Schoen-Yau’s proof of positive mass theorem

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Abstract
This is a project paper for Schoen and Yau’s classic proof of the positive mass theorem for dimension 3.

1 Introduction
Let \((M, \bar{g})\) be a space-time, the positive mass theorem states that if \(M\) has non-negative local mass density then the total mass of \(M\) as viewed from spatial infinity (the ADM mass) must be positive unless \(M\) is the flat Minkowski space-time. Mathematically, the theorem can be described as follows: Suppose \(N \hookrightarrow M\) is an oriented three dimensional space-like hypersurface in \(M\) with induced metric \(g\) and second fundamental form \(h_{ij}\). Then the local mass density \(\mu\) and current density \(J^i\) can be expressed as

\[
\mu = \frac{1}{2} [R - \sum_{i,j} h_{ij} h_{ij} + (\sum_i h_{ij})^2]
\]

\[
J^i = \sum_j \nabla_j [h_{ij} - (\sum_k h_{kj}) g^{ij}]
\]

where \(R\) is the scalar curvature of \(N\). We assume that \(\mu\) and \(J^i\) satisfies the dominant energy condition

\[
\mu \geq (\sum_i J^i J_i)^{\frac{1}{2}} \tag{1.1}
\]

Furthermore, we assume that \(g\) is asymptotically flat, i.e. there exists a compact subset \(K\) of \(N\) so that \(N \setminus K\) consists of finitely number of components \(N_1, N_2, \ldots, N_r\) with each \(N_k\) diffeomorphic to \(R^4\) minus a ball (we call each \(N_k\) the end of \(N\)). We also assume that \(g\) is of \(C^5\) and be asymptotically flat in the sense that each boundary component of \(N\) has positive mean curvature with respect to the outward normal, and each \(N_k\) admits a coordinate system \((x^1, x^2, x^3)\) in which \(g\) has the expansion \(g = g_{ij}dx^i dx^j\) with \(g_{ij}\) satisfying

\[
g_{ij} = (1 + \frac{M_k}{2r}) \delta_{ij} + h_{ij}, \quad |h_{ij}| \leq \frac{k_1}{1 + r^2},
\]

\[
|h_{ij}| \leq \frac{k_2}{1 + r^3}, \quad |\partial h_{ij}| \leq \frac{k_3}{1 + r^4}. \tag{1.2}
\]
where $r = \left(\sum_{i=1}^{3} (x^i)^2\right)^{\frac{1}{2}}$ and $\partial$ is the Euclidean gradient

$$M_k = \lim_{r \to \infty} \frac{1}{16\pi} \int_{S_r} \sum_{i,j} (\partial_j g_{ij} - \partial_i g_{jj}) d\sigma_i$$

is the total mass of $N_k$. Here we only consider the case when $\sum_i h_i^i = 0$ in this project survey, the general case can be reduced to this one (see [7]). First we note that, in the case $\sum_i h_i^i = 0$, the dominant energy condition (1.1) implies that $R \geq 0$. So, we can state the theorem as

**Theorem 1.1.** Let $g$ be a asymptotically flat metric on an oriented 3-manifold $N$. If $R \geq 0$ on $N$, then the total mass of each end is nonnegative. Furthermore, if in addition $h_{ij}$ satisfies

$$|\partial^3 h_{ij}| + |\partial^4 h_{ij}| + |\partial^5 h_{ij}| \leq \frac{k_4}{1 + r^5}$$

(1.3)

for some positive constant $k_4$ on an end $N_k$ and the total mass of $N_k$ is zero, then $g$ is flat, $N$ is isomorphic to $\mathbb{R}^3$ with stand metric.

### 2 Ideals of Proof

The proof of Theorem 1.1 can be divided into two parts. First, they prove the nonnegativity of the total mass by contradiction by using the minimal surface theory. Then they prove the uniqueness of $N$ with an evolution of the metric $g$. First, suppose some $N_k$ has negative mass $M_k < 0$, then the proof of nonnegativity of $M_k$ can be divided into three steps:

- **step 1:** replace the initial metric $g$ with a conformally equivalent metric $\tilde{g}$ which is still asymptotically flat and satisfies $\tilde{R} \geq 0$ on $N$, $\tilde{R} > 0$ outside a compact subset of $N_k$, and having negative total mass $N_k$.

- **step 2:** construct a complete area minimizing surface $M$ properly embedded in $N$ so that $S \cap (N \setminus N_k)$ is compact and $S \cap N_k$ lies between two parallel Euclidean 2-planes in the 3-space defined by $x^1, x^2, x^3$.

- **step 3:** use a result in [2] and the asymptotically flat property to show that the minimal surface constructed in step 2 does not exist to get a contradiction.

For the uniqueness of $N$, suppose $M_k = 0$ for some $k$, the proof also consists of three steps:

- **step 1:** use the assumption $M_k = 0$ and $R \geq 0$ to show that $N$ is scalar flat.
• step 2: define a family of metric \( g(t) \) on \( N \) by
\[
 g(t) = \sum_{i,j=1}^{3} (g_{ij} + tS_{ij}) dx^i dx^j,
\]
where \( S_{ij} \) is the Ricci curvature of \( g \). Use the condition that \( M_k = 0 \) and 
\( R \geq 0 \) to find an asymptotically flat metric \( \phi^4 g(t) \) with mass
\[
 M(t) = -\frac{1}{32\pi} \int_N R(t) \phi(t) \sqrt{g(t)} dx.
\]

• step 3: show that \( M'(0) = \frac{1}{32\pi} \int_N \|Ric\|^2 dx = 0 \) to prove that 
\( Ric \equiv 0 \).
Since \( \dim N = 3 \), this implies that \( g \) is the Euclidean metric.

3 None-negativity of \( M_k \)

**Step 1** Suppose that \( x^1, x^2, x^3 \) are asymptotically flat coordinates describing \( N_k \) on \( \mathbb{R}^3 \setminus B_{\sigma_0}(0) \), where \( B_{\sigma_0}(0) = \{ x \in \mathbb{R}^3 ||x| = (\sum_{i=1}^{3} (x^i)^2)^{\frac{1}{2}} < \sigma_0 \} \).

Let \( \Delta \) be the Laplacian operator on \( N_k \). We use the assumption \( M_k < 0 \) to construct the conformal metric \( \tilde{g} \) as needed. First, let us calculate the asymptotic expansion of \( \Delta \frac{1}{r} \) on \( \mathbb{R}^3 \setminus B_{\sigma_0}(0) \). By (1.2), we know
\[
 \sqrt{g} = det^\frac{1}{2}(g_{ij}) = (1 + \frac{6M_k}{r} + O(r^{-2}))^{\frac{1}{2}} = 1 + \frac{3M_k}{r} + O(r^{-2})
\]
\[
 g^{ij} = (1 - \frac{M_k}{2r})^4 \delta_{ij} + O(r^{-2}) = 1 - \frac{2M_k}{r} \delta_{ij} + O(r^{-2})
\]

So we have
\[
 \Delta \frac{1}{r} = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{3} \frac{\partial}{\partial x^i} (\sqrt{g} g^{ij} \frac{\partial(1)}{\partial x^j})
\]
\[
 = (1 - \frac{3M_k}{r} + O(r^{-2})) \sum_{i,j=1}^{3} \frac{\partial}{\partial x^i} ((1 + \frac{3M_k}{r} + O(r^{-2}))((1 - \frac{2M_k}{r}) \delta_{ij} + O(r^{-2}))(-\frac{x^j}{r^3}))
\]
\[
 = \sum_{i=1}^{3} \frac{\partial}{\partial x^i} ((-\frac{2M_k}{r} + \frac{3M_k}{r})(-\frac{x^i}{r^3}) + O(r^{-5}))
\]
\[
 = \frac{M_k}{r^4} + O(r^{-5})
\]

Since \( M_k < 0 \), so there exists a number \( \sigma > \sigma_0 \) such that \( \Delta \frac{1}{r} < 0 \) for 
\( r > \sigma \). Using this fact, we can construct a metric whose scalar curvature is strictly positive outside a compact set on \( N_k \). In fact, let \( t_0 = -\frac{M_k}{8\sigma_0} \).
and let $\zeta(t)$ be a $C^5$ function which satisfies

$$
\zeta(t) = \begin{cases} 
   \frac{t}{3t_0} & t \leq t_0 \\
   \frac{3t_0}{t} & t > 2t_0 
\end{cases}
$$

$$
\zeta'(t) \geq 0, \quad \zeta''(t) \leq 0, \quad \text{for } t \in (0, +\infty)
$$

(3.2)

Define a $C^5$ function $\varphi(x) : N \rightarrow \mathbb{R}$ such that

$$
\varphi = \begin{cases} 
   1 + \frac{3t_0}{2} & \text{on } N \setminus N_k \\
   1 + \zeta \left( \frac{-M_k}{4r} \right) & \text{on } \mathbb{R}^3 \setminus B_{\sigma_0}(0) = N_k.
\end{cases}
$$

Due to (3.1) and (3.2), we have

$$
\Delta \varphi \leq 0 \quad \text{on } N, \\
\Delta \varphi < 0 \quad \text{for } r > 2\sigma.
$$

(3.3)

Then the function $\varphi$ will give us the conformal metric needed. In fact, define $\widetilde{g} = \varphi^4 g$. Then

- $\widetilde{g}$ is asymptotically flat on $N$ since it is a multiple on $N \setminus N_k$ since it is a constant multiple of $g$ on them and it is asymptotically flat on $N_k$ since

  $$
  \widetilde{g}_{ij} = \left(1 - \frac{M_k}{4r}\right)^4 \left(1 + \frac{M_k}{2r}\right)^4 \delta_{ij} + \mathcal{O}(r^{-2})
  $$

  $$
  = \left(1 + \frac{M_k}{4r}\right)^4 \delta_{ij} + \mathcal{O}(r^{-2})
  $$

  on $N_k$ when $r$ is large

- $N_k$ has new mass $\widetilde{M_k} = \frac{M_k}{2} < 0$;

- $\widetilde{R} \geq 0$ on $N$ and $\widetilde{R} > 0$ for $r > 2\sigma$ on $N_k$ (recall $\widetilde{R} = \varphi^{-5}[-8\Delta \varphi + R \varphi]$, see page 156 of [5]).

So we can always assume $R \geq 0$ on $N$ and $R > 0$ outside a compact set of $N_k$ by replacing $R$ by $\widetilde{R}$.

**Step 2** We construct a complete minimal surface $S$ properly embedded in $N$ so that $S \cap (N \setminus N_k)$ is compact and $S \cap N_k$ lies between two parallel Euclidean 2-planes in the 3-space defined by $x^1, x^2, x^3$.

First note that for $\sigma > 2\sigma_0$, let $C_\sigma$ be the circle of radius $\sigma$ lying in the $x^1 - x^2$ plane with center origin. From the theory of minimal surface (see chapter 4-6 in [1]), there is a smooth imbedded oriented area minimal surface $S_\sigma$ among all competing surfaces regardless of topological type having boundary curve $C_\sigma$. Our plan is to prove some compactness result for the set $A = \{S_\sigma | \sigma > 2\sigma_0\}$ and extract a sequence $\sigma_i \rightarrow \infty$ such that $S_{\sigma_i}$ converges to the required surface $S$. 

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Claim 1: there exists a compact set \( K_0 \subset \sigma \) so that
\[
S_\sigma \cap (N \setminus N_k) \subset K_0 \quad \text{for every} \quad \sigma > 2\sigma_0.
\] (3.4)
To prove this claim, we need first the lemma (see [8])

**Lemma 3.1.** Let \( E \) be a convex set bounded by bounded convex surface \( H \), suppose an \( P \) is an interior point of a connected minimal surface \( S \subset E \) and is contained in \( H \), then all of \( S \) is a subset of \( H \).

We use this lemma to prove claim 1: Let \( N_{k'} \) be any other end with an asymptotically flat coordinate system \( y^1, y^2, y^3 \) associating \( N_{k'} \) with \( \mathbb{R}^3 \setminus B_{r_0} \). In this coordinate system, the metric \( g \) has the form \( g = g_{ij}^i dy^i dy^j \) with \( g_{ij} \) satisfying (1.2). We calculate the Laplacian if \( |y|^2 \). By (1.2)
\[
\Delta |y|^2 = g^{ij} \nabla_i |y|^2\nabla_j |y|^2
\]
\[
= (\delta_{ij} + O(|y|^{-1})) (\frac{\partial^2 |y|^2}{\partial y_i \partial y_j} - (\nabla_{\omega \rho} \frac{\partial}{\partial y_i \partial y_j})(|y|^2))
\]
\[
= (\delta_{ij} + O(|y|^{-1}))(2\delta_{ij} - (\Gamma^k_{ij} \frac{\partial |y|^2}{\partial y^k}))
\]
\[
= 6 + O(|y|^{-1})
\]
So there exists some \( \tau_1 > \tau_0 \) such that \( |y|^2 \) is convex for \( |y| > \tau_1 \). Now fix a \( \sigma_1 > 2\sigma_0 \) and choose \( \tau > \tau_1 \) such that \( S_{\sigma_1} \cap N_{k'} \subset B_{r_0}(0) \). Since \( \partial S_\sigma = C_\sigma \) is contained in \( N_k \) which does not intersect \( N_{k'} \), we know from Lemma 3.1 that \( S_{\sigma_1} \cap N_{k'} \) will never contact \( \partial B_{r_0}(0) \), thus is whole contained in \( B_r(0) \) for any \( \sigma > 2\sigma_0 \). Since \( k' \) is arbitrary, we have established claim 1.

Claim 2: the height of \( S_\sigma \) is bounded, i.e. there exists a number \( h > 0 \) such that
\[
S_\sigma \cap N_k \subset E_h \quad \text{for every} \quad \sigma > 2\sigma_0
\] (3.5)
where \( E_h = \{ x \in \mathbb{R}^3 \ | \ |x^3| < h \} \).

Proof of claim 2: We calculate the asymptotic behavior of Laplacian on \( S_\sigma \cap N_k \) of \( x^3 \) and use maximum principle to estimate \( x^3 \). Suppose \( x^3 \) attains its maximum \( \bar{h} \) on \( S_\sigma \cap N_k \) at \( x_0 \in S_\sigma \). If \( \bar{h} \leq \sigma_0 \), then we are done. Now suppose \( \bar{h} > \sigma_0 \). The tangent space of \( S_\sigma \) at \( x_0 \) is spanned by \( \frac{\partial}{\partial x^3}(x_0) \), \( \frac{\partial}{\partial x^2}(x_0) \). Extend \( \frac{\partial}{\partial x^3}(x_0) \), \( \frac{\partial}{\partial x^2}(x_0) \) to be a frame field \( \{ v_1, v_2 \} \) of \( TS_\sigma \) near \( x_0 \). Let \( \bar{g} \) and \( \bar{\nabla} \) be the induced metric and connection on \( S_\sigma \) respectively. Since
\[
\nabla_{v_i} v_j = (\nabla_{v_i} v_j)^T = \nabla_{v_i} v_j - g(\nabla_{v_i} v_j, \nu)\nu
\]
\[
\nabla_{ij} x^3 = v_i v_j (x^3) - (\nabla_{v_i} v_j)(x^3)
\]
\[
= v_i v_j (x^3) - (\nabla_{v_i} v_j)(x^3) + (g(\nabla_{v_i} v_j, \nu)\nu)(x^3)
\]
\[
= \nabla_{ij} x^3 + (g(\nabla_{v_i} v_j, \nu)\nu)(x^3) \quad \text{(at} x_0 \text{)}
\]
\[
= \nabla_{ij} x^3 + (b_{ij} \nu)(x^3) \quad \text{(at} x_0 \text{)}
\]
where $\nu$ is the unit normal vector field of $S_\sigma$, $b_{ij}$ is the second fundamental form of $S_\sigma$. From (1.2), we know
\[
\nabla_{ij}x^3 = \frac{\partial x^3}{\partial x^i x^j} - (\nabla x_i x_j)(x^3)
= -\Gamma^3_{ij}
= \frac{M_k x^j}{r^3} \delta_{i3} + \frac{M_k x^i}{r^3} \delta_{j3} - \frac{M_k x^3}{r^3} \delta_{ij} + O\left(\frac{1}{r^3}\right)
\]
(3.6)
Since $S_\sigma$ is a minimal surface, we have
\[
\bar{g}^{ij} b_{ij} = 0
\]
(3.7)
Thus, we get
\[
\bar{\Delta} x^3 = \bar{g}^{ij} \nabla_{ij} x^3 = -\frac{2M_k \bar{h}}{r^3} + O\left(\frac{1}{r^3}\right)
\]
(3.8)
Since $M_k < 0$, we have $\bar{\Delta} x^3 > 0$ at $x_0$ for $\bar{h}$ sufficiently large which contradicts the fact that $x_0$ is a maximum point of $x^3$. Similar argument gives an estimate of the minimum of $x^3$ on $S_\sigma \cap K_0$.

From (3.4) and (3.5) and regularity theory for minimal surface, we can find a sequence $\sigma_i \to \infty$ so that $S_{\sigma_i} \to S$, an imbedded $C^2$ surface, uniformly in $C^2$ norm on compact subset of $N$. Moreover, we know $S \cap (N \setminus N_k) \subset K_0$ is compact and $S \cap N_k \subset E_h$ which is the region between two parallel plane in $\mathbb{R}^3$.

**Step 3** Use asymptotically flat property and the condition on scalar curvature to derive a contradiction. We estimate $\int_S K$, where $K$ is the Gauss curvature of $S$.

Let $e_1, e_2, e_3$ be a local orthonormal vector field on $N$. Let $K_{ij}$ the sectional curvature of the section spanned by $e_i, e_j$. Choose $e_1, e_2, e_3$ so that $e_1, e_2$ is tangent to $S$ and $e_3 = \nu$ normal to $S$. Let $b$ be the second fundamental form of $S$, i.e. $b_{ij} = g(\nabla e_i \nu, e_j)$. Then the condition that $S$ is a minimal surface is
\[
\text{Trace}(b) = b_{11} + b_{22} = 0
\]
(3.9)
The second variation inequality for $S$ to be area minimizing is
\[
\int_S f [\Delta f + (Ric(\nu) + \|A\|^2)f] \leq 0
\]
(3.10)
for any $C^2$ function $f$ with compact support on $S$, where $\|A\|^2 = \sum_{i,j=1}^2 b_{ij}^2$ is the length of the second fundamental form of $S$. Integration by parts, we get
\[
\int_S (Ric(\nu) + \|A\|^2)f^2 \leq \int_S \|\nabla f\|^2
\]
(3.11)
for any $C^2$ function $f$ with compact support on $S$ (by approximation argument, (3.11) holds for any Lipschitz function $f$ with compact support on $S$). From Gaussian equation and (3.9), we know

$$K = K_{12} + b_{11}b_{22} - b_{12}^2$$

$$= R - K_{13} - K_{23} - \frac{1}{2}\|A\|^2$$

$$= R - \text{Ric}(\nu) - \frac{1}{2}\|A\|^2 \quad (3.12)$$

Substitute this into (3.10), we get

$$\int_S \left( R - K + \frac{1}{2}\|A\|^2 \right) f^2 \leq \int_S \|\nabla f\|^2 \quad (3.13)$$

Now, choose a suitable cutoff function for $f$ to get our needed estimates. For $\sigma > \sigma_0$ define exhaustion sets $S(\sigma) = [S \cap (N \setminus N_k)] \cup [S \cap B_\sigma(0)]$ and cutoff functions $\varphi$

$$\varphi = \begin{cases} 
1 & \text{on } S(\sigma) \\
\log \frac{\sigma^2}{\log \sigma} & \text{on } S(\sigma^2) \setminus S(\sigma) \\
0 & \text{outside } S(\sigma^2).
\end{cases}$$

Let $g$ be a Lipschitz function on $S$ satisfying $|g| \leq 1$ and $g = 1$ outside a compact set of $S$ (Here, we only use the case when $g \equiv 1$, while for the steps we omitted, the general $g$ is used). Setting $f = \varphi g$ in (3.11) and applying the Cauchy-Schwarz inequality yields

$$\int_S (\text{Ric}(\nu) + \|A\|^2) \varphi^2 g^2 \leq \int_S \|\varphi \nabla g + g \varphi \nabla g\|^2$$

$$\leq 2 \int_S \varphi^2 \|\nabla g\|^2 + \int_S g^2 \|\nabla \varphi\|^2$$

$$\leq 2 \int_S \varphi^2 \|\nabla g\|^2 + \frac{2}{(\log \sigma)^2} \int_{S(\sigma^2) \setminus S(\sigma)} \|\nabla r\|^2 \frac{\|\nabla \varphi\|^2}{r^2} \quad (3.14)$$

Due to the asymptotically flat property (1.2), there is a constant $C_1$ with $\|\nabla r\|^2 \leq C_1$. Thus the above inequality implies that

$$\int_{S(\sigma^2)} \|A\|^2 g^2 \leq 2 \int_S \|\nabla g\|^2 + \frac{2C_1}{(\log \sigma)^2} \int_{S(\sigma^2) \setminus S(\sigma)} \frac{1}{r^2} + \int_S \|\text{Ric}\|g^2 \quad (3.14)$$

Now we estimate the second and third term on the right hand side.

First note that

$$\text{Area}(S(\sigma)) \leq C_2 \sigma^2 \quad (3.15)$$

for some constant $C_2$ independent of $\sigma > \sigma_0$. To see this, note if $S$ has transverse intersection with $\partial B_\sigma(0)$ then this intersection is a union of
oriented $C^2$ Jordan curves on $\partial B_\sigma(0)$ which bounds $S(\sigma)$. Since these curves also bound a domain $\Omega \subset \partial B_\sigma(0)$, so by the area minimizing property of $S$ we have

$$\text{Area}(S(\sigma)) \leq \text{Area}(\Omega) \leq \text{Area}(\partial B_\sigma(0))$$

Since (1.2) implies that $dV_g$ is uniformly equivalently to the volume element in Euclidean metric on $\mathbb{R}^3 \setminus B_\sigma(0)$, there exists some constant $C_2$ such that

$$\text{Area}(\partial B_\sigma(0)) \leq C_2 \sigma^2$$

from which we get (3.15) when $S \cap \partial B_\sigma(0)$ is transverse. Since this is true for $\sigma$ outside a set of measure zero, (3.15) follows for any $\sigma > \sigma_0$ by approximation. We can use (3.15) to estimate the second and third integrals in (3.14). For $a > 2$, by integrating by parts, we have

$$\int_S \frac{1}{1 + r^a} = \int_{S(\sigma_0)} \frac{1}{1 + r^a} + \int_{\sigma_0}^{+\infty} \frac{d}{dt} \int_{S(\sigma)} \frac{1}{1 + r^a} \, dt$$

$$\leq \text{Area}(S(\sigma_0)) + \int_{\sigma_0}^{+\infty} \frac{1}{1 + t^a} (\frac{d}{dt} \text{Area}(S(t))) \, dt$$

$$= \text{Area}(S(\sigma_0)) + \int_{\sigma_0}^{+\infty} \frac{1}{1 + t^a} \text{Area}(S(t)) \, dt$$

$$\leq C_2 \sigma_0^2 + \int_{\sigma_0}^{+\infty} \frac{aC_2 t^{a+1}}{(1 + t^a)^2} \, dt$$

$$< +\infty \quad (3.16)$$

By (1.2), we have $\text{Ric}(\nu) = O(\frac{1}{t^2})$, thus from the above results, we know

$$\int_S \|\text{Ric}\| g^2 \leq \int_S \|\text{Ric}\| \leq +\infty \quad (3.17)$$

Similarly, by integrating by parts we get

$$\int_{S(\sigma_2) \setminus S(\sigma)} \frac{1}{r^2} \leq 2C_2 \log \frac{\sigma^2}{\sigma} + C_3 \quad (3.18)$$

for some positive constant $C_3$ independent of $\sigma$. Thus, combine (3.14), (3.17), (3.18) and let $\sigma \to \infty$, we get

$$\int_S \|A\|^2 g^2 \leq 2 \int_S \|\nabla g\|^2 + \int_S \|\text{Ric}\| g^2 < +\infty \quad (3.19)$$

for any Lipschitz $g$ with $|g| \leq 1$ and $g \equiv 1$ outside a compact set of $S$. Take $g \equiv 1$ on $S$, we get $\int_S \|A\|^2 < +\infty$. 

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On the other hand, from (3.12), we know $|K| \leq |R| + |Ric| + \frac{1}{2} \|A\|^2$. Since $|R|$ and $|Ric|$ is of $O(\frac{1}{r^3})$ by (1.2). We know from (3.16) that
\begin{equation}
\int_S |K| < +\infty \tag{3.20}
\end{equation}
Moreover, by taking $f = \varphi$ in (3.13) and using dominate convergence theorem and (3.18), we get
\begin{align*}
\int_{S(\sigma)} (R - K + \frac{1}{2} \|A\|^2) + \int_{S(\sigma_2) \setminus S(\sigma)} (R - K + \frac{1}{2} \|A\|^2) \frac{\log^2 \sigma^2}{\log^2 \sigma} \\
\leq \frac{1}{\log \sigma} \int_{S(\sigma_2) \setminus S(\sigma)} \frac{\|\nabla r\|^2}{r^2} \\
\leq \frac{C_2}{\log \sigma} \int_{S(\sigma_2) \setminus S(\sigma)} \frac{1}{r^2}
\end{align*}
Letting $\sigma \to +\infty$ and use dominate convergence theorem and (3.18), we get
\begin{equation}
\int_S (R - K + \frac{1}{2} \|A\|^2) \leq 0 \tag{3.21}
\end{equation}
Since $R \geq 0$ and $R > 0$ outside a compact subset of $S$, we conclude
\begin{equation}
\int_S K > 0 \tag{3.21}
\end{equation}
On the other hand, the Cohn-Vossen inequality says that $\int_S K < 2$, where $\chi(S)$ is the Euler characteristic of $S$. This together with (3.21) shows that $S$ is topologically $\mathbb{R}^2$. From this and (3.20), the result in [2] says that $S$ is conformally equivalent to the complex plane, i.e. there exists a conformal diffeomorphism $F : C \to S$. Then by using the Gauss-Bonnet theorem with boundary and estimating the boundary terms, we can prove that
\begin{equation}
\int_S K \leq 0 \tag{3.22}
\end{equation}
which contradicts to (3.21). Thus, we have $M_k \geq 0$. This completes the first part of Theorem 1.1.

## 4 Uniqueness of $N$

We now begin to prove the uniqueness part of Theorem 1. Suppose (1.3) is satisfied and $M_k = 0$ for some end $N_k$. Then by the Corollary 3.1 of [6], we know any asymptotically flat metric satisfying $M_k = 0$, $R \geq 0$ must have $R \equiv 0$ on $N$. Define a family of metric
\begin{equation}
g(t) = \sum_{i,j=1}^{3} (g_{ij} + t S_{ij}) dx^i dx^j, \quad t \in (-\epsilon, +\epsilon) \tag{9}
\end{equation}
where $S_{ij}$ is the Ricci curvature of $g$. Then $g(0) = g$ and $g(t)$ is asymptotically flat for $t$ small since $g$ satisfies (1.3). Let $R(t)$ be the scalar curvature of $g(t)$, so that $R(0) = R \equiv 0$. Differentiate $R(t)$ at $t = 0$, we get (see p228 of [4])

$$R'(0) = \left. \frac{dR(t)}{dt} \right|_{t=0} = -\Delta_R + \nabla_i \nabla_j S^{ij} - \|Ric\|^2$$

where $Ric = (S_{ij})$ is the Ricci tensor, $\Delta$ is the Laplacian operator on $N$, $\nabla_i \nabla_j S^{ij}$ is the sum of second covariant derivative of $S^{ij}$, $\|Ric\|^2 = S^{ij}S_{ij}$. Since $R \equiv 0$, we have $\Delta R \equiv 0$ and by contracting the Bianchi identity twice, we get $\nabla_i \nabla_j S^{ij} = 2\Delta R = 0$. Thus, we have

$$R'(0) = -\|Ric\|^2$$

(4.2)

Since $R(0) \equiv 0$, we know from 1.3 that for $t$ sufficiently small, we have

$$\frac{1}{8} \int_N (R(t^{-\frac{3}{2}}))^2 \, dx \leq \varepsilon_0$$

for some $\varepsilon_0 > 0$ constant independent of $t$ for small $t$, where $R(t) = -\max \{-R(t), 0\}$ is the negative part of $R(t)$. Thus, from Lemma 3.3 of [6], we can find a function $\varphi_t$ so that the metric $\varphi_t^4 g(t)$ is asymptotically flat and scalar flat, the mass $M(t)$ of this metric is

$$M(t) = -\frac{1}{32\pi} \int_N R(t) \varphi(t) \sqrt{g(t)} dx$$

(4.3)

where $\sqrt{g(t)} dx = \sqrt{\det(g_{ij}(t))} dx$ is the volume element of $g(t)$. Due to (1.2) (1.3), we can use the estimate of Lemma 3.2 of [6] to show that $M(t)$ is differentiable at $t = 0$ and we can differentiate it under the integral sign in (4.3) so that

$$M'(0) = -\frac{1}{32\pi} \int_N R(0)(\varphi(t) \sqrt{g(t)})' dx - \frac{1}{32\pi} \int_N R'(0) \varphi(0) \sqrt{g(0)} dx$$

Since $R(0) = R \equiv 0$ and $\varphi_0 \equiv 1$ (since by Lemma 3.3 of [6] that by $\varphi_0$ is unique). We may use (4.2) to conclude that

$$M'(0) = -\frac{1}{32\pi} \int_N \|Ric\|^2 dV_0$$

(4.4)

If $Ric$ is not identically zero, (4.4) implies that $M'(0) > 0$. Hence $M(t_0) < 0$ for $t_0 < 0$ close to 0. However, since by construction of $\varphi_t$, the metric $\varphi_0^4 g(t_0)$ is asymptotically flat, scalar flat and this implies that $M(t_0) \geq 0$ by the first part of Theorem 1. This contradiction shows that $Ric \equiv 0$ and this implies that $g$ is flat since $\dim N = 3$. This completes the second part of Theorem 1.1.
5 References

References


