MATH 599 PROBLEM SET 5

DUE THURSDAY APRIL 11

1. Consider the Lorentzian manifold $M = I \times \mathbb{R}^3$, equipped with the metric
$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2),$$
where $I \subset \mathbb{R}$ is an open interval, and $a(t) > 0$ is the so-called scale factor. This is one of the Robertson-Walker cosmological models. We assume that a perfect fluid with density $\rho = \rho(t) > 0$ and pressure $p = p(t) \geq 0$ is present. That is, we assume that M satisfies the Einstein field equations with the stress-energy tensor given in its *contravariant form* with respect to the coordinate frame $(\partial_t, \partial_x, \partial_y, \partial_z)$ by
$$S = \begin{pmatrix} \rho + p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}.$$
   (a) Show that the vector field $T = \partial_t \equiv -(\mathrm{d}t)^\#$ is a timelike geodesic congruence, and compute its vorticity ω, shear σ, and expansion scalar θ. We take T as future directed.
   (b) Write down the Raychaudhuri equation, and deduce that the scale factor $a(t)$ cannot be constant. Assuming that $\theta(c) > 0$ for some $c \in I$, and assuming that I is large enough, conclude that there is a point $t < c$ at which $\theta(t)$ becomes $-\infty$.
   (c) From the t-component of the contracted Bianchi identity, derive an equation for $\dot{\rho}$. Assuming the model $p = \kappa\rho$ with $\kappa > -1$ a constant, solve this equation for ρ in terms of a. Then substitute ρ into the Raychaudhuri equation, and obtain a solution $a(t)$.
   (d) Choose I as large as possible, so that $a(t)$ remains positive. Show that the resulting Lorentzian manifold M is inextendible, and that all causal geodesics are incomplete.
   (e) Compute the Kretschmann scalar and identify its singularities.
2. Let M be a Lorentzian manifold of dimension n, and let $\phi : \Sigma \to M$ be a spacelike submanifold of dimension $n-2$ embedded into M. We will identify $\phi(\Sigma) \subset M$ with Σ. Let $\gamma : [a, b] \to M$ be a null geodesic with $\gamma(a) = p \in \Sigma$, $\gamma(b) = q \in M \setminus \Sigma$, and $\gamma'(a) \perp T_p\Sigma$. Such a geodesic will be called a null geodesic *normal to* Σ. Then we define the *null second fundamental form* χ *of* Σ *at* p *relative to* $L = \gamma'(a)$ by
$$\chi(X, Y) = \langle \nabla_X Y, L \rangle,$$
for $X, Y \in \mathfrak{X}(\Sigma)$. Finally, a point $r = \gamma(s)$ is called *conjugate to* (or a *focal point of*) Σ along γ if there exists a nontrivial Jacobi field $X \in \mathfrak{X}^\perp(\gamma)$ along γ such that
   - $X(s) = 0$,
   - $X(a) \in T_p\Sigma$,
   - $\langle \nabla_L X, V \rangle = -\chi(X, V)$ for all $V \in T_p\Sigma$.

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Prove that if there exists a point $\gamma(s)$ conjugate to Σ along γ for some $s \in (a, b)$, then there exists a strictly timelike curve connecting Σ and q.

3. Let S be a subset of a time-oriented Lorentzian manifold M. Prove the following.
   (a) $I^+(I^+(S)) = I^+(S)$.
   (b) $I^+(\overline{S}) = \overline{I^+(S)}$.
   (c) $J^+(S) \subset \overline{I^+(S)}$.
   (d) $I^+(S) = \mathrm{int}(J^+(S))$.
4. Show that any extendable Lorentzian manifold (i.e., a properly and isometrically embedded submanifold of a Lorentzian manifold) contains an incomplete timelike geodesic.