$ outside a compact set of S (Here, we only use the case when $g \equiv 1$, while for the steps we omitted, the general g is used). Setting $f = \varphi g$ in (3.11) and applying the Cauchy-Schwarz inequality yields

$$\begin{aligned} \int_S (Ric(\nu) + \|A\|^2)\varphi^2 g^2 &\leq \int_S \|\varphi \nabla g + g \nabla \varphi\|^2 \\ &\leq 2\int_S \varphi^2 \|\nabla g\|^2 + \int_S g^2 \|\nabla \varphi\|^2 \\ &\leq 2\int_S \varphi^2 \|\nabla g\|^2 + \frac{2}{(\log \sigma)^2} \int_{S_{(\sigma^2)} \setminus S_{(\sigma)}} \frac{\|\nabla r\|^2}{r^2} \end{aligned}$$

Due to the asymptotically flat property (1.2), there is a constant C_1 with $\|\nabla r\|^2 \leq C_1$. Thus the above inequality implies that

$$\int_{S_{(\sigma^2)}} \|A\|^2 g^2 \leq 2\int_S \|\nabla g\|^2 + \frac{2C_1}{(\log \sigma)^2} \int_{S_{(\sigma^2)} \setminus S_{(\sigma)}} \frac{1}{r^2} + \int_S \|Ric\| g^2 \tag{3.14}$$

Now we estimate the second and third term on the right hand side.

First note that

$$Area(S_{(\sigma)}) \leq C_2 \sigma^2 \tag{3.15}$$

for some constant C_2 independent of $\sigma > \sigma_0$. To see this, note if S has transverse intersection with $\partial B_\sigma(0)$ then this intersection is a union of oriented C^2 Jordan curves on $\partial B_\sigma(0)$ which bounds $S_{(\sigma)}$. Since these curves also bound a domain $\Omega \subset \partial B_\sigma(0)$, so by the area minimizing property of S we have

$$Area(S_{(\sigma)}) \leq Area(\Omega) \leq Area(\partial B_\sigma(0))$$

Since (1.2) implies that dV_g is uniformly equivalently to the volume element in Euclidean metric on $\mathbb{R}^3 \setminus B_{\sigma_0}(0)$, there exists some constant C_2 such that

$$Area(\partial B_\sigma(0) \leq C_2\sigma^2$$

from which we get (3.15) when $S \cap \partial B_\sigma(0)$ is transverse. Since this is true for σ outside a set of measure zero, (3.15) follows for any $\sigma > \sigma_0$ by approximation. We can use (3.15) to estimate the second and third integrals in (3.14). For $a > 2$, by integrating by parts, we have

$$\begin{aligned}\int_S \frac{1}{1+r^a} &= \int_{S_{(\sigma_0)}} \frac{1}{1+r^a} + \int_{\sigma_0}^{+\infty} (\frac{d}{dt}\int_{S_{(\sigma_t)}} \frac{1}{1+r^a})dt \\ &\leq Area(S_{(\sigma_0)}) + \int_{\sigma_0}^{+\infty} \frac{1}{1+t^a}(\frac{d}{dt}Area(S_{(t)}))dt \\ &= Area(S_{(\sigma_0)}) + \frac{1}{1+t^a}Area(S_{(t)})|_{\sigma_0}^{+\infty} + \int_{\sigma_0}^{+\infty} \frac{at^{a-1}}{(1+t^a)^2}Area(S_{(t)})dt \\ &\leq C_2\sigma_0^2 + \int_{\sigma_0}^{+\infty} \frac{aC_2t^{a+1}}{(1+t^a)^2}dt \\ &< +\infty \end{aligned} \tag{3.16}$$

By (1.2), we have $Ric(\nu) = \mathcal{O}(\frac{1}{r^3})$, thus from the above results, we know

$$\int_S \|Ric\|g^2 \leq \int_S \|Ric\| \leq +\infty \tag{3.17}$$

Similarly, by integrating by parts we get

$$\int_{S_{(\sigma^2)} \setminus S_{(\sigma)}} \frac{1}{r^2} \leq 2C_2 \log \frac{\sigma^2}{\sigma} + C_3 \tag{3.18}$$

for some positive constant C_3 independent of σ. Thus, combine (3.14), (3.17), (3.18) and let $\sigma \to \infty$, we get

$$\int_S \|A\|^2 g^2 \leq 2\int_S \|\nabla g\|^2 + \int_S \|Ric\|g^2 < +\infty \tag{3.19}$$

for any Lipschitz g with $|g| \leq 1$ and $g \equiv 1$ outside a compact set of S. Take $g \equiv 1$ on S, we get $\int_S \|A\|^2 < +\infty$.

On the other hand, from (3.12), we know $|K| \leq |R| + |Ric| + \frac{1}{2}\|A\|^2$. Since $|R|$ and $|Ric|$ is of $\mathcal{O}(\frac{1}{r^3})$ by (1.2). We know from (3.16) that

$$\int_S |K| < +\infty \tag{3.20}$$

Moreover, by taking $f = \varphi$ in (3.13) and using dominate convergence theorem and (3.18), we get

$$\begin{aligned}
&\int_{S_{(\sigma)}} (R - K + \frac{1}{2}\|A\|^2) + \int_{S_{(\sigma^2)} \setminus S_{(\sigma)}} (R - K + \frac{1}{2}\|A\|^2) \frac{\log^2 \frac{\sigma^2}{r}}{\log^2 \sigma} \\
&\leq \frac{1}{\log^2 \sigma} \int_{S_{(\sigma^2)} \setminus S_{(\sigma)}} \frac{\|\nabla r\|^2}{r^2} \\
&\leq \frac{C_2}{\log^2 \sigma} \int_{S_{(\sigma^2)} \setminus S_{(\sigma)}} \frac{1}{r^2}
\end{aligned}$$

Letting $\sigma \to +\infty$ and use dominate convergence theorem and (3.18), we get

$$\int_S (R - K + \frac{1}{2}\|A\|^2) \leq 0$$

Since $R \geq 0$ and $R > 0$ outside a compact subset of S, we conclude

$$\int_S K > 0 \tag{3.21}$$

On the other hand, the Cohn-Vossen inequality says that $\displaystyle\int_S K < 2$, where $\chi(S)$ is the Euler characteristic of S. This together with (3.21) shows that S is topologically $\mathbb{R}^2$. From this and (3.20), the result in [2] says that S is conformally equivalent to the complex plane, i.e. there exists a conformal diffeomorphism $F : \mathbb{C} \to S$. Then by using the Gauss-Bonnet theorem with boundary and estimating the boundary terms, we can prove that

$$\int_S K \leq 0 \tag{3.22}$$

which contradicts to (3.21). Thus, we have $M_k \geq 0$. This completes the first part of Theorem 1.1.

4 Uniqueness of N

We now begin to prove the uniqueness part of Theorem1. Suppose (1.3) is satisfied and $M_k = 0$ for some end N_k. Then by the Corollary 3.1 of [6], we know any asymptotically flat metric satisfying $M_k = 0$, $R \geq 0$ must have $R \equiv 0$ on N. Define a family of metric

$$g(t) = \sum_{i,j=1}^{3} (g_{ij} + tS_{ij}) dx^i dx^j, \qquad t \in (-\epsilon, +\epsilon)$$

where S_{ij} is the Ricci curvature of g. Then $g(0) = g$ and $g(t)$ is asymptotically flat for t small since g satisfies (1.3). Let $R(t)$ be the scalar curvature of $g(t)$, so that $R(0) = R \equiv 0$. Differentiae $R(t)$ at $t = 0$, we get(see p228 of [4])

$$R'(0) = \frac{dR(t)}{dt}|_{t=0} = -\Delta_R + \nabla_i \nabla_j S^{ij} - \|Ric\|^2 \tag{4.1}$$

where $Ric = (S_{ij})$ is the Ricci tensor, Δ is the Laplacian operator on N, $\nabla_i \nabla_j S^{ij}$ is the sum of second covariant derivative of S^{ij}, $\|Ric\|^2 = S^{ij} S_{ij}$. Since $R \equiv 0$, we have $\Delta R \equiv 0$ and by contracting the Bianchi identity twice, we get $\nabla_i \nabla_j S^{ij} = 2\Delta R = 0$. Thus, we have

$$R'(0) = -\|Ric\|^2 \tag{4.2}$$

Since $R(0) \equiv 0$, we know from 1.3 that for t sufficiently small, we have

$$\frac{1}{8} \int_N (R(t)^-)^{\frac{3}{2}}))^{\frac{2}{3}} \leq \varepsilon_0$$

for some $\varepsilon_0 > 0$ constant independent of t for small t, where $R(t)^-$ =max$\{-R(t), 0\}$ is the negative part of $R(t)$. Thus, from Lemma 3.3 of [6], we can find a function φ_t so that the metric $\varphi_t^4 g(t)$ is asymptotically flat and scalar flat, the mass $M(t)$ of this metric is

$$M(t) = -\frac{1}{32\pi} \int_N R(t)\varphi(t)\sqrt{g(t)}dx \tag{4.3}$$

where $\sqrt{g(t)}dx = \sqrt{det(g_{ij}(t))}dx$ is the volume elemnet of $g(t)$. Due to (1.2) (1.3), we can use the estimate of Lemma3.2 of [6] to show that $M(t)$ is differentiable at $t = 0$ and we can differentiate it under the integral sign in (4.3) so that

$$M'(0) = -\frac{1}{32\pi} \int_N R(0)(\varphi(t)\sqrt{g(t)})'dx - \frac{1}{32\pi} \int_N R'(0)\varphi(0)\sqrt{g(0)}dx$$

Since $R(0) = R \equiv 0$ and $\varphi_0 \equiv 1$ (since by Lemma 3.3 of [6] that by φ_0 is unique). We may use (4.2) to conclude that

$$M'(0) = \frac{1}{32\pi} \int_N \|Ric\|^2 dV_0 \tag{4.4}$$

If Ric is not identically zero, (4.4) implies that $M'(0) > 0$. Hence $M(t_0) < 0$ for $t_0 < 0$ close to 0. However, since by construction of φ_t, the metric $\varphi_{t_0}^4 g(t_0)$ is asymptotically flat, scalar flat and this implies that $M(t_0) \geq 0$ by the first part of Theorem1. This contradiction shows that $Ric \equiv 0$ and this implies that g is flat since dim$N$=3. This completes the second part of Theorem 1.1.

5 References

References

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